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On natural approaches related to classical trigonometric inequalities

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Abstract: In this paper, we establish sharp inequalities for trigonometric functions. We prove in particular for $0 < x < \frac{\pi}{2}$ and any $n \geq 5$

$$0 < P_n(x) \ < \ (\sin x)^2 - x^3 \cot x < P_{n-1}(x) + \left[\left(\frac{2}{\pi}\right)^{2n} - \sum_{k=3}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n}$$

where $P_n(x) = \sum_{3=k}^{n} a_k x^{2k+1}$ is a n-polynomial, with positive coefficients ($k \geq 5$), $a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k}\right)$, B_{2k} are Bernoulli numbers. This improves a lot of lower bounds of $\frac{\sin(x)}{x}$ and generalizes inequalities chains. Moreover, bounds are obtained for other trigonometric inequalities as Huygens and Cusa inequalities as well as for the function

$$g_n(x) = \left(\frac{\sin(x)}{x}\right)^2 \left(1 - \frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1 - (\frac{2x}{\pi})^2}\right) + \frac{\tan(x)}{x}, \ n \geq 1$$

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1. Introduction

Inequalities involving trigonometric functions are used in many applications in various fields of mathematics. The method to compare functions to their corresponding Taylor polynomials has been successfully applied to prove and approximate a lot of trigonometric inequalities [1].

A method called the natural approach, introduced by Mortici in [2], uses the idea of comparing functions to their corresponding Taylor polynomials. This method has been successfully applied to prove and approximate a wide category of trigonometric inequalities.

Let us consider the double inequality

$$(\cos x)^{\frac{1}{3}} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}. \tag{1}$$

The left-hand side is known as Adamovic-Mitrinovic inequality (see [1,3]), while the right-hand side is known as Cusa inequality. The latter one which was proved by Huygens was used in order to estimate the number π, [3].

In this paper we provide another natural approach by comparing functions with their corresponding Taylor polynomials. This approach is analog to that given by [2]. As a corollary, that permits us to extend and sharpen results related to trigonometric inequalities and give generalizations and refinements.

In particular, the following inequalities have recently been established

Statement 1. [4, Theorem 1 p.4] for $0 < x < \pi/2$ one has

$$-\frac{x^4}{15}+\frac{23x^6}{1890}-\frac{41x^8}{37800} < \cos x-\left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15}+\frac{23x^6}{1890}-\frac{41x^8}{37800}+\frac{53x^{10}}{831600}. \quad (2)$$

Notice that this statement is finer that provided by Mortici [2] :

$$-\frac{x^4}{15} < \cos x-\left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15}+\frac{23x^6}{1890}.$$

Statement 2. [4, Theorem 3 p.7] for $0 < x < \pi/2$ one has

$$\frac{2\sin x}{x}+\frac{\tan x}{x} > 3+\left(\frac{3}{20}x^4-\frac{3}{140}x^6+\frac{3}{2240}x^8-\frac{1}{19800}x^{10}\right)(\cos x)^{-1}.$$

Statement 3. [5, Theorem] Neuman-Sandor for $0 < x < \pi/2$ one has

$$\frac{2\sin x}{x}+\frac{\tan x}{x} > \frac{2x}{\sin x}+\frac{x}{\tan x} > 3.$$

Statement 4. Cheng and Paris proved (Theorem 3.4 (3.23) of [6])

$$3+(\frac{3}{20}+\frac{1}{280}x^2+\frac{23}{33600}x^4)\,x^3\tan x < \frac{2\sin x}{x}+\frac{\tan x}{x}.$$

Statement 5. [4, Theorem 4 p.9] for $0 < x < \pi/2$ one has

$$2+\frac{1}{\cos x}\left(\frac{8x^4}{45}-\frac{4x^6}{105}+\frac{19x^8}{4725}-\frac{37x^{10}}{133650}\right) <$$

$$\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x} < 2+\frac{1}{\cos x}\left(\frac{8x^4}{45}-\frac{4x^6}{105}+\frac{19x^8}{4725}\right).$$

In this paper, we aim to refine all the inequalities mentioned above.

2. Main results

In [7] one proved for $0 < x < \frac{\pi}{2}$

$$\cos x+x^3\left(1-\frac{x^2}{63}\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 < \cos x+\frac{x^3\sin x}{15}. \quad (3)$$

In the sequel we provide lower bound of 1 which is finer than 3 and appears to be sharper than many known bounds .
WTo that end, we will consider a function

$$f(x)=(\sin x)^2-x^3\cot x-\frac{x^6}{15}+\frac{x^8}{945}$$

defined for $0 < x < \pi/2$ and we will provide good estimations.

2.1. Estimation for the function $f(x)$

At first, consider the following technical lemma which has been proved by [[8] , Lemma 3.1], but the proof we give here is slightly different.

Lemma 1. *For $0 < x < \pi/2$ consider the function*

$$f(x)=(\sin x)^2-x^3\cot x-\frac{x^6}{15}+\frac{x^8}{945}$$

then $f(x)$ can be expanded as power series $f(x) = \sum_{k\geq 5} a^k x^{2k}$,

$$a_k = \frac{2^{2k-2}\,|B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!} = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)\,k}\right),$$

where B_{2k} are the Bernoulli numbers. Moreover, the coefficients $a_k, k \geq 5$ are all positive:

$$a_5 = \frac{1}{255150},\ a_6 = \frac{2}{15436575},\ a_7 = \frac{103}{8300667375}, \dots$$

Proof. The following series expansions can be found in [9]

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}\mid B_{2k}\mid}{(2k)!} x^{2k-1}, \qquad x \in (0,\pi)$$

and

$$\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k}, \qquad x \in (0, \frac{\pi}{2})$$

$$(\sin x)^2 - x^3 \cot x = (\frac{1}{15}x^6 - \frac{1}{945}x^8 + \frac{1}{2835}x^{10} + \frac{8}{467775}x^{12} + \frac{206}{91216125}x^{14} + \frac{139}{638512875}x^{16} + \frac{10861}{488462349375}x^{18} + \frac{438628}{194896477400625}x^{20} + O\left(x^{22}\right)).$$

To prove the positivity of the coefficients a_k, k even we will use the following inequality for Bernoulli numbers established by D'Aniello [10]:

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} <\mid B_{2k} \mid< \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}.$$

For any odd value of k, we have

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)\,k}\right) = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{1}{(2k-1)\,k}\right) > 0.$$

Consider the even case, then

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| + \frac{-1}{(2k-1)\,k}\right).$$

By definition of Bernoulli numbers

$$S_n(p) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p+1-k}.$$

Then for k even

$$a_k = \frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}| - \frac{1}{(2k-1)\,k}\right) > \frac{2^{2k-2}}{(2k-2)!}\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)} - \frac{1}{(2k-1)\,k}\right)$$

and

$$a_k < \frac{2^{2k-2}}{(2k-2)!}\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)\,k}\right).$$

Therefore

$$\frac{2^{2k-2}\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)}}{(2k-2)!} + \frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!} < a_k = \frac{2^{2k-2}\,|B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!}$$
$$< \frac{2^{2k-2}\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)}}{(2k-2)!} + \frac{(-1)^{k+1}\,2^{2k-1}}{(2k)!}.$$

and
$$\sum_{k\geq 5}\left(\frac{2^{2k-1}}{2\pi^{2k-2}(2^{2k-2}-1)}+\frac{(-1)^{k+1}2^{2k-1}}{(2k)!}\right)x^{2k}<f(x).$$
Thus it implies the left hand of (4) since for any integer $n\geq 5$ we have
$$\cos x\leq \cos x+\sin x\,(\frac{x^3}{15}-\frac{x^5}{945})+\sin x\sum_{5\leq k\leq n}a_k x^{2k-3}$$
$$\leq \cos x+\sin x\,(\frac{x^3}{15}-\frac{x^5}{945})+\sin x\sum_{5\leq k\leq\infty}a_k x^{2k-3}=(\frac{\sin x}{x})^3.$$

Notice that
$$(\sin x)^2-x^3\cot x=(\sin x)^2-x^3\left(\frac{1}{x}-\sum_{k=1}^{\infty}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k-1}\right)=$$
$$(\sin x)^2-x^2+\sum_{k=1}^{\infty}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k+2}<(\sin x)^2-x^2+\sum_{k=1}^{\infty}\frac{2^{2k+1}x^{2k+2}}{\pi^{2k}\,(2^{2k}-2)}.$$
In the other hand we know that for $k>1$
$$(2k)!>\sqrt{4\pi k}\,(\frac{2k}{e})^{2k}\,e^{\frac{1}{24k+1}}.$$

It implies
$$\left(\frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)>\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{e})^{2k-2}e^{\frac{1}{48k-23}}}{\pi^{2k-2}(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)>$$
$$\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{\pi e})^{2k-2}}{(2^{2k-2}-1)}-\frac{1}{(2k-1)\,k}\right)>\left(\frac{2\sqrt{4\pi(k-1)}(\frac{2k-2}{\pi e})^{2k-2}}{2^{2k-2}}-\frac{1}{(2k-1)\,k}\right)>$$
$$\left(2\sqrt{4\pi(k-1)}(\frac{k-1}{\pi e})^{2k-2}-\frac{1}{(2k-1)\,k}\right).$$
Thanks to *Maple* we may easily verify that the last expression is non negative as soon as $k>6$. This means that a_k are non negative. □

Our first result Theorem 1 motivated us to further refine the Adamovic-Mitrinovic inequality. It permits us to deduce the lower bound of $(\frac{\sin x}{x})^3$ which appears to be finer than the corresponding bounds in Statement 1.

Theorem 1. *For* $0<x<\frac{\pi}{2}$ *for any* $n\geq 5$ *the following inequalities hold*
$$\sin x\sum_{k=5}^{n}a_k x^{2k-3}\;\leq\;(\frac{\sin x}{x})^3-\cos x-\sin x\,(\frac{x^3}{15}+\frac{x^5}{945})\leq$$
$$\sin x\sum_{k=5}^{n-1}a_k x^{2k-3}+\sin x\left[\left(\frac{2}{\pi}\right)^{2n}-\sum_{k=5}^{n-1}a_k\left(\frac{2}{\pi}\right)^{2n-2k}\right]x^{2n-3}\tag{4}$$
where
$$a_k=\frac{2^{2k-2}}{(2k-2)!}\left(|B_{2k-2}|+\frac{(-1)^{k+1}}{(2k-1)\,k}\right),\quad k\geq 5$$
and B_{2k} *are the Bernoulli numbers.*

Proof. By Lemma 1 the function
$$f(x)=(\sin x)^2-x^3\cot x-\frac{x^6}{15}+\frac{x^8}{945}$$

is positive. We also need the following Lemma which gives a upper and lower bound for the preceding function.

Lemma 2. *consider the real analytic functions f defined on the interval $[a,b]$:*

$$f(x)=\sum_{k=0}^{\infty} a_k(x-a)^k$$

where $a_k \in IR, a_k \geq 0$ for all $k \in IN$ Then $f(x)$ may be bounded by Taylor's approximation for any $n \geq 1$

$$\sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k \leq f(x) \leq \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k + \frac{1}{(b-a)^n} R_n(b)(x-a)^n,$$

where $R_n(x)=f(x)-\sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k$.

We may find an elegant proof of this Lemma in [15, Malesevic-Rasajski-Lutovac, Theorem 4] . Moreover, if we suppose in addition $f^{(n)}$ is increasing in $[a,b]$ we easily deduce

$$\sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k + \frac{f^{(n)}(a+)}{n!}(x-a)^n \leq f(x) \leq$$

$$\sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k + \frac{f^{(n)}(a+)}{n!}(x-a)^n + \frac{1}{(b-a)^n} R_n(b)(x-a)^n.$$

The coefficients $\frac{f^{(n)}(a+)}{n!}$ and $\frac{1}{(b-a)^n} R_n(b)$ are the best possible constants. This result may also be deduced from [8, Theorem 3.2]

Applying the preceding to the function $f(x)=(\sin x)^2-x^3\cot x=\sum_{k\geq 5} a_k x^{2k}$ we then derive for $0<x<\pi/2$ and $n \geq 5$ the following inequalities

$$\sum_{k\geq 5}^{n} a_k x^{2k} \leq f(x) \leq \sum_{k\geq 5}^{n-1} a_k x^{2k} + \left[\left(\frac{2}{\pi}\right)^{2n} - \sum_{k\geq 5}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n}$$

hold. Therefore,

$$\sin x \sum_{k\geq 5}^{n} a_k x^{2k-3} \leq f(x)\frac{\sin x}{x^3} \leq \sin x \sum_{k\geq 5}^{n-1} a_k x^{2k-3} +$$

$$\sin x \left[\left(\frac{2}{\pi}\right)^{2n} - \sum_{k\geq 5}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n-3}.$$

This means

$$\sin x \sum_{k\geq 5}^{n} a_k x^{2k-3} \leq \left(\frac{\sin x}{x}\right)^3 - \cos x - \frac{x^3 \sin x}{15} + \frac{x^5 \sin x}{945} \leq$$

$$\sin x \sum_{k\geq 5}^{n-1} a_k x^{2k-3} + \sin x \left[\left(\frac{2}{\pi}\right)^{2n} - \sum_{k\geq 5}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n-3}.$$

This proves the theorem. □

Some particular cases of Theorem 1 are given below. Let us introduce some examples of the inequalities obtained for $n=5,6,7,8,....$

Putting $n=5$, we obtain the following

Proposition 1. *For $0 < x < \pi/2$ the following inequalities hold*

$$\cos x + x^3\left(1 - \frac{x^2}{63} + \frac{x^4}{189}\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 <$$
$$\cos x + x^3\left(1 - \frac{x^2}{63}\right)\frac{\sin x}{15} + \frac{1024}{\pi^{10}}x^7 \sin x \tag{5}$$

Moreover, for $0 < x < \pi/2$ the following inequalities hold

$$\left[\frac{1+\cos x}{2}\right]^2 < \cos x + x^3\left(1 - \frac{x^2}{63} + \frac{x^4}{189}\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3. \tag{6}$$

Taking $n = 6$, one has for $0 < x < \pi/2$ the following inequality

$$\cos x + x^3\left(1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8}{31185}x^6\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 < \tag{7}$$

$$\cos x + x^3\left(1 - \frac{x^2}{63} + \frac{x^4}{189}\right)\frac{\sin x}{15} + \left(\frac{4096}{\pi^{12}} - \frac{4}{2835\pi^2}\right)x^9 \sin x.$$

Taking $n = 7$, one has for $0 < x < \pi/2$ the following inequality

$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525}\right)\right)\sin x < \frac{(\sin x)^3}{x^3} < \tag{8}$$
$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495}\right)\right)\sin x + \left(\frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{5\pi^2}\right)x^{11}\sin x.$$

Putting $n = 8$, one has for $0 < x < \pi/2$ the following inequality

$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} - \frac{139x^8}{675675}\right)\right)\sin x < \frac{(\sin x)^3}{x^3} < \tag{9}$$
$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525}\right)\right)\sin x +$$
$$\left(\frac{65536}{\pi^{16}} - \frac{64}{2825\pi^6} - \frac{128}{467775\pi^4} - \frac{824}{91216125\pi^2}\right)x^{13}\sin x.$$

Etc...

Remark 1. Notice that when the degree n of the polynomial increases, the function $\sin x \sum_{5\le k\le n} a_k x^{2k-3}$ approaches the upper bound $(\frac{\sin x}{x})^3 - \cos x - \frac{x^3}{15} + \frac{x^5}{945}$ since the coefficients $a_k > 0$. In other words, the precision increases with n and allows us to have a good estimate of the error.
Indeed, consider the difference

$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} + bx^4\right)\right)\sin x - \frac{(\sin x)^3}{x^3}$$

For $b = -\frac{8}{495}$ we have

$$\cos x + \frac{x^3}{15}\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495}\right)\right)\sin x - \frac{(\sin x)^3}{x^3}$$
$$< -\frac{206x^{12}}{91216125} + \frac{304x^{14}}{1915538625} - \frac{373x^{16}}{78153975900} < 0$$

For $a=\frac{1}{3}$, $b=-\frac{8}{495}$ and $c=-\frac{206}{96525}$ we get

$$\cos x+\frac{x^3}{15}\left(1-\frac{x^2}{63}\left(1-\frac{x^2}{3}+bx^4+cx^6\right)\right)\sin x-\frac{(\sin x)^3}{x^3}$$

$$<-\frac{139x^{14}}{638512875}+\frac{13723x^{16}}{976924698750}-\frac{559483x^{18}}{1559171819205000}<0$$

For $a=\frac{1}{3}$, $b=-\frac{8}{495}$, $c=-\frac{206}{96525}$, $d=-\frac{139}{675675}$ we get

$$\cos x+\frac{x^3}{15}\left(1-\frac{x^2}{63}\left(1-\frac{x^2}{3}-\frac{8x^4}{495}-\frac{206x^6}{96525}-\frac{139x^8}{675675}+dx^{10}\right)\right)\sin x-\frac{(\sin x)^3}{x^3}$$

$$<-\frac{10861x^{16}}{488462349375}+\frac{567257x^{18}}{389792954801250}-\frac{13763x^{20}}{359808881355000}<0$$

2.2. Another estimation for $f(x)$

The following result provided new bounds for the function $f(x)=(\sin x)^2-x^3\cot x$ It has been proved by [[8] - Theorem 3.3, p.10]

Lemma 3. *For $0<x<\pi/2$ and $n\geq 0$ the following inequalities hold*

$$(\sin x)\sum_{k=1}^{n}b_{k+2}x^{2k+1}<(\sin x)^2-x^3\cot x<(\sin x)\sum_{k=1}^{n-1}b_{k+2}x^{2k+1}+$$

$$\left[\left(\frac{2}{\pi}\right)^{2n+5}-\sum_{k=1}^{n-1}b_{k+2}\left(\frac{2}{\pi}\right)^{2n-2k}\right]x^{2n+3}\sin x$$

where

$$b_k=\frac{2(2k-1)(2k+1)(2^{2k-1}-1)\mid B_{2k}\mid+(-1)^k}{(2k+1)!}.$$

Moreover, all the coefficients b_k are non negative for $k\geq 2$.

Let $f(x)=(\sin x)^2-x^3\cot x$ then following [8] one has the expansion

$$f(x)=\left(\frac{1}{15}+\frac{19x^2}{1890}+\frac{167x^4}{113400}+\frac{479x^6}{2494800}+...\right)x^5\sin x.$$

Here is an improvement of this Lemma which will be useful to refine the bounds of the function.

Lemma 4. *For $0<x<\pi/2$ and $n\geq 0$ the following inequalities hold*

$$(sinx)\left[\sum_{k=1}^{n}b_{k+2}x^{2k+1}-\left(\frac{x^5}{15}-\frac{x^7}{945}\right)\left(1+\sum_{k=1}^{\infty}2\,\frac{(2^{2k-1}-1)\,|B_{2k}|\,x^{2k}}{(2k)!}\right)\right]<$$

$$(\sin x)^2-x^3\cot x-\frac{x^6}{15}+\frac{x^8}{945}<(sinx)\left[\sum_{k=1}^{n-1}b_{k+2}x^{2k+1}-\left(\frac{x^5}{15}-\frac{x^7}{945}\right)\left(1+\sum_{k=1}^{\infty}2\,\frac{(2^{2k-1}-1)\,|B_{2k}|\,x^{2k}}{(2k)!}\right)\right]+$$

$$\left[\left(\frac{2}{\pi}\right)^{2n+5}-\sum_{k=1}^{n-1}\left(\frac{2}{\pi}\right)^{2n-2k}\right]x^{2n+3}\sin x$$

where

$$b_k = \frac{2(2k-1)(2k+1)(2^{2k-1}-1) \mid B_{2k} \mid +(-1)^k}{(2k+1)!}.$$

Moreover, all the coefficients b_k *are non negative for* $k \geq 2$.

Indeed, since [[11], p.145]

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty} 2\, \frac{\left(2^{2\,k-1}-1\right) |B_{2k}|\, x^{2\,k-1}}{(2\,k)!}$$

this Lemma may be deduced from Lemma 1.5 in writing

$$-\frac{x^4}{15} + \frac{x^6}{945} = (\sin x)(-\frac{x^4}{15} + \frac{x^6}{945}) \left(\frac{1}{x} + \sum_{k=1}^{\infty} 2\, \frac{\left(2^{2\,k-1}-1\right) |B_{2k}|\, x^{2\,k-1}}{(2\,k)!} \right).$$

Now, we will proceed as above, we will use Taylor's approximation for this function to provide bounds for $\left(\frac{\sin x}{x}\right)^3 - \cos x$. By Lemma 3 we derive

Theorem 2. *For* $0 < x < \pi/2$ *the following inequalities hold*

$$(sinx)^2 \sum_{k=1}^{n} b_{k+2} x^{2k-2} < \frac{(\sin x)^3}{x^3} - \cos x < (sinx)^2 \sum_{k=1}^{n-1} b_{k+2} x^{2k-2} +$$

$$\left[\left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=1}^{n-1} b_{k+2} \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n} (\sin x)^2$$

where

$$b_k = \frac{2(2k-1)(2k+1)(2^{2k-1}-1) \mid B_{2k} \mid +(-1)^k}{(2k+1)!},$$

B_{2k} *are the Bernoulli numbers.*

2.3. Bounds of Adamovic-Mitrinovic inequalities

By Theorem 1 we are able to improve the left hand inequality 3. Indeed, one has the following

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < -\sin x\, P_n(x) < -\frac{x^3}{15}\left(1 - \frac{x^2}{63}\right) \sin(x) = -\sin x P_5(x)$$

where $P_n(x), n \geq 5$ is the n-polynomial

$$P_n(x) = \sum_{5 \leq k \leq n} a_k x^{2k-2} = \sum_{5 \leq k \leq n} \left(\frac{2^{2k-2}}{(2k-2)!} \left(\frac{(-1)^{k+1}}{(k)\,(2\,k-1)} + \mid B_{2k} \mid \right) x^{2k-2}.$$

Turn to statement 1. In [[4], Theorem 1] the authors proved for $0 < x < \pi/2$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} < \cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600}$$

The authors used the following frame

$$\sum_{k=2}^{2n} (-1)^k A(k) x^{2k} < \cos x - \left(\frac{\sin x}{x}\right)^3 < \sum_{k=2}^{2n+1} (-1)^k A(k) x^{2k},$$

where $A(k) = \frac{3^{2k+3}-32k^3-96k^2-88k-27}{4(2k+3)!}$.

By increasing the degree n of $P_n(x)$ in 1 , one can improve the precision so that the left bound obtained is better than the one provided by [4].

In this case, taking $n = 7$ is enough to prove statement 1

$$P_7(x) = \frac{x^3}{15}\left(1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185}\right)\sin x.$$

recall that we have for $x \in (0, \pi/2)$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}.$$

Then we obtain thanks to *Maple*

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} + \frac{x^3}{15}\left(1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185}\right)\sin x$$

$$> -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} + \frac{x^3}{15}\left(1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}\right) =$$

$$\frac{47x^{12}}{157172400} + \frac{19x^{14}}{261954000} - \frac{x^{16}}{294698250} = -\frac{x^{12}\left(-705 - 171\,x^2 + 8\,x^4\right)}{2357586000}.$$

It is easy to see that the last expression is non negative for $0 < x < \pi/2$.
Then for $x \in (0, \pi/2)$ the following inequalities hold

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} > -\frac{x^3}{15}\left(1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185}\right)\sin x >$$

$$\cos x - \left(\frac{\sin x}{x}\right)^3.$$

Thus, for the left hand our estimate appears to be finer than that provided by [4]. However, this is not the case for the right hand. indeed, we may verify that the expression is non positive

$$-\frac{x^3\sin x}{15} + \frac{x^5\sin x}{945} - \frac{x^7\sin x}{2835} - \left(\frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{467775\pi^2}\right)x^9\sin x +$$

$$\frac{x^4}{15} - \frac{23x^6}{1890} + \frac{41x^8}{37800} < 0,$$

which means that the following holds for $0 < x < \pi/2$

$$-\frac{x^3\sin x}{15} + \frac{x^5\sin x}{945} - \frac{x^7\sin x}{2835} - \left(\frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{467775\pi^2}\right)x^9\sin x <$$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} < \cos x - \left(\frac{\sin x}{x}\right)^3.$$

Other examples We interest here in the right part on the inequality. By the Taylor approximation we get the bound [[11], p.145]

$$(\sin x]^2 < x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + ... = \sum_{1}^{2n+1}(-1)^{k+1}\frac{2^{2k+1}x^{2k}}{(2k)!}.$$

Then Theorem 2 implies

$$\frac{(\sin x)^3}{x^3} - \cos x < (sinx)^2 \left[\sum_{k=1}^{n-1} b_{k+2} x^{2k-2} + \left[\left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=1}^{n-1} b_{k+2}\left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n}\right] <$$

$$(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - ...) \left[\sum_{k=1}^{n-1} b_{k+2} x^{2k-2} + \left[\left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=1}^{n-1} b_{k+2}\left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n}\right].$$

Putting $n = 4$ one gets thanks to *Maple*

$$\frac{(\sin x)^3}{x^3} - \cos x < -\frac{x^4}{15} + \frac{23x^6}{1890} - \left(-\frac{11}{28350} + \frac{2048}{\pi^{11}} - \frac{152}{945\pi^4} - \frac{167}{28350\pi^2}\right) x^8 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800}$$

since $-\left(-\frac{11}{28350} + \frac{2048}{\pi^{11}} - \frac{152}{945\pi^4} - \frac{167}{28350\pi^2}\right) + \frac{41x^8}{37800} = -0.0032400 < 0.$

Putting $n = 5$ one gets

$$\frac{(\sin x)^3}{x^3} - \cos x < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \left(\frac{29}{113400} - \frac{8192}{\pi^{13}} + \frac{608}{945\pi^6} + \frac{334}{14175\pi^4} + \frac{479}{623700\pi^2}\right) x^{10} <$$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600}$$

since

$$\left(\frac{29}{113400} - \frac{8192}{\pi^{13}} + \frac{608}{945\pi^6} + \frac{334}{14175\pi^4} + \frac{479}{623700\pi^2}\right) - \frac{53}{831600} = -0.0016403 < 0.$$

Thus Theorem 2 provides a finer bound than the one given by [4].

Remark 2. By the same way we may improve another bound of [4]: for $0 < x < \pi/2$

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}.$$

We must then use in Theorem 1 the polynomial $P_n(x)$ with a degree of order $n = 9$ to get a better estimate. Using *Maple* consider

$$\frac{x^3}{15}\sin x\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525}\right)\right) - \left(\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}\right) >$$

$$\frac{x^3}{15}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}\right)\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525}\right)\right) -$$

$$\left(\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}\right) =$$

$$-\frac{x^{12}}{5443200} - \frac{14929x^{14}}{70053984000} + \frac{197x^{16}}{12770257500} - \frac{103x^{18}}{229864635000} = -\frac{x^{12}\left(1351350 + 1567545\,x^2 - 113472\,x^4 + 3296\,x^6\right)}{7355668320000} > 0.$$

That means for $0 < x < \pi/2$ the following hold

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^3}{15}\sin x\left(1 - \frac{x^2}{63}\left(1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525}\right)\right) <$$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}.$$

A conjecture More generally, all the examples studied above naturally suggest that we may expect that the following inequalities hold

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^6 \sin x}{15} + \frac{x^8 \sin x}{945} - \sin x \sum_{k=5}^{n} a_k x^{2k-3} < \sum_{k=2}^{2n+1} (-1)^k A(k) x^{2k},$$

where

$$a_k = \frac{2^{2\,k-2}}{(2\,k-2)!}\left(|B_{2\,k-2}| + \frac{(-1)^{k+1}}{(2\,k-1)\,k}\right), \quad A(k) = \frac{3^{2k+3} - 32k^3 - 96k^2 - 88k - 27}{4(2k+3)!}.$$

We may also expect (following Theorem 2 that

$$\frac{(\sin x)^3}{x^3} - \cos x < (sinx)^2 \sum_{k=1}^{n-1} b_{k+2} x^{2k-2} +$$

$$\left[\left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=1}^{n-1} b_{k+2} \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n} (\sin x)^2 < - \sum_{k=2}^{2n} (-1)^k A(k) x^{2k}$$

hold.

Other refinements

We may prove the following frame which also improves the one of Mortici

$$-\frac{x^6}{945} + \frac{x^8}{1890} - \frac{x^{10}}{19800} < \frac{(\sin x)^3}{x^3} - \cos x - \frac{x^3}{15} \sin x$$

$$< -\frac{x^6}{945} + \frac{x^8}{1890} - \frac{x^{10}}{19800} + \frac{2903 x^{12}}{1135134000} < 0$$

The following are stronger

$$-\frac{x^8}{5670} + \frac{113 x^{10}}{1247400} - \frac{5293 x^{12}}{486486000} < \frac{(\sin x)^3}{x^3} - \cos x$$

$$-\frac{x^3}{15} \sin x \left(1 - \frac{x^2}{63} \cos x\right) < -\frac{x^8}{5670} + \frac{113 x^{10}}{1247400} < 0$$

For $a \leq 1$ we have

$$\frac{(\sin x)^3}{x^3} - \cos x - \frac{x^3 \sin x}{15} + \frac{x^5 \sin x}{945} - \frac{a x^7 \sin x}{2835}$$

$$> \left(\frac{1}{2835} - \frac{a}{2835}\right) x^8 + \left(-\frac{13}{311850} + \frac{a}{17010}\right) x^{10} + \left(\frac{571}{243243000} - \frac{a}{340200}\right) x^{12} > 0$$

For $a = 1$ we have

$$\frac{(\sin x)^3}{x^3} - \cos x - \frac{1}{15} x^3 \sin x + \frac{1}{945} x^5 \sin x - \frac{1}{2835} x^7 \sin x$$

$$> \frac{8 x^{10}}{467775} - \frac{2 x^{12}}{3378375} - \frac{31 x^{14}}{1915538625} + \frac{x^{16}}{724990500} - \frac{43789 x^{18}}{1039447879470000} > 0$$

In particular

$$\frac{8 x^{10}}{467775} - \frac{2 x^{12}}{3378375} > 0.$$

3. Huygens inequality

Notice that we have an equivalence between inequalities

$$\cos x < (\frac{\sin x}{x})^3 \iff 1 < (\frac{\sin x}{x})^2 \;\; \frac{\tan x}{x}, \qquad 0 < x < \pi/2.$$

By using the arithmetic-geometric mean inequality, Baricz and Sandor have pointed out that this inequality implies

$$\frac{2 \sin x}{x} + \frac{\tan x}{x} > 3 \quad \textit{and} \quad (\frac{\sin x}{x})^2 + \frac{\tan x}{x} > 2$$

for $0 < x < \pi/2$.

The following inequality which is due to Huygens

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > 3, \qquad x \in (0,\pi)(ii) \tag{10}$$

is a consequence of (1.1).
Mortici [2] showed

$$3 + \frac{3x^4}{20\cos x} - \frac{3x^6}{140\cos x} < \frac{2\sin x}{x} + \frac{\tan x}{x} < 3 + \frac{3x^4}{20\cos x}$$

which improves 9. Neuman-Sandor [5] proved the following

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \frac{2x}{\sin x} + \frac{x}{\tan x} > 3.$$

Cheng and Paris proved (Theorem 3.4 (3.23) of [6])

$$3 + (\frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4)\; x^3 \tan x < \frac{2\sin x}{x} + \frac{\tan x}{x}.$$

The following result improves the one of Mortici [[2],p.]

Theorem 3. *For* $0 < x < \pi/2$ *the following inequalities holds*

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x}\,(3 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8) >$$

$$3 + \frac{3x^4}{20\cos x} - \frac{3x^6}{140\cos x} > 3.$$

Proof. Notice that for $0 < x < \pi/2$ we have the expansion [[11] p.140]

$$\frac{1}{\cos x} = 1 + \sum_{k\geq 1} \frac{E_{2k}}{(2k)!} x^{2k} = 1 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + ...,$$

where E_{2k} are Euler numbers. Then we have obviously the left hand of inequalities since

$$\frac{1}{\cos x} > 1 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + ...$$

Write

$$\frac{2\sin x}{x} + \frac{\tan x}{x} = \frac{\sin x}{x}\left(2 + \frac{1}{\cos x}\right) = \frac{\sin x}{x}\left(2 + \sum_{k\geq 1} \frac{E_{2k}}{(2k)!} x^{2k}\right) >$$

$$\frac{\sin x}{x}\left(3 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277x^8}{144}\right).$$

□

We need the lemma

Lemma 5. *For* $0 < x < \pi/2$ *the following trigonometric inequality holds*

$$\frac{1}{\cos x} < \left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right)\left(1 - 2\,\frac{x}{\pi}\right)^{-1}.$$

Indeed, the Euler numbers verify the following frame, [12] [AS, p.805]

$$\frac{4^{k+1}}{\pi^{2n+1}(1 + 3^{-2n-1})} < \frac{E_{2k}}{(2k)!} < \frac{4^{k+1}}{\pi^{2n+1}}.$$

We then deduce

$$\frac{1}{\cos x} = 1 + \sum_{k\geq 1} \frac{E_{2k}}{(2k)!} x^{2k} < 1 + \sum_{k\geq 1} \frac{4^{k+1}}{\pi^{2n+1}} x^{2k} =$$

$$1 + \frac{2}{\pi} \sum_{k\geq 1} (\frac{x}{\pi})^{2k} = 1 + \frac{2}{\pi} \left[\frac{1}{1-\frac{2}{\pi}} - 1 \right] =$$

$$\left[1 + \frac{\frac{2}{\pi}}{1-\frac{2}{\pi}} - \frac{2}{\pi} \right] = \left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right) \left(1 - 2\,\frac{x}{\pi}\right)^{-1}.$$

On the other hand, recall that for $0 < x < \pi/2$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} < \sin x.$$

Therefore, thanks to *Maple* we can estimate the difference

$$\frac{\sin x}{x} \left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) - 3 - \frac{3x^4}{20\cos x} + \frac{3x^6}{140\cos x} >$$

$$\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) \left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) -$$

$$3 + \left(-\frac{3x^4}{20} + \frac{3x^6}{140}\right) \left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right) \left(1 - 2\,\frac{x}{\pi}\right)^{-1} =$$

$$\Big[\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) \left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) \left(1 - 2\,\frac{x}{\pi}\right) -$$

$$3\left(1 - 2\,\frac{x}{\pi}\right) + \left(-\frac{3x^4}{20} + \frac{3x^6}{140}\right) \left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right) \Big] \left(1 - 2\,\frac{x}{\pi}\right)^{-1} =$$

$$\frac{277}{362880}\,\frac{x^{15}}{\pi} - \frac{277}{725760}\,x^{14} - \frac{4817}{151200}\,\frac{x^{13}}{\pi} + \frac{4817}{302400}\,x^{12} + \frac{191363}{302400}\,\frac{x^{11}}{\pi} - \frac{191363}{604800}\,x^{10} -$$

$$\frac{7421}{2016}\,\frac{x^9}{\pi} + \frac{7421}{4032}\,x^8 + \left(-\frac{359}{360\pi} + \frac{3}{35\pi^2}\right) x^7 + \frac{359}{720}\,x^6 - \frac{3}{5}\,\frac{x^5}{\pi^2} \left(1 - 2\,\frac{x}{\pi}\right)^{-1} > 0$$

for $0 < x < \pi/2$.

The right inequality of Theorem 3 is then proved.

Turn now to statement 2

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > 3 + \left(\frac{3}{20}\,x^4 - \frac{3}{140}\,x^6 + \frac{3}{2240}\,x^8 - \frac{1}{19800}\,x^{10}\right) (\cos x)^{-1}.$$

Theorem 4. *For $0 < x < \pi/2$ the following inequalities hold*

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) >$$

$$3 + \left(\frac{3}{20}\,x^4 - \frac{3}{140}\,x^6 + \frac{3}{2240}\,x^8 - \frac{1}{19800}\,x^{10}\right) (\cos x)^{-1}.$$

Proof. Write the difference

$$h(x) = \frac{\sin x}{x} \left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) - 3 - \left(\frac{3x^4}{20} - \frac{3x^6}{140} + \frac{3x^8}{2240} - \frac{x^{10}}{19800}\right) \frac{1}{\cos x}.$$

By Lemma 5
$$\frac{1}{\cos x} < \left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right)\left(1 - 2\,\frac{x}{\pi}\right)^{-1}$$
implies
$$h(x) > \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^7}{5040}\right)\left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) -$$
$$3 - \left(\frac{3x^4}{20} - \frac{3x^6}{140} + \frac{3x^8}{2240} - \frac{x^{10}}{19800}\right)\left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right)\left(1 - 2\,\frac{x}{\pi}\right)^{-1} =$$
$$\left[\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^7}{5040}\right)\left(1 - 2\,\frac{x}{\pi}\right)\left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) -\right.$$
$$\left.3\left(1 - 2\,\frac{x}{\pi}\right) - \left(\frac{3x^4}{20} - \frac{3x^6}{140} + \frac{3x^8}{2240} - \frac{x^{10}}{19800}\right)\left(1 - 2\,\frac{x}{\pi} + 4\,\frac{x}{\pi^2}\right)\right]\left(1 - 2\,\frac{x}{\pi}\right)^{-1} =$$
$$\left(\frac{50521}{9144576000\pi} - \frac{50521}{4572288000\pi^2}\right)x^{17} - \frac{50521}{18289152000}\,x^{16} +$$
$$\left(\frac{115679}{217728000\pi} - \frac{115679}{108864000\pi^2}\right)x^{15} - \frac{115679}{435456000}\,x^{14} + \left(-\frac{42329}{1555200\pi} + \frac{42329}{777600\pi^2}\right)x^{13} +$$
$$\frac{42329}{3110400}\,x^{12} + \left(\frac{12072211}{19958400\pi} - \frac{1097657}{907200\pi^2}\right)x^{11} - \frac{12072211}{39916800}\,x^{10} + \left(-\frac{18539}{5040\pi} + \frac{7421}{1008\pi^2}\right)x^9 +$$
$$\frac{18539}{10080}\,x^8 + \left(-\frac{359}{360\pi} + \frac{481}{252\pi^2}\right)x^7 + \frac{359}{720}\,x^6 + \frac{3x^5}{5\pi^2} =$$
$$0.0000006391\,x^{17} - 0.0000027623\,x^{16} + 0.00006146\,x^{15} - 0.00026565\,x^{14} -$$
$$0.0031484\,x^{13} + 0.013609\,x^{12} + 0.06995\,x^{11} - 0.30243\,x^{10} - 0.42497\,x^9 +$$
$$1.8392\,x^8 - 0.12404\,x^7 + 0.49861\,x^6 + 0.060792\,x^5 > 0.$$

□

That means the following inequalities hold and implying statement 2
$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \sin x\left(3 + \frac{x^2}{2} + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8\right)x^{-1} >$$
$$3 + \left(\frac{3}{20}\,x^4 - \frac{3}{140}\,x^6 + \frac{3}{2240}\,x^8 - \frac{1}{19800}\,x^{10}\right)(\cos x)^{-1}.$$

Theorem 5. *For $0 < x < \pi/2$ the following inequalities holds*
$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x}\,(3 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8) > \frac{2x}{\sin x} + \frac{x}{\tan x} > 3.$$

Proof. We need this lemma

Lemma 6. *Consider the function*
$$g(x) = (\sin x)^2\left(3 + 1/2\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8\right) - 2\,x^2 - x^2\cos x$$
defined for $0 < x < \pi/2$.
Then $g(x)$ is non negative in this interval.

Indeed , since $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6!}$. *then we have*

$$g(x) > \left(x - \frac{x^3}{6} + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7\right)^2 \left(3 + \frac{x^2}{2} + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8\right)$$
$$-2\,x^2 - x^2\left(1 - \frac{1}{2}\,x^2 + \frac{1}{24}\,x^4 - \frac{1}{720}\,x^6\right) >$$
$$-\frac{374501}{604800}\,x^{12} + \frac{76129}{43200}\,x^{10} + \frac{761}{1680}\,x^8 + \frac{2}{15}\,x^6 > 0.$$

This implies that $\frac{g(x)}{x \sin x} > 0$. *Or equivalently*

$$\frac{\sin x}{x}\,(3 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8) - \left(\frac{2x}{\sin x} + \frac{x}{\tan x}\right) > 0.$$

This completes the proof.
□

Theorem 6. *For* $0 < x < \pi/2$ *the following inequalities holds*

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x}\,(3 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 + \frac{61}{120}\,x^6 + \frac{277}{144}\,x^8) >$$
$$3 + (\frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4)x^3\tan x.$$

Theorem 6 implies obviously statement 4.

Proof. For $0 < x < \pi/2$ recall that

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{9!} - \frac{x^{11}}{11!} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{9!},$$
$$1 - \frac{x^2}{2} + \frac{x^4}{4!} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Therefore since $\frac{1}{\cos x} \geq 1 + \frac{x^2}{2}$. we have the inequality thanks to *Maple*

$$\frac{\sin x}{x}\left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) - \left(3 + \left(\frac{3}{20} + \frac{x^2}{280} + \frac{23x^4}{33600}\right)x^3\tan x\right) =$$
$$\left[\frac{\sin 2x}{2x}\left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) - 3\cos x - \left(\frac{3}{20} + \frac{x^2}{280} + \frac{23x^4}{33600}\right)x^3\sin x\right]\frac{1}{\cos x} >$$
$$\frac{1}{2x}\left(2\,x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \frac{4x^9}{2835} - \frac{8x^{11}}{155925}\right)\left(3 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{120} + \frac{277x^8}{144}\right) -$$
$$3 + \frac{3x^2}{2} - \left(\frac{3}{20} + \frac{x^2}{280} + \frac{23x^4}{33600}\right)x^3\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}\right) =$$
$$-\frac{277}{5613300}\,x^{18} + \frac{108154933}{80472268800}\,x^{16} - \frac{6306127}{261954000}\,x^{14} + \frac{1048466449}{4191264000}\,x^{12} -$$
$$\frac{24287083}{19958400}\,x^{10} + \frac{97187}{60480}\,x^8 + \frac{151}{360}\,x^6 + \frac{x^4}{8} > 0.$$

Theorem 5 is then proved. □

4. Wilker inequality

The following inequality

$$\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}>2, \qquad x\in(0,\frac{\pi}{2})$$

due to Wilker [13] was intensively studied by many authors, e.g, [14–16]

Mortici [4] proved

$$2+\left(\frac{8}{45}x^4-\frac{8}{105}x^6\right)\left(\frac{1}{\cos x}\right)<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}<2+\left(\frac{8x^4}{45\cos(x)}\right).$$

Theorem 7. *For $0<x<\pi/2$ the following inequalities holds for $1\le m\le n$ and $p\le n$*

$$\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{n}a^kx^{2k-2}\right)\times$$

$$\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right)<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}<$$

$$\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}+\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^kx^{2k-2}\right)\times$$

$$\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right).$$

where B_{2k} are the Bernoulli numbers.

Proof. Remark at first we may write obviously

$$\left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin(2x)}\right)=\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

Lemma 7. *For $0<x<\pi/2$ the following inequalities holds for any integer $m\ge 5$*

$$\left(\frac{\sin(x)}{x}\right)^2<1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}+\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^kx^{2k-2},$$

$$1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{n}a^kx^{2k-2}<\left(\frac{\sin(x)}{x}\right)^2.$$

By Lemma 1 and Theorem 1 we may deduce that

$$\frac{f(x)}{x^2}=\left(\frac{\sin(x)}{x}\right)^2-x\cot x-\frac{x^4}{15}+\frac{x^6}{945}=\sum_{k\ge 5}a^kx^{2k-2},$$

where

$$a_k=\frac{2^{2\,k-2}}{(2\,k-2)!}\left(|B_{2\,k-2}|+\frac{(-1)^{k+1}}{(2\,k-1)\,k}\right).$$

Thus for any integer $n\ge 5$

$$\left(\frac{\sin(x)}{x}\right)^2=x\cot x+\frac{x^4}{15}-\frac{x^6}{945}+\sum_{k\ge 5}a^kx^{2k-2}\ge x\cot x+\frac{x^4}{15}-\frac{x^6}{945}+\sum_{k=5}^{n}a^kx^{2k-2}$$

since coefficients $a_k > 0$..
Therefore since by [[9], p.]

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - .... \sum_{k=n}^{\infty} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1}, \qquad x \in (0, \pi)$$

and by [D'Agnello]

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} <\mid B_{2k} \mid< \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}. \tag{11}$$

Then we deduce the inequalities

$$\sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k-1} < \cot x - \frac{1}{x} + \sum_{k=1}^{n} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k-1} < \sum_{k=n+1}^{\infty} \frac{-2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k-1}.$$

It follows for any integer $m \geq 5$

$$\left(\frac{\sin(x)}{x}\right)^2 < 1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)} x^{2k} + \sum_{k=1}^{m} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=3}^{\infty} a^k x^{2k-2},$$

$$1 - \sum_{k=m+1}^{\infty} \frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)} x^{2k} - \sum_{k=1}^{m} \frac{2^{2k} \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=3}^{n} a^k x^{2k-2} < \left(\frac{\sin(x)}{x}\right)^2.$$

Let us consider now expansions trigonometric functions with power series. We will use the Taylor expansions of $\sin(x)$,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where $0 < \theta < 1$. *It is easy to remark that*

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

for $0 < x < \frac{\pi}{2}$. *We then deduce bounds for* $\left(\frac{\sin x}{x}\right)^2$

$$\sum_{k=1}^{2p} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2} < \left(\frac{\sin x}{x}\right)^2 < \sum_{k=1}^{2p+1} (-1)^{k+1} \frac{2^{2k+1}}{(2k)!} x^{2k-2}.$$

On the other hand, we know that

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + ... = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2) \mid B_{2k} \mid}{(2k)!} x^{2k-1}.$$

We then derive the following inequalities for any $n \geq 1$
□

Lemma 8. *For* $0 < x < \pi/2$ *the following inequalities holds for any integer* $p \geq 1$

$$2 + \sum_{k=1}^{p} \frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!} x^{2k} + \sum_{k=p+1}^{n} \frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)} x^{2k} < 1 + \frac{2x}{\sin 2x} <$$

$$2 + \sum_{k=1}^{p} \frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!} x^{2k} + 2\left(\frac{2x}{\pi}\right)^{2p+2} \frac{1}{1 - (\frac{2x}{\pi})^2}.$$

Indeed,

$$1 + \frac{2x}{\sin 2x} = 2 + \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!} x^{2k} > 2 + \sum_{k=1}^{n} \frac{2^{2k}(2^{2k}-2) \mid B_{2k} \mid}{(2k)!} x^{2k}.$$

Then D'Agnolo inequalities (..)

$$2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}<1+\frac{2x}{\sin 2x}<$$

$$2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\sum_{k=p+1}^{\infty}\frac{2^{2k}}{\pi^{2k}}x^{2k}=$$

$$2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}.$$

We may also prove the following frame

$$\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}<1+\frac{2x}{\sin 2x}<\frac{2}{1-(\frac{2x}{\pi})^2}.$$

Finally, we get a lower bound for the product

$$\left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin 2x}\right)>\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-2)}x^{2k}-\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{n}a^kx^{2k-2}\right)\times$$

$$\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right).$$

The upper bound is

$$\left(\frac{\sin(x)}{x}\right)^2\left(1+\frac{2x}{\sin 2x}\right)<\left(1-\sum_{k=m+1}^{\infty}\frac{2^{2k+1}}{\pi^{2k}(2^{2k}-1)}x^{2k}+\sum_{k=1}^{m}\frac{2^{2k}\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=3}^{\infty}a^kx^{2k-2}\right)\times$$

$$\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right).$$

Theorem 7 is then proved.

Examples Let $0<x<\pi/2$. By Theorem 7 we are able to precise the lower bound of the Huygens inequality in putting different values of n,p.

- Taking $n=3$ and $p=2$ we find again a result of [2]

$$2+\frac{8x^4}{45}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

- Taking $n=4$ and $p=3$ we find again

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

- Taking $n=5$ and $p=3$ we have

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

- Taking $n=6$ and $p=4$ we have

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

This permits to find again statement 5 a result of [[4], p.9]. indeed, since (see lemma 9 below) for $0<x<\pi/2$

$$\cos x>1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^10}{10!}+\frac{x^12}{12!}-\frac{x^14}{14!}$$

we then deduce

$$(\cos x)\left(2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}\right)>\frac{8}{45}\,x^4-\frac{4}{105}\,x^6+\frac{19}{4725}\,x^8-\frac{37}{133650}\,x^{10}.$$

Taking $n=7$ and $p=4$ we obtain

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}+\frac{152912x^{12}}{42567525}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

By the same way, using again the lower of $\cos x$ we find a result of [4, p.10]

$$(\cos x)\left(2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592x^{10}}{66825}+\frac{152912x^{12}}{42567525}\right)>$$

$$\frac{8}{45}\,x^4-\frac{4}{105}\,x^6+\frac{19}{4725}\,x^8-\frac{37}{133650}\,x^{10}+\frac{283}{20638800}\,x^{12}-\frac{3503}{6810804000}\,x^{14}.$$

Etc...

In the sequel we will find upper and lower bounds of Huygens inequalities which appear to be finer than known previous. Consider at first

Lemma 9. *For $0<x<\pi/2$ the following inequalities holds for any integer $p\geq 1$*

$$\sum_{k=1}^{2p}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}<\left(\frac{\sin x}{x}\right)^2<\sum_{k=1}^{2p+1}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2},$$

$$\sum_{k=1}^{2p+1}(-1)^{k}\frac{1}{(2k)!}x^{2k}<\cos x<\sum_{k=1}^{2p}(-1)^{k}\frac{1}{(2k)!}x^{2k}.$$

Let us consider expansions trigonometric functions with power series. We will use the Taylor expansions of $\sin x,\ \cos x$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+....+(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}+(-1)^k\frac{\sin\theta x}{(2k+1)!}x^{2k+1},$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+....+(-1)^k\frac{x^{2k}}{(2k)!}+(-1)^{k+1}\frac{\cos\theta x}{(2k+2)!}x^{2k+2}$$

where $0<\theta<1$. It is easy to remark that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}<\frac{\sin x}{x}<1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}$$

for $0<x<\frac{\pi}{2}$. We then deduce bounds for $\left(\frac{\sin x}{x}\right)^2\cos x$

$$\sum_{k=1}^{2p}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}<\left(\frac{\sin x}{x}\right)^2<\sum_{k=1}^{2p+1}(-1)^{k+1}\frac{2^{2k+1}}{(2k)!}x^{2k-2}.$$

$$\sum_{k=1}^{2p+1}(-1)^{k}\frac{1}{(2k)!}x^{2k}<\cos x<\sum_{k=1}^{2p}(-1)^{k}\frac{1}{(2k)!}x^{2k}.$$

We then derive the following which improves Theorem 7

Theorem 8. *For $0 < x < \pi/2$ the following inequalities holds for any $q \geq 1, 1 \leq p \leq n$*

$$\left(\sum_{k=1}^{2q}(-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k-2}\right)\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\sum_{k=p+1}^{n}\frac{2^{2k+1}(2^{2k}-2)}{\pi^{2k}(2^{2k}-1)}x^{2k}\right)$$

$$<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}<$$

$$\left(\sum_{k=1}^{2q+1}(-1)^{k+1}\frac{2^{2k-1}}{(2k)!}x^{2k-2}\right)\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+2\left(\frac{2x}{\pi}\right)^{2p+2}\frac{1}{1-(\frac{2x}{\pi})^2}\right),$$

where B_{2k} are the Bernoulli numbers.

Corollary 1. *For $0 < x < \pi/2$ the following inequalities holds for any $n, p, \quad 1 \leq p \leq n$*

$$\left(\frac{\sin(x)}{x}\right)^2\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}\right)<$$

$$\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}<$$

$$\left(\frac{\sin(x)}{x}\right)^2\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}+\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right).$$

where B_{2k} are the Bernoulli numbers.

Corollary 1 means that the following inequalities hold

$$\left(\frac{\sin(x)}{x}\right)^2\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}-2\sum_{k=1}^{n}\frac{1}{2^k-1}\left(\frac{2x}{\pi}\right)^{2k}\right)<$$

$$g_n(x)=\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^{2n+2}}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}<$$

$$\left(\frac{\sin(x)}{x}\right)^2\times\left(2+\sum_{k=1}^{p}\frac{2^{2k}(2^{2k}-2)\mid B_{2k}\mid}{(2k)!}x^{2k}\right)<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}.$$

The function $g_n(x)$ is growing as n increasing. We have

$$\left(\frac{\sin(x)}{x}\right)^2\left(1-\frac{2\left(\frac{2x}{\pi}\right)^2}{1-(\frac{2x}{\pi})^2}\right)+\frac{\tan(x)}{x}<g_n(x)<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}.$$

Moreover, we may compute the limit when x tends to $\frac{\pi}{2}$

$$lim_{x\to\frac{\pi}{2}}g_n(x)=\frac{2(5+4n)}{\pi^2}.$$

Examples Let $0 < x < \pi/2$. - Taking $n = 3$ and $p = 2$ we find

$$2+\frac{8x^4}{45}<\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-\left(\frac{\sin x}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^8}{1-(\frac{2x}{\pi})^2}\right)<$$

$$\frac{2306x^{10}}{467775}-\frac{2x^8}{63}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<\frac{16x^6}{315}+\frac{8x^4}{45}+2.$$

- Taking $n = 4$ and $p = 3$ we find

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}-\left(\frac{\sin(x)}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^{10}}{1-(\frac{2x}{\pi})^2}\right)<$$

$$\frac{61232x^{12}}{30405375}-\frac{868x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2.$$

Taking $n = 5$ and $p = 3$ we obtain

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}-\left(\frac{\sin(x)}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^{12}}{1-(\frac{2x}{\pi})^2}\right)<$$

$$\frac{1566172x^{14}}{1915538625}-\frac{480604x^{12}}{91216125}+\frac{592x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<$$
$$\frac{592x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2.$$

We then improve statement 5 ([4, p.9]) since by Lemma 9

$$\left(\frac{8}{45}x^4+\frac{16}{315}x^6+\frac{104}{4725}x^8+\frac{592}{66825}x^{10}\right)\cos(x)<\frac{8}{45}x^4-\frac{4}{105}x^6+\frac{19}{4725}x^8$$

$$\left(\frac{8}{45}x^4+\frac{16}{315}x^6+\frac{104}{4725}x^8+\frac{592}{66825}x^{10}+\frac{152912}{42567525}x^{12}\right)\cos(x)>$$
$$\frac{8}{45}x^4-\frac{4}{105}x^6+\frac{19}{4725}x^8-\frac{37}{133650}x^{10}+\frac{283}{20638800}x^{12}.$$

Taking $n = 6$ and $p = 3$ we obtain

$$2+\frac{8x^4}{45}+\frac{16x^6}{315}+\frac{104x^8}{4725}+\frac{592}{66825}x^{10}<\left(\frac{\sin(x)}{x}\right)^2+\frac{\tan(x)}{x}-\left(\frac{\sin(x)}{x}\right)^2\left(\frac{2\left(\frac{2x}{\pi}\right)^{14}}{1-(\frac{2x}{\pi})^2}\right)<$$

$$\frac{161934166x^{16}}{488462349375}-\frac{123992x^{14}}{58046625}+\frac{152912x^{12}}{42567525}+\frac{592x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2<$$
$$\frac{152912x^{12}}{42567525}+\frac{592x^{10}}{66825}+\frac{104x^8}{4725}+\frac{16x^6}{315}+\frac{8x^4}{45}+2$$

The last estimate improves ([4, p.10]).

Etc...

Wu and Srivastava [15, Lemma 3] proved the following dual inequality

$$(\frac{x}{\sin(x)})^2+\frac{x}{\tan(x)}>2, \qquad 0<x<\frac{\pi}{2}$$

Mortici [4] proved

$$(\frac{x}{\sin(x)})^2+\frac{x}{\tan(x)}>2+\frac{2x^4}{45}.$$

$$(\frac{x}{\sin(x)})^2+\frac{x}{\tan(x)}>\frac{2\sin(x)}{x}+\frac{\tan(x)}{x}$$

Mortici refined a result of Neuman and Sandor [5, Theorem 2.3], who showed

$$\frac{3x}{\sin x}+\cos x>4, \qquad 0<x<\frac{\pi}{2}$$

establishing that

$$\frac{3x}{\sin x}+\cos x>4+\frac{x^4}{10}+\frac{x^6}{210}.$$

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