Open Journal of
Mathematical Sciences

PSR Press

Article

Some arguments for the wave equation in quantum theory 3

Tristram de Piro

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter (550), Woodstock Road, Oxford, OX2 6GG, England.; t.depiro@curvalinea.net

Academic Editor: Merve Ilkhan Kara
Received: Received: 20 April 2022; Accepted: 30 April 2023; Published: 13 June 2023.

Abstract: In this paper, we proved that solutions (ρ, J) exist for the 1-dimensional wave equation on $[-\pi, \pi]$. When (ρ, J) is extended to a smooth solution $(\rho, \overline{J})$ of the continuity equation on a vanishing annulus $Ann(1, \epsilon)$ containing the unit circle S^1, a corresponding causal solution $(\rho, \overline{J}'\overline{E}, \overline{B})$ to Maxwell's equations can be obtained from Jefimenko's equations. The power radiated in a time cycle from any sphere $S(r)$ with $r > 0$ is $O\left(\frac{1}{r}\right)$, which ensure that no power is radiated at infinity over a cycle.

Keywords: Wave equation; Continuity equation; Maxwell's equations; Jefimenko's equations; Radiation.

MSC: 35Q40; 35Q60; 83A05.

---

1. Introduction

This paper is divided into three parts. In the first part, in Lemma 1, we prove a simple result showing that we can produce a pair of solutions (Ψ, J) to the 1-dimensional wave equation on the circle with given initial conditions Ψ_0 and $\frac{\partial \Psi}{\partial t}|0$ such that $\frac{\partial \Psi}{\partial t} + \frac{\partial J}{\partial x} = 0$. This last equation allows us, in the second part, in Lemma 6, to extend the pair (Ψ, J) to a smooth pair $(\rho, \overline{J})$ satisfying the continuity equation $\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot \overline{J} = 0$ in three dimensions, which restricts to the pair (Ψ, J) on the unit circle but may not satisfy the three-dimensional wave equation, see [8]. We also prove some results about possible flows on the unit circle that satisfy the continuity equation. Some good references for fluid dynamics arguments are [1] and [4]. In the third part, we use a result from [8] to construct a solution $(\rho, \overline{J}, \overline{E}, \overline{B})$ to Maxwell's equations from the pair $(\rho, \overline{J})$, using Jefimenko's equations, which is referred to as the causal solution in [2]. In Lemma 8, we calculate the power radiated $P(r, t)$ over the sphere $S(r)$ of radius r for a fundamental solution $\rho^1 = \cos(mx)\cos(mt)$, $J^1 = \sin(mx)\sin(mt)$ to the wave equation on the circle and note that it does not satisfy the no radiating condition, in the sense that $\lim r \rightarrow \infty P(r, t) = 0$, even when averaged over a cycle $(t, t + \frac{\pi}{m})$. The calculation does not involve any approximations. However, in Lemma 12, we find that we can satisfy the no radiation condition over a cycle, $\lim_{r \rightarrow \infty} \int_{t}^{t+\frac{\pi}{m}} P(r, t)dt = 0$, by taking a linear combination of fundamental solutions, setting three of the parameters to be equal, and reversing the sign of the fourth. This result relies on Poynting's theorem and the fact that the mechanical energy of a charge and current configuration restricted to a circle can be computed over any ball $B(r)$ with $r > 1$. However, there are still some issues with computing the mechanical energy, which we hope can be resolved accurately in [13].

The result is interesting because the Larmor formula, see [3], predicts that the power radiated at infinity by an accelerating charge, $P = \frac{\mu_0 q^2 a^2}{6\pi c}$, is non-zero unless the particle travels in a straight line. However, for a collection of particles, we obtain effective cancellation when averaged over a cycle. Rutherford also used Larmor's result to suggest that the electrons in an atomic orbit couldn't orbit the nucleus in circular or elliptical orbits as they would lose energy and spiral into the nucleus. The result in the paper suggests, however, that there are atomic configurations of charge and current that retain energy over a cycle and, therefore, wouldn't collapse as Rutherford predicted.

We show that we can generalize the results of the paper to the 1-dimensional wave equation with velocity c. Analogously, in the paper [8], we proved that for any initial conditions ρ_0, $\frac{\partial \rho}{\partial t}|_0$, there exist solutions $(\rho, \overline{J})$ to the three-dimensional wave equations with velocity c, a connecting relation with velocity c, and the continuity equation;

$$\begin{cases} \square^2\rho = 0, \square^2\bar{J} = 0, \\ \bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial\bar{J}}{\partial t} = \bar{0}, \\ \frac{\partial\rho}{\partial t} + \bigtriangledown \cdot \bar{J} = 0, \end{cases} \tag{1}$$

where $\square^2$ denotes the d'Alembertian operator, we show that there exist fields $\{\bar{E}_S, \bar{B}_S\}$ in every inertial frame S, with $div(\bar{E}_S \times \bar{B}_S) = 0$, and $(\rho_S, \bar{J}_S, \bar{E}_S, \bar{B}_S)$ solutions to Maxwell's equations for the transformed charge and current $(\rho_S, \bar{J}_S)$. In particular, the power radiated $P(r,t)$ over any sphere $S(r)$ is identically zero, without averaging, and irrespective of the inertial frame. This seems to be a stronger type of radiation than that considered in this paper and would probably have to be generated over a sphere using a cavity magnetron, rather than a circular antenna.

In the paper [9], we proved that an atomic system that satisfies the no radiating condition and is in thermal equilibrium must also satisfy the Eqs. (1). However, we have not yet shown the converse, i.e., that there is an atomic system that satisfies the Eqs. (1) and is in thermal equilibrium. Nevertheless, the results of this paper, together with ongoing work in [12] and [13], yield a method for finding the appropriate initial conditions. The interested reader can look at Lemma 15 for more details, and [5] for more information on the idea of thermal equilibrium, a concept, along with pressure equilibrium, which is important in chemistry.

In the final part of the paper, we define the notion of classically non-radiating in a cycle, and use Rutherford's idea of radiating systems losing energy, to show that, in any inertial frame, the main system considered in this paper is classically non-radiating. The proof uses the notion of thermal equilibrium.

2. The wave equation on a circle

We will use standard notation throughout this paper. Specifically, we let $C^\infty([-\pi,\pi])$, $C(\mathcal{R})$, and $C^\infty(\mathcal{R})$ have their conventional meanings. We define T to be the rectangular region $[-\pi,\pi] \times \mathcal{R}$ and T^0 to be its interior $(-\pi,\pi) \times \mathcal{R}$. We then define the function spaces $C(T)$ and $\mathcal{S}(T)$ as follows:

- $C(T)$ consists of functions G that are continuous on T and have continuous partial derivatives G_t with respect to the time variable $t \in \mathcal{R}$.
- $\mathcal{S}(T)$ consists of functions G in $C(T)$ that have smooth partial derivatives G_t with respect to t and are smooth in the interior T^0.

These function spaces will play an important role in our analysis of the wave equation.

Definition 1. For any real $\{\Psi,\Psi_0,\Psi_1\} \subset C^\infty([-\pi,\pi])$, we define the Fourier coefficients, for $m \in \mathcal{Z}$ by;

$$\mathcal{F}(\Psi)(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_0(x)e^{-imx}dx,$$

$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_0(x)\cos(mx)dx,$$

$$b_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_0(x)\sin(mx)dx,$$

$$a'_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_1(x)\cos(mx)dx,$$

$$b'_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_1(x)\sin(mx)dx.$$

Lemma 1. *For any real $\{\Psi_0,\Psi_1\} \subset C^\infty([-\pi,\pi])$, there exists a unique real $\Psi \in \mathcal{S}(T)$ solving the rescaled wave equation;*

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} = 0$$

with $\Psi(0,x)=\Psi_0(x)$*, and* $\frac{\partial\Psi}{\partial t}(0,x)=\Psi_1(x)$ *for* $x\in[-\pi,\pi)$*. Moreover, using the terminology of Definition 1,* Ψ *is given explicitly by the series;*

$$a_0+ta_0'+2\sum_{m\in\mathcal{Z}_{>0}}a_m\cos(mx)\cos(mt)+2\sum_{m\in\mathcal{Z}_{>0}}b_m\sin(mx)\cos(mt)$$
$$+2\sum_{m\in\mathcal{Z}_{>0}}\frac{a_m'}{m}\cos(mx)\sin(mt)+2\sum_{m\in\mathcal{Z}_{>0}}\frac{b_m'}{m}\sin(mx)\sin(mt).$$

If $\Psi\in\mathcal{S}(T)$ *denotes such a solution, with related charge density* $\rho(x,t)=\Psi(x,t)$ *and current* $J(x,t)=\int_{-\pi}^{x}-\frac{\partial\rho}{\partial t}dx$*, then* $\{\rho,J\}$ *satisfy the continuity equation;* $\frac{\partial\rho}{\partial t}+\frac{\partial J}{\partial x}=0$*.*

In particular if $P(t)=\int_{-\pi}^{\pi}\rho(x,t)dx$*, then* $P(t)=P(0)$ *is constant. Moreover, when* Ψ_0 *and* Ψ_1 *are symmetric, we have the additional relation;* $\frac{\partial J}{\partial t}=-\frac{\partial\rho}{\partial x}$ *and* J *also satisfies the wave equation with* $J\in\mathcal{S}(T)$*, and with* $\{\rho,J\}$ *given explicitly by;*

$$\rho(x,t)=a_0+a_0't+2\sum_{m\in\mathcal{Z}>0}a_m\cos(mx)\cos(mt)+2\sum_{m\in\mathcal{Z}>0}\frac{a_m'}{m}\cos(mx)\sin(mt),$$
$$J(x,t)=-a_0'\pi-a_0'x+2\sum_{m\in\mathcal{Z}>0}a_m\sin(mx)\sin(mt)-2\sum_{m\in\mathcal{Z}>0}\frac{a_m'}{m}\sin(mx)\cos(mt).$$

Proof. For the first part, suppose there exists $\Psi\in S(T)$ satisfying the hypotheses, then taking Fourier coefficients of the equation, for $m\in\mathcal{Z}$, and using integration by parts, we have that;

$$\mathcal{F}\left(\frac{\partial^2\Psi}{\partial t^2}-\frac{\partial^2\Psi}{\partial x^2}\right)(m)=\frac{d^2\mathcal{F}(\Psi)(m,t)}{dt^2}+m^2\mathcal{F}(\Psi)(m,t)=0\,.$$

Solving the resulting ODE's for $m\neq 0$, we have that;

$$\begin{cases}\mathcal{F}(\Psi)(m,t)=A_me^{imt}+B_me^{-imt}=(A_m+B_m)\cos(mt)+i(A_m-B_m)\sin(mt),\\ \mathcal{F}(\Psi)(0,t)=A+Bt,\end{cases}$$

where $(A_m+B_m)=\mathcal{F}(\Psi)(m,0)=\mathcal{F}(\Psi_0)(m)$, $(imA_m-imB_m)=\mathcal{F}(\frac{\partial\Psi}{\partial t})(m,0)=\mathcal{F}(\Psi_1)(m)$, $A=\mathcal{F}(\Psi_0)(0)$, $B=\mathcal{F}(\Psi_1)(0)$. It follows that;

$$\begin{cases}\mathcal{F}(\Psi)(m,t)=\mathcal{F}(\Psi)(m,0)\cos(mt)+\frac{\mathcal{F}(\Psi_1)(m)}{m}\sin(mt), & (m\neq 0)\\ \mathcal{F}(\Psi)(0,t)=\mathcal{F}(\Psi_0)(0)+\mathcal{F}(\Psi_1)(0)t.\end{cases}\tag{2}$$

By the inversion theorem, Ψ is unique, and, using the fact that Ψ is real, symmetry properties of the Fourier coefficients $\{a_m, b_m, a'_m, b'_m\}$ and (2);

$$\begin{aligned}
\Psi(x,t) &= \sum_{m\in\mathcal{Z}} \mathcal{F}(\Psi)(m,t)e^{ixm} \\
&= \mathcal{F}(\Psi_0)(0) + \mathcal{F}(\Psi_1)(0)t + \sum_{m\in\mathcal{Z}_{\neq 0}} (\mathcal{F}(\Psi)(m,0)\cos(mt) \\
&\quad + \frac{\mathcal{F}(\Psi_1)(m)}{m}\sin(mt))(\cos(mx) + i\sin(mx)) \\
&= a_0 + a'_0 t + \sum_{m\in\mathcal{Z}_{\neq 0}} ((a_m - ib_m)\cos(mt)(\cos(mx) + i\sin(mx)) \\
&\quad + \sum_{m\in\mathcal{Z}_{\neq 0}} \left(\frac{a'_m - ib'_m}{m}\right)\sin(mt))(\cos(mx) + i\sin(mx)) \\
&= a_0 + a'_0 t + \sum_{m\in\mathcal{Z}_{\neq 0}} a_m\cos(mx)\cos(mt) + \sum_{m\in\mathcal{Z}_{\neq 0}} b_m\sin(mx)\cos(mt) \\
&\quad + \sum_{m\in\mathcal{Z}_{\neq 0}} \frac{a'_m}{m}\cos(mx)\sin(mt) + \sum_{m\in\mathcal{Z}_{\neq 0}} \frac{b'_m}{m}\sin(mx)\sin(mt)
\end{aligned}$$

$$\begin{aligned}
&= a_0 + a'_0 t + 2\sum_{m\in\mathcal{Z}_{>0}} a_m\cos(mx)\cos(mt) + 2\sum_{m\in\mathcal{Z}_{>0}} b_m\sin(mx)\cos(mt) \\
&\quad +2\sum_{m\in\mathcal{Z}_{>0}} \frac{a'_m}{m}\cos(mx)\sin(mt) + 2\sum_{m\in\mathcal{Z}_{>0}} \frac{b'_m}{m}\sin(mx)\sin(mt)
\end{aligned}$$

as required. It is easily checked that the above series also defines $\Psi \in S(T)$ with the required properties, settling the existence question.

For the second part, $\{\rho, J\}$ satisfy the continuity equation, by the definition of J and the fundamental theorem of calculus. By inspection of the series for Ψ, it is clear that $J \in S(T)$. Moreover;

$$P'(t) = \int_{-\pi}^{\pi} \frac{\partial\rho}{\partial t}(x,t)dx = \int_{-\pi}^{\pi} -\frac{\partial J}{\partial x}(x,t)dx = J(-\pi) - J(\pi) = 0,$$

so that $P(t) = P(0)$ is constant. Differentiating under the integral sign, using the fact that ρ satisfies the wave equation, and using the fundamental theorem of calculus again, we have;

$$\frac{\partial J}{\partial t} = \int_{-\pi}^{x} -\frac{\partial^2\rho}{\partial t^2}dt = \int_{-\pi}^{x} -\frac{\partial^2\rho}{\partial x^2}dx = -\frac{\partial\rho}{\partial x} + \left.\frac{\partial\rho}{\partial x}\right|_{-\pi}.$$

If Ψ_0 and Ψ_1 are symmetric, we have the above coefficients $\{b_m, b'_m\}$ are zero, and $\frac{\partial\rho}{\partial x}$ expands as a series in $\sin(mx)$. It follows that;

$$\left.\frac{\partial\rho}{\partial x}\right|_{-\pi} = 0; \quad \text{and} \quad \frac{\partial J}{\partial t} = -\frac{\partial\rho}{\partial x},$$

so that, combined with the continuity equation, and the fact that the partial derivatives commutes, we obtain that;

$$\frac{\partial^2 J}{\partial t^2} = -\frac{\partial^2\rho}{\partial x\partial t} = -\frac{\partial^2\rho}{\partial t\partial x} = \frac{\partial^2 J}{\partial x^2}.$$

Moreover, a simple calculation, using the definition of J shows that;

$$\begin{aligned}
\rho(x,t) &= a_0 + a'_0 t + 2\sum_{m\in\mathcal{Z}>0} a_m\cos(mx)\cos(mt) + 2\sum_{m\in\mathcal{Z}>0} \frac{a'_m}{m}\cos(mx)\sin(mt), \\
J(x,t) &= -a'_0\pi - a'_0 x + 2\sum_{m\in\mathcal{Z}>0} a_m\sin(mx)\sin(mt) - 2\sum_{m\in\mathcal{Z}>0} \frac{a'_m}{m}\sin(mx)\cos(mt).
\end{aligned}$$

□

Remark 1. Observe that in the case of the wave equation, the solutions are bounded backwards in time, that is there exists a constant C_t, such that $|\Psi(x,t')| \leq C_t$, for all $t' \leq t$. This is an important consideration when we come to discuss the radiation condition behind Jefimenko's equations.

3. Extending Charge and Current

Lemma 2. *Let $\{\Psi, J\}$ be as in Lemma 1, and $S(1) \subset z = 0$ be the circle of radius 1, centred at $(0,0,0)$ then, if ρ is defined on $S(1)$ by $\rho(1,\theta,t) = \Psi(\theta,t)$, for $\theta \in [-\pi,\pi)$, $t \in \mathcal{R}$, and $\overline{J}$ is any smooth extension to the annulus $Ann(1,\epsilon)$, $0<\epsilon<1$, of $\overline{K}$, defined on $S(1)$ by $\overline{K}(1,\theta,t) = J(\theta,t)(-\sin(\theta),\cos(\theta),0)$, $t \in \mathcal{R}$, then $\{\rho, \overline{J}\}$ satisfy the continuity equation $\frac{\partial\rho}{\partial t} + div(\overline{J}) = 0$ on $S(1) \times \mathcal{R}$.*

Proof. Using Lemma 1, it is sufficient to prove that, for $-\pi \leq \theta < \pi$, $t \in \mathcal{R}$;

$$div(\overline{J})|_{(1,\theta,0,t)} = J'(\theta,t). \tag{3}$$

Omitting the t for ease of notation, and letting $\overline{J} = (J())$, we have that;

$$div(-J(\theta)\sin(\theta), J(\theta)\cos(\theta), 0) = \frac{\partial}{\partial x}(-J(\theta)\sin(\theta)) + \frac{\partial}{\partial y}(J(\theta)\cos(\theta)).$$

We have;

$$\begin{aligned}
\frac{\partial}{\partial x}(-J(\theta)\sin(\theta)) &= -\left(\frac{\partial J}{\partial x}\sin(\theta) + J(\theta)\frac{\partial \sin(\theta)}{\partial x}\right) \\
&= -\left(J'(\theta)\frac{\partial\theta}{\partial x}\sin(\theta) + J(\theta)\frac{\partial y}{\partial x}\right), \qquad \text{as } r = 1 \ \text{ and } \ \sin(\theta) = \frac{y}{r} = y \\
&= -\left(J'(\theta)\sin(\theta)\frac{\partial\theta}{\partial x}\right), \qquad \text{as } \frac{\partial y}{\partial x} = 0 \\
&= -(J'(\theta)\sin(\theta) - y) = J'(\theta)\sin(\theta)y, \\
&\qquad \text{as } \theta = \tan^{-1}\left(\frac{y}{x}\right) \ \text{ and } \ \frac{\partial\theta}{\partial x} = \frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2} = \frac{-y}{x^2+y^2} = -y, \ \text{ with } \ r = 1.
\end{aligned}$$

Similarly;

$$\begin{aligned}
\frac{\partial}{\partial y}(J(\theta)\cos(\theta)) &= \left(\frac{\partial J}{\partial y}\cos(\theta) + J(\theta)\frac{\partial \cos(\theta)}{\partial y}\right) \\
&= \left(J'(\theta)\frac{\partial\theta}{\partial y}\cos(\theta) + J(\theta)\frac{\partial x}{\partial y}\right), \qquad \text{as } r = 1 \ \text{ and } \ \cos(\theta) = \frac{x}{r} = x \\
&= \left(J'(\theta)\cos(\theta)\frac{\partial\theta}{\partial y}\right), \qquad \text{as } \frac{\partial x}{\partial y} = 0 \\
&= (J'(\theta)\cos(\theta)x) = J'(\theta)\cos(\theta)x \\
&\qquad \text{as } \theta = tan^{-1}\left(\frac{y}{x}\right) \ \text{ and } \ \frac{\partial\theta}{\partial y} = \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2} = x, \ \text{ with } \ r = 1.
\end{aligned}$$

It follows that;

$$div(-J(\theta)\sin(\theta), J(\theta)\cos(\theta), 0) = J'(\theta)\sin(\theta)y + J'(\theta)\cos(\theta)x = J'(\theta)(\sin^2(\theta) + \cos^2(\theta)) = J'(\theta),$$

as $x = r\cos(\theta) = \cos(\theta)$, $x = r\sin(\theta) = \sin(\theta)$, when $r = 1$. □

Lemma 3. *Let $D(1)$ be the closed punctured disc, with radius 1, and let ρ on $D(1)$ be constant, then any smooth circular flow, with velocity $v(r) = \frac{w(r)}{\rho}\hat{\theta}\ \overline{J}(r,t) = w(r)(-\sin(\theta),\cos(\theta))$ satisfies the continuity equation.*

Conversely, any smooth circular flow, $\{\rho, \overline{J}\}$, independent of time, satisfying the continuity equation, requires the density to depend only on r, with an equivalent flow $\{1, \overline{J}\}$, obtained with constant density 1, by changing the velocity from $\overline{v}$ to $\rho\overline{v}$, where $\overline{J} = \rho\overline{v}$.

Proof. We have that $\frac{\partial \rho}{\partial t} = 0$, and

$$\begin{aligned} div(\bar{J}) &= \frac{\partial(-w(r)\sin(\theta))}{\partial x} + \frac{\partial(w(r)\cos(\theta))}{\partial y} \\ &= -\frac{\partial w}{\partial x}\sin(\theta) - w\frac{\partial \sin(\theta)}{\partial x} + \frac{\partial w}{\partial y}\cos(\theta) + w\frac{\partial \cos(\theta)}{\partial y} \\ &= -w'(r)\frac{\partial r}{\partial x}\sin(\theta) - w\frac{\partial \sin(\theta)}{\partial x} + w'(r)\frac{\partial r}{\partial y}\cos(\theta) + w\frac{\partial \cos(\theta)}{\partial y} \\ &= -w'(r)\frac{x}{r}\sin(\theta) - \frac{w(r)(-\sin(\theta)\cos(\theta))}{r} + w'(r)\frac{y}{r}\cos(\theta) - w(r)\frac{(\sin(\theta)\cos(\theta))}{r} \\ &= -w'(r)\cos(\theta)\sin(\theta) - \frac{w(r)(-\sin(\theta)\cos(\theta))}{r} + w'(r)\sin(\theta)\cos(\theta - w(r)\frac{(\sin(\theta)\cos(\theta))}{r} = 0 \end{aligned}$$

with $x = r\cos(\theta)$, $y = r\sin(\theta)$,([1]).

Conversely, suppose that $div(\bar{J}) = 0$, with $\bar{J}(r,\theta,t) = \rho(r,\theta)(-\sin(\theta),\cos(\theta))$. Then

$$\begin{aligned} \frac{\partial(-\rho(r,\theta)\sin(\theta))}{\partial x} + \frac{\partial(\rho(r,\theta)\cos(\theta))}{\partial y} &= -\frac{\partial \rho}{\partial x}\sin(\theta) - \rho\frac{\partial \sin(\theta)}{\partial x} + \frac{\partial \rho}{\partial y}\cos(\theta) + \rho\frac{\partial \cos(\theta)}{\partial y} \\ = -\left(\frac{\partial \rho}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\sin(\theta) - \rho\left(\frac{-\sin(\theta)\cos(\theta)}{r}\right) &+ \left(\frac{\partial \rho}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\cos(\theta) + \rho\left(\frac{-\sin(\theta)\cos(\theta)}{r}\right), \end{aligned}$$

using footnote 2, $\frac{\partial \theta}{\partial x} = \frac{-\sin(\theta)}{r}$, $\frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r}$. Therefore

$$div(\bar{J}) = \left(\frac{\partial \rho}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\cos(\theta) - \left(\frac{\partial \rho}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\sin(\theta).$$

It follows that

$$div(\bar{J}) = \frac{\partial \rho}{\partial r}\frac{y}{r}\cos(\theta) + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial y}\cos(\theta) - \frac{\partial \rho}{\partial r}\frac{x}{r}\sin(\theta) - \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x}\sin(\theta)$$

with $y = r\sin(\theta)$ and $x = r\cos(\theta)$, using footnote 1 again; $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos(\theta)$, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin(\theta)$. Hence

$$div(\bar{J}) = \left(\frac{\partial \rho}{\partial r}\sin(\theta)\cos(\theta) + \frac{\partial \rho}{\partial \theta}\frac{\cos^2(\theta)}{r}\right) - \left(\frac{\partial \rho}{\partial r}\sin(\theta)\cos(\theta) + \frac{\partial \rho}{\partial \theta}\frac{\sin^2(\theta)}{r}\right) = \frac{\partial \rho}{\partial \theta}\frac{1}{r} = 0.$$

If $r \neq 0$, $\frac{\partial \rho}{\partial \theta} = 0$, so ρ is independent of θ. □

Lemma 4. *Let $\{\Psi, J\}$ be as in Lemma 2, with Ψ non constant and independent of time, then any extension $\{\rho, \bar{J}\}$ of $\{\Psi, J\}$ to $Ann(1,\epsilon)$, $0 < \epsilon < 1$ which satisfies the continuity equation is not a circular flow.*

Proof. By Lemma 3, any circular flow satisfying the continuity equation on $Ann(1,\epsilon)$ has a density ρ, depending only on r. In particular, $\rho|_{S(1)}$ is constant, contradicting the hypothesis.
□

Lemma 5. *Determination of flows for density independent of time Suppose that $\rho(\theta, r)$ is smooth and independent of time on the annulus, defined by;*

$$Ann(1,\epsilon,\delta) = \{(\theta, r) : -\pi \leq \theta < \pi, 1 - \epsilon < r < 1 + \delta\},$$

---

[1] We use $\frac{\partial r}{\partial x} = \frac{\frac{1}{2}2x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{\frac{1}{2}2y}{(x^2+y^2)^{\frac{1}{2}}} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}} = \frac{y}{r}$ with $r = (x^2+y^2)^{\frac{1}{2}}$, and with $\theta = tan^{-1}(\frac{y}{x})$; $\frac{\partial \theta}{\partial x} = \frac{-y}{x^2}\frac{1}{1+(\frac{y}{x})^2} = \frac{-y}{(x^2+y^2)} = \frac{-y}{r^2} = \frac{-r\sin(\theta)}{r^2} = \frac{-\sin(\theta)}{r}$, $\frac{\partial \theta}{\partial y} = \frac{1}{x}\frac{1}{1+(\frac{y}{x})^2} = \frac{x}{(x^2+y^2)} = \frac{x}{r^2} = \frac{r\cos(\theta)}{r^2} = \frac{\cos(\theta)}{r}$, $\frac{\partial \sin(\theta)}{\partial x} = \cos(\theta)\frac{\partial \theta}{\partial x} = \frac{-\cos(\theta)\sin(\theta)}{r}$ $\frac{\partial \cos(\theta)}{\partial y} = -\sin(\theta)\frac{\partial \theta}{\partial y} = \frac{-\sin(\theta)\cos(\theta)}{r}$

with smooth $\overline{J}(\theta,r)$, satisfying the continuity equation, and

$$\overline{J}(\theta,r)=(J_1(\theta,r),J_2(\theta,r))=w_1(\theta,r)\hat{\hat{r}}+w_2(\theta,r)\hat{\hat{\theta}}.$$

Then for a given $\epsilon>0, 1-\epsilon<r_0<1+\delta$, smooth boundary condition g_ϵ on $S(1-\epsilon)$, and smooth w_2 on $Ann(1,\epsilon,\delta)$, we obtain that $r_0w_1(r_0,\theta_0)=\frac{(1-\epsilon)g_{1-\epsilon}(\theta_0)}{r_0}+\frac{1}{r_0}\int\limits_{1-\epsilon}^{r_0}-{\frac{\partial w_2}{\partial\theta}}^{\theta_0}dr$ and, for $\epsilon=\delta=0$, on the $D(0,1)$, smooth w_2 on $D(0,1)$, with $\lim\limits_{r\to 0}-{\frac{\partial w_2}{\partial\theta}}^{\theta_1}=\lim\limits_{r\to 0}-{\frac{\partial w_2}{\partial\theta}}^{\theta_2}$ for all $\{\theta_1,\theta_2\}\subset[-\pi,\pi)$. with we obtain that, for $r_0\neq 0$; $w_1(r_0,\theta_0)=\frac{1}{r_0}\int\limits_{0}^{r_0}-{\frac{\partial w_2}{\partial\theta}}^{\theta_0}dr$ and $w_1(0,0)=-w_2(0,0)$ with w_1 continuous.

Proof. Suppose that $\rho(\theta,r)$ is smooth and independent of time on the annulus, defined by

$$Ann(1,\epsilon,\delta)=\{(\theta,r):-\pi\leq\theta<\pi,1-\delta<r<1+\epsilon\},$$

with smooth $\overline{J}(\theta,r)$, satisfying the continuity equation, and $\overline{J}(\theta,r)=(J_1(\theta,r),J_2(\theta,r))=w_1(\theta,r)\hat{\hat{r}}+w_2(\theta,r)\hat{\hat{\theta}}$, where $\hat{\hat{r}}=(\cos(\theta),\sin(\theta))$ and $\hat{\hat{\theta}}=(-\sin(\theta),\cos(\theta))$. It follows that

$$\begin{aligned}\overline{J}(\theta,r)&=w_1(\theta,r)(\cos(\theta),\sin(\theta))+w_2(\theta,r)(-\sin(\theta),\cos(\theta))\\&=(w_1\cos(\theta)-w_2\sin(\theta),w_1\sin(\theta)+w_2\cos(\theta)).\end{aligned}$$

Using the hypotheses, that $div(\overline{J})=\frac{\partial\rho}{\partial t}=0$, we have that

$$\begin{aligned}\frac{\partial J_1}{\partial x}&=\frac{\partial w_1}{\partial x}\cos(\theta)+w_1\frac{\partial(\cos(\theta))}{\partial x}-\frac{\partial w_2}{\partial x}\sin(\theta)-w_2\frac{\partial(\sin(\theta))}{\partial x}\\&=\frac{\partial w_1}{\partial x}\cos(\theta)+w_1-\sin(\theta)\frac{-y}{r^2}-\frac{\partial w_2}{\partial x}\sin(\theta)-w_2\cos(\theta)\frac{-y}{r^2}\\&=\frac{\partial w_1}{\partial x}\cos(\theta)+w_1\frac{\sin(\theta)y}{r^2}-\frac{\partial w_2}{\partial x}\sin(\theta)+w_2\frac{\cos(\theta)y}{r^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial J_2}{\partial y}&=\frac{\partial w_1}{\partial y}\sin(\theta)+w_1\frac{\partial(\sin(\theta))}{\partial y}+\frac{\partial w_2}{\partial y}\cos(\theta)+w_2\frac{\partial(\cos(\theta))}{\partial y}\\&=\frac{\partial w_1}{\partial y}\sin(\theta)+w_1\cos(\theta)\frac{x}{r^2}+\frac{\partial w_2}{\partial y}\cos(\theta)+w_2-\sin(\theta)\frac{x}{r^2}\\&=\frac{\partial w_1}{\partial y}\sin(\theta)+w_1\cos(\theta)\frac{x}{r^2}+\frac{\partial w_2}{\partial y}\cos(\theta)-w_2\frac{\sin(\theta)x}{r^2}.\end{aligned}$$

Therefore

$$div(\overline{J})=grad(w_1)\centerdot\hat{\hat{r}}+\frac{w_1}{r^2}(\overline{v_\theta}\centerdot flip(\overline{r}))+grad(w_2)\centerdot\hat{\hat{\theta}}+\frac{w_2}{r^2}(\overline{w_\theta}\centerdot flip(\overline{r})),\tag{4}$$

where $\overline{v_\theta}=(\sin(\theta),\cos(\theta))$, $\overline{w_\theta}=(\cos(\theta),-\sin(\theta))$, and $flip(\overline{r})=(y,x)$. We find $\{\alpha,\beta\}$ such that $\alpha\hat{\hat{\theta}}+\beta\hat{\hat{r}}=\overline{v_\theta}$. This is equivalent to finding $\{\alpha,\beta\}$ such that $M_\theta\overline{v_{\alpha,\beta}}=\overline{v_\theta}$, where $(M_\theta)_{1,2}=(M_\theta)_{2,1}=\cos(\theta)$ and $(M_\theta)_{1,1}=-(M_\theta)_{2,2}=-\sin(\theta)$ and $\overline{v_{\alpha,\beta}}=(\alpha,\beta)$. We have,

$$\overline{v_{\alpha,\beta}}=M_\theta^{-1}\overline{v_\theta}=\frac{1}{-\sin^2(\theta)-\cos^2(\theta)}N_\theta\overline{v_\theta}=-(\sin^2(\theta)-\cos^2(\theta),-2\sin(\theta)\cos(\theta))=(\cos(2\theta),\sin(2\theta)),$$

where $(N_\theta)_{1,2}=(N_\theta)_{2,1}=-\cos(\theta)$ and $(N_\theta)_{1,1}=-(N_\theta)_{2,2}=\sin(\theta)$.

It follows that $\alpha=\cos(2\theta)$, $\beta=\sin(2\theta)$. Similarly, we find $\{\alpha,\beta\}$ such that $\alpha\hat{\hat{\theta}}+\beta\hat{\hat{r}}=\overline{w_\theta}$. Again, this is equivalent to finding $\{\alpha,\beta\}$ such that $M_\theta\overline{v_{\alpha,\beta}}=\overline{w_\theta}$. We have

$$\overline{v_{\alpha,\beta}}=M_\theta^{-1}\overline{w_\theta}=\frac{1}{-\sin^2(\theta)-\cos^2(\theta)}N_\theta\overline{w_\theta}=-(2\sin(\theta)\cos(\theta),-\cos^2(\theta)+\sin^2(\theta))=(-\sin(2\theta),\cos(2\theta)).$$

It follows that $\alpha = \sin(2\theta)$, $\beta = \cos(2\theta)$. Substituting in (4), we obtain

$$\begin{aligned} div\left(\bar{J}\right) &= grad\left(w_1\right) \centerdot \hat{\bar{r}} + \frac{w_1}{r^2}\left(\cos\left(2\theta\right)\hat{\bar{\theta}} + \sin\left(2\theta\right)\hat{\bar{r}}\right) \centerdot flip\left(\bar{r}\right) \\ &\quad + grad\left(w_2\right) \centerdot \hat{\bar{\theta}} + \frac{w_2}{r^2}\left(-\sin\left(2\theta\right)\bar{\theta} + \cos\left(2\theta\right)\bar{r}\right) \centerdot flip\left(\bar{r}\right) \\ &= \left(grad\left(w_1\right) + \frac{w_1}{r^2} flip\left(\bar{r}\right)\sin\left(2\theta\right) + \frac{w_2}{r^2} flip\left(\bar{r}\right)\cos\left(2\theta\right)\right) \centerdot \hat{\bar{r}} \\ &\quad + \left(grad\left(w_2\right) - \frac{w_2}{r^2} flip\left(\bar{r}\right)\sin\left(2\theta\right) + \frac{w_1}{r^2} flip\left(\bar{r}\right)\cos\left(2\theta\right)\right) \centerdot \hat{\bar{\theta}}. \end{aligned} \tag{5}$$

We have that $\bar{r} = r\hat{\bar{r}}$ and determine $\{\alpha, \beta\}$ such that $flip(\bar{r}) = \alpha\bar{\theta} + \beta\hat{\bar{r}}$. Similarly to the above, we obtain

$$\overline{v_{\alpha,\beta}} = -(r\sin^2(\theta) - r\cos^2(\theta), -2r\sin(\theta)\cos(\theta)) = (r\cos(2\theta), r\sin(2\theta)) = (r\cos(2\theta), r\sin(2\theta)),$$

so that $\alpha = r\cos(2\theta)$, $\beta = r\sin(2\theta)$, and $flip(\bar{r}) = r\cos(2\theta)\hat{\bar{\theta}} + r\sin(2\theta)\hat{\bar{r}}$. Substituting into (5), and using the fact that $\{\hat{\bar{r}}, \hat{\bar{\theta}}\}$ are orthonormal, we obtain

$$\begin{aligned} div\left(\bar{J}\right) &= \left(grad\left(w_1\right) + \left(\frac{w_1\sin\left(2\theta\right)}{r^2} + \frac{w_2\cos\left(2\theta\right)}{r^2}\right)\left(r\cos\left(2\theta\right)\hat{\bar{\theta}} + r\sin\left(2\theta\right)\bar{r}\right)\right) \centerdot \hat{\bar{r}} \\ &\quad + \left(grad\left(w_2\right) + \left(\frac{-w_2\sin\left(2\theta\right)}{r^2} + \frac{w_1\cos\left(2\theta\right)}{r^2}\right)\left(r\cos\left(2\theta\right)\hat{\bar{\theta}} + r\sin\left(2\theta\right)\bar{r}\right)\right) \centerdot \hat{\bar{\theta}}, \\ &= grad\left(w_1\right) \centerdot \hat{\bar{r}} + \left(\frac{\left(w_1\sin\left(2\theta\right)\right) r\sin\left(2\theta\right)}{r^2} + \frac{\left(w_2\cos\left(2\theta\right)\right) r\sin\left(2\theta\right)}{r^2}\right) \\ &\quad + grad\left(w_2\right) \centerdot \hat{\bar{\theta}} + \left(\frac{\left(w_1\cos\left(2\theta\right)\right) r\cos\left(2\theta\right)}{r^2} - \frac{\left(w_2\sin\left(2\theta\right)\right) r\cos\left(2\theta\right)}{r^2}\right). \end{aligned} \tag{6}$$

Writing $grad(w_1) = \alpha\hat{\bar{r}} + \beta\hat{\bar{\theta}}$ and $grad(w_2) = \gamma\hat{\bar{r}} + \delta\hat{\bar{\theta}}$ with $\alpha = grad(w_1) \centerdot \hat{\bar{r}}$ and $\delta = grad(w_2) \centerdot \hat{\bar{\theta}}$, we obtain from (6) and $div(\bar{J}) = 0$ that

$$\alpha + \delta = \frac{-w_1}{r} = \frac{-w_1}{r}. \tag{7}$$

We have that

$$\begin{aligned} \alpha &= grad(w_1) \centerdot \hat{\bar{r}} = \frac{\partial w_1}{\partial x}\cos(\theta) + \frac{\partial w_1}{\partial y}\sin(\theta), \\ \delta &= grad(w_2) \centerdot \hat{\bar{\theta}} = \frac{\partial w_2}{\partial x} - \sin(\theta) + \frac{\partial w_2}{\partial y}\cos(\theta). \end{aligned} \tag{8}$$

We have, using the calculations for $\{\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\}$, from the previous lemma, and the chain rule;

$$\begin{cases} \frac{\partial w_1}{\partial x} = \frac{\partial w_1}{\partial\theta}\frac{-\sin(\theta)}{r} + \frac{\partial w_1}{\partial r}\cos(\theta), \\ \frac{\partial w_1}{\partial y} = \frac{\partial w_1}{\partial\theta}\frac{\cos(\theta)}{r} + \frac{\partial w_1}{\partial r}\sin(\theta), \\ \frac{\partial w_2}{\partial x} = \frac{\partial w_2}{\partial\theta}\frac{-\sin(\theta)}{r} + \frac{\partial w_2}{\partial r}\cos(\theta), \\ \frac{\partial w_2}{\partial y} = \frac{\partial w_2}{\partial\theta}\frac{\cos(\theta)}{r} + \frac{\partial w_2}{\partial r}\sin(\theta). \end{cases} \tag{9}$$

It follows from (8) and (9) that

$$\begin{aligned} \alpha &= \left(\frac{\partial w_1}{\partial\theta}\frac{-\sin(\theta)}{r} + \frac{\partial w_1}{\partial r}\cos(\theta)\right)\cos(\theta) + \left(\frac{\partial w_1}{\partial\theta}\frac{\cos(\theta)}{r} + \frac{\partial w_1}{\partial r}\sin(\theta)\right)\sin(\theta), \\ \delta &= \left(\frac{\partial w_2}{\partial\theta}\frac{-\sin(\theta)}{r} + \frac{\partial w_2}{\partial r}\cos(\theta)\right) - \sin(\theta) + \left(\frac{\partial w_2}{\partial\theta}\frac{\cos(\theta)}{r} + \frac{\partial w_2}{\partial r}\sin(\theta)\right)\cos(\theta). \end{aligned}$$

Simplifying and substituting into (7), we obtain $\frac{\partial w_1}{\partial r} + \frac{1}{r}\frac{\partial w_2}{\partial\theta} = \frac{-w_1}{r}$ and rearranging $r\frac{\partial w_1}{\partial r} + w_1 = -\frac{\partial w_2}{\partial\theta}$. Fixing θ_0 and multiplying by $p(\theta_0, r)$, we obtain

$$p(r, \theta_0) r \frac{dw_1^{\theta_0}}{dr} + p(r, \theta_0) w_1^{\theta_0} = -p(r, \theta_0)\frac{\partial w_2}{\partial\theta}^{\theta_0}.$$

Letting
$$s(r,\theta_0) = p(r,\theta_0)r, \tag{10}$$
we have that $[s(r,\theta_0)w_1(r,\theta_0)]' = s'(r,\theta_0)w_1 + sw_1'(r,\theta_0)$ so, equating coefficients, we require
$$s'(r,\theta_0) = p(r,\theta_0). \tag{11}$$
Using (10),(11), and, assuming $p(r,\theta_0) \neq 0$, we obtain $\frac{s'(r,\theta_0)}{s(r,\theta_0)} = \frac{1}{r}\ ln(s(r,\theta_0)) = ln(r) + d(\theta_0)$ and, taking exponentials; $s(r,\theta_0) = A(\theta_0)r$ where $A(\theta_0) = e^{d(\theta_0)}$;
$$[A(\theta_0)r^{c(\theta_0)}w_1(r,\theta_0)]' = -p(r,\theta_0)\frac{\partial w_2}{\partial\theta}^{\theta_0},$$
where $p(r,\theta_0) = A(\theta_0)$. Integrating, and using the fundamental theorem of calculus, we obtain
$$[A(\theta_0)rw_1(r,\theta_0)]_{1-\epsilon}^{r_0} = \int\limits_{1-\epsilon}^{r_0} -A(\theta_0)\frac{\partial w_2}{\partial\theta}^{\theta_0} dr.$$

Case 1; For a given $\epsilon, \delta > 0$, and $1-\epsilon < r_0 < 1+\delta$, we obtain
$$r_0 w_1(r_0,\theta_0) - (1-\epsilon)w_1(1-\epsilon,\theta_0) = \int\limits_{1-\epsilon}^{r_0} -\frac{\partial w_2}{\partial\theta}^{\theta_0} dr,$$
so free to choose a smooth w_2 on $Ann(\epsilon,\delta,1)$ and a smooth boundary condition for w_1 on $S^1(1-\epsilon)$, to obtain w_1.

Case 2; Letting $\epsilon = 1, \delta = 0$, and, assuming w_1 is defined at $(0,0)$, we must have that, for $r_0 > 0$;
$$r_0 w_1(r_0,\theta_0) = \int\limits_0^{r_0} -\frac{\partial w_2}{\partial\theta}^{\theta_0} dr.$$
By L'Hopital's rule, the fact that $r(0) = 0$, $a(0) = 0$, where $a(r) = \int\limits_0^r -\frac{\partial w_2}{\partial\theta}^{\theta_0} dr'$, and the Fundamental Theorem of Calculus, $a'(0) = -\frac{\partial w_2}{\partial\theta}(0,0)$, we have that
$$\lim_{r_0\to 0}\frac{1}{r_0}\int\limits_0^{r_0} -\frac{\partial w_2}{\partial\theta}^{\theta_0} dr = \frac{a'(0)}{r'(0)} = -\frac{\partial w_2}{\partial\theta}^{\theta_0}(0),$$
so defining $w_1(0,0) = -w_2(0,0)$ generates a continuous solution, provided $\lim\limits_{r\to 0} -\frac{\partial w_2}{\partial\theta}^{\theta_1} = \lim\limits_{r\to 0} -\frac{\partial w_2}{\partial\theta}^{\theta_2}$, for all $\{\theta_1,\theta_2\} \subset [-\pi,\pi)$, this is true if w_2 is analytic in (x,y),([2]). □

Remark 2. It follows we are free to choose a smooth w_2 on $Ann(\epsilon,1)$ and a smooth boundary condition for w_1 on $S^1(1-\epsilon)$, to obtain w_1, so we are free to choose a smooth w_2 on $D(0,1)$, to obtain a smooth w_1.

Lemma 6. *Given $\{\Psi, J\}$ as in Lemma 1, $0 < \epsilon < 1$ and notation as in Lemma 2, there exists a smooth pair $(\rho_1, \overline{J})$ supported on $Ann(1,\epsilon)\times(-\epsilon,\epsilon)$, such that $(\rho_1,\overline{J})$ satisfies the continuity equation in $\mathcal{R}^3\times\mathcal{R}$ and $\rho_1|_{S^1\times\{0\}} = \rho$, $\overline{J}|_{S^1\times\{0\}} = \overline{K}$.*

---

[2] In this case, we can write; $w_{2,r}(x,y) = \sum\limits_{m,n=0}^{\infty} w_2^{(m,n)}(0,0)\frac{x^m y^n}{m!n!}$, so that $w_2(r,\theta) = \sum\limits_{m,n=0}^{\infty} w_2^{(m,n)}(0,0)\frac{r\cos(\theta)^m r\sin(\theta)^n}{m!n!}$. Then $\lim\limits_{r\to 0}\left(\frac{1}{r}\int\limits_0^r w_2(r,\theta)\right) = \lim\limits_{r\to 0}\left(\frac{1}{r}\int\limits_0^r \sum\limits_{m,n=0}^{\infty} w_2^{(m,n)}(0,0)\frac{r\cos(\theta)^m r\sin(\theta)^n}{m!n!}\right) = \sum\limits_{m,n=0}^{\infty} w_2^{(m,n)}(0,0)\lim\limits_{r\to 0}\left(\frac{1}{r}\int\limits_0^r \frac{r\cos(\theta)^m r\sin(\theta)^n}{m!n!}\right) = \sum\limits_{m,n=0}^{\infty} w_2^{(m,n)}(0,0)\left(\frac{r\cos(\theta)^m r\sin(\theta)^n}{m!n!}\right)\Big|_{r=0} = w_{2,r}(0,0)$ for all θ, using the fundamental theorem of calculus again. For the general case, we need to check higher derivatives and use the product rule.

Proof. Let $\Phi_\epsilon(r) = e^{-\frac{1}{1-\frac{(r-1)^2}{\epsilon^2}}}$, if $r \in (1-\epsilon, 1+\epsilon)$ and $\Phi_\epsilon(r) = 0$ otherwise, $r > 0$, then Φ_ϵ is smooth on $\mathcal{R}_{>0}$ and supported on $(1-\epsilon, 1+\epsilon)$. Let $\Phi_{1,\epsilon}(z) = e^{-\frac{1}{1-\frac{z^2}{\epsilon^2}}}$, if $z \in (-\epsilon, \epsilon)$ and $\Phi_{1,\epsilon}(z) = 0$ otherwise, $z \in \mathcal{R}$, then $\Phi_{1,\epsilon}$ is smooth on $\mathcal{R}$ and supported on $(-\epsilon, \epsilon)$. Define $(\rho_1, \overline{J})$ on $\mathcal{R}^2 \times \mathcal{R}$ by $\rho_1(r,\theta,t) = \Phi_\epsilon(r)\rho(1,\theta,t) = \Phi_\epsilon(r)\Psi(\theta,t)$ for $r > 0, \theta \in [-\pi, \pi), t \in \mathcal{R}$, $\rho_1(0,0,t) = 0, t \in \mathcal{R}$,$\overline{J}(r,\theta,t) = \Phi_\epsilon(r)\overline{K}(1,\theta,t) = \Phi_\epsilon(r)J(\theta,t)(-\sin(\theta, \cos(\theta, 0)$, for $r > 0, \theta \in [-\pi, \pi), t \in \mathcal{R}$, $\overline{J}(0,0,t) = \overline{0}, t \in \mathcal{R}$.

Then, using Lemma 2 and the facts that, for any given $r > 0$, $\Phi_\epsilon(r)\Psi(\theta,t)$ and $\Phi_\epsilon(r)J(\theta,t)$ satisfy the conditions of Lemma 1, we have that $(\rho_1, \overline{J})$ satisfy the continuity equation on $\mathcal{R}^2 \times \mathcal{R}$.

Now define $(\rho_1, \overline{J})$ on $\mathcal{R}^3 \times \mathcal{R}$ by

$$\rho_1(r,\theta,z,t) = \Phi_{1,\epsilon}(z)\rho_1(r,\theta,t),$$

for $r > 0, \theta \in [-\pi, \pi), z \in \mathcal{R}, t \in \mathcal{R}$,

$$\rho_1(0,0,z,t) = 0, \ \ z \in \mathcal{R}, \ \ t \in \mathcal{R},$$

$$\overline{J}(r,\theta,z,t) = \Phi_{1,\epsilon}(z)\overline{J}(r,\theta,t),$$

for $r > 0, \theta \in [-\pi, \pi), z \in \mathcal{R}, t \in \mathcal{R}$

$$\overline{J}(0,0,z,t) = \overline{0}, \ \ z \in \mathcal{R}, \ \ t \in \mathcal{R}.$$

Clearly, $(\rho_1, \overline{J})$ is supported on $Ann(1,\epsilon) \times (-\epsilon, \epsilon)$, and if $\overline{J} = (j_1, j_2, j_3)$, we have that $j_3(x,y,z,t) = 0$, so that $\frac{\partial j_3}{\partial z} = 0$ and

$$\bigtriangledown \cdot \overline{J} = \frac{\partial j_1}{\partial x} + \frac{\partial j_2}{\partial y}. \tag{12}$$

We have that, for any $z \in \mathcal{R}$, $(\Phi_{1,\epsilon}(z)\rho_1(r,\theta,t), \Phi_{1,\epsilon}(z)\overline{J}(r,\theta,t))$ satisfies the continuity equation on $\mathcal{R}^2 \times \mathcal{R}$, so combining the result with (12), we obtain that $(\rho_1, \overline{J})$ satisfies the continuity equation on $\mathcal{R}^3 \times \mathcal{R}$. □

Remark 3. If we require the pair $(\rho_1, \overline{J})$ to be real analytic, we can replace $\Phi_\epsilon(r)$ by $\Phi_{\epsilon,an}(r) = e^{-\frac{(r-1)^2}{\epsilon^2}}$, if $r > 0$, and $\Phi_{1,\epsilon}$ by $\Phi_{1,\epsilon,an}(z) = e^{-\frac{z^2}{\epsilon^2}}$, if $z \in \mathcal{R}$, in the proof, leaving $(\rho_1, \overline{J})$ to be undefined at $(0,0,z,t)$, for $z \in \mathcal{R}$, $t > 0$.

4. The No Radiation Condition

We consider the charge density ρ on $S(1)$, defined as in Lemma 2, with corresponding current $\overline{K}$, which we also denote by $\overline{J}$, coming from the wave equation, so that $\{\rho, \overline{J}\}$ satisfy the continuity equation $\frac{\partial \rho}{\partial t} + div(\overline{J}) = 0$ on $S(1)$, for small extensions of $\{\rho, \overline{J}\}$. In [8], it was shown that one then obtain an electromagnetic solution $(\rho, \overline{J}, \overline{E}, \overline{B})$, satisfying Maxwell's equations, given by the Jefimenko Equations;

$$\overline{E}(\overline{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\overline{r'},t_r)}{\mathfrak{r}^2}\hat{\bar{\mathfrak{r}}} + \frac{\dot{\rho}(\overline{r'},t_r)}{c\mathfrak{r}}\hat{\bar{\mathfrak{r}}} - \frac{\dot{\overline{J}}(\overline{r'},t_r)}{c^2\mathfrak{r}}\right]d\tau', \ \ \overline{B}(\overline{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\overline{J}(\overline{r'},t_r)}{\mathfrak{r}^2} + \frac{\dot{\overline{J}}(\overline{r'},t_r)}{c\mathfrak{r}}\right] \times \hat{\bar{\mathfrak{r}}}d\tau', \tag{13}$$

where $\mathfrak{r} = |\overline{r} - \overline{r'}|$, $\hat{\bar{\mathfrak{r}}} = \frac{\overline{r}-\overline{r'}}{|\overline{r}-\overline{r'}|}$, $t_r = t - \frac{\mathfrak{r}}{c}$, τ' is the measure with respect to the integrand variable $\overline{r'}$. We determine the conditions on $\{\rho, \overline{J}\}$ for which the no radiation condition holds, that is; $lim_{r\rightarrow\infty}P(r) = 0$ where $P(r) = \int_{S^3(r)} (\overline{E} \times \overline{B}).d\overline{S}(r)$ and $\{\overline{E}, \overline{B}\}$ are these (causal) fields, see [2].

Lemma 7. *Let $(\rho, \overline{J})$ be solutions to the three dimensional wave equations with velocity c, the connecting relation, with velocity c, and the continuity equation*

$$\begin{cases} \square^2\rho = 0, \\ \square^2\overline{J} = 0, \\ \bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}, \\ \frac{\partial \rho}{\partial t} + \bigtriangledown \cdot \overline{J} = 0, \end{cases} \tag{14}$$

where $\square^2$ *denotes the d'Alembertian operator. Then for any solutions* $(\overline{E},\overline{B})$*, such that* $(\rho,\overline{J},\overline{E},\overline{B})$ *satisfy Maxwell's equations, in particular for the causal solutions* $(\overline{E},\overline{B})$ *given by Jefimenko's equation, we have that;*

$$\square^2\overline{E}=\overline{0},\qquad \square^2\overline{B}=\overline{0},$$

and, the same result holds for the transformed current and charge, $(\rho_S,\overline{J}_S)$*, in every inertial frame* S*.*

Proof. By the proof in [8], we can find a pair $(\overline{E},\overline{B})$ in the base frame, with $\square^2\overline{E}=\overline{0}$, and $\overline{B}=\overline{0}$, (13), such that $(\rho,\overline{J},\overline{E},\overline{B})$ satisfy Maxwell's equations. If $(\overline{E}',\overline{B}')$ is any pair such that $(\rho,\overline{J},\overline{E}',\overline{B}')$ satisfy Maxwell's equations, then, taking the difference, $(0,\overline{0},\overline{E}-\overline{E}',\overline{B}-\overline{B}')$ is a vacuum solution to Maxwell's equations, so that, see [2];

$$\square^2(\overline{E}-\overline{E}')=\overline{0},\qquad \square^2(\overline{B}-\overline{B}')=\overline{0}.$$

From (13), we obtain that;

$$\square^2\overline{E}'=\overline{0},\qquad \square^2\overline{B}'=\overline{0},$$

as well. The last claim follows from the above proof and the results in [8]. □

Remark 4. We conjecture that: For any $\{\rho,\overline{J}\}$ satisfying the conditions from Lemma 7, and corresponding causal $\{\overline{E},\overline{B}\}$ from Jefimenko's equations, that $\{\overline{E},\overline{B}\}$ satisfies the no radiation condition iff $\{\overline{E}+\overline{E}_0,\overline{B}+\overline{B}_0\}$ satisfies the no radiation condition, for certain corresponding pairs $\{\overline{E}_0,\overline{B}_0\}$, satisfying Maxwell's equations in vacuum. This is still work in progress, see [10].

In this case, as $\overline{B}=\overline{0}$, we may obtain for the causal fields $(\overline{E}',\overline{B}')$ from Lemma 7, that they satisfy the no radiation condition, and the same is true in every inertial frame S.

Lemma 8. *Keeping the order in (13) and writing;*

$$\overline{E}(\overline{r},t)=\overline{E}_1(\overline{r},t)+\overline{E}_2(\overline{r},t)+\overline{E}_3(\overline{r},t),\qquad \overline{B}(\overline{r},t)=\overline{B}_1(\overline{r},t)+\overline{B}_2(\overline{r},t).$$

we have that

$$\lim_{r\to\infty}P(r)=\lim_{r\to\infty}\int_{S(r)}(\overline{E}_2\times\overline{B}_2+\overline{E}_3\times\overline{B}_2).d\overline{S}(r).$$

Proof. We have that $\overline{E}\times\overline{B}=(\overline{E}_1+\overline{E}_2+\overline{E}_3)\times(\overline{B}_1+\overline{B}_2)$. A simple calculation shows that, as $c>1$, that for $|\overline{r}|>1,\overline{r}|\in S^3(r)$;

$$\begin{aligned}
|\overline{E}_1\times\overline{B}_1|_{\overline{r},t} &\leq \frac{\mu_0}{16\pi^2\epsilon_0}\int\limits_{S^1}\int\limits_{S^1}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|p|(\overline{r'},t-t_r)|\overline{J}|(\overline{s'},t-t_r))}{(r-1)^4}d\theta d\phi\\
&\leq 4\pi^2\frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|p|(\overline{r'},t-t_r)|\overline{J}|(\overline{s'},t-t_r))}{(r-1)^4},\\
|\overline{E}_1\times\overline{B}_2| &\leq \frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|p|(\overline{r'},t-t_r)|\dot{\overline{J}}|(\overline{s'},t-t_r))}{(r-1)^3}d\theta d\phi\\
&\leq 4\pi^2\frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|p|(\overline{r'},t-t_r)|\dot{\overline{J}}|(\overline{s'},t-t_r))}{(r-1)^3},\\
|\overline{E}_2\times\overline{B}_1| &\leq \frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|\dot{p}|(\overline{r'},t-t_r)\overline{J}|(\overline{s'},t-t_r))}{(r-1)^3}d\theta d\phi\\
&\leq 4\pi^2\frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|\dot{p}|(\overline{r'},t-t_r)|\overline{J}|(\overline{s'},t-t_r))}{(r-1)^3},\\
|\overline{E}_3\times\overline{B}_1| &\leq \frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|\overline{J}|(\overline{r'},t-t_r)|\dot{\overline{J}}|(\overline{s'},t-t_r))}{(r-1)^3}d\theta d\phi\\
&\leq 4\pi^2\frac{\mu_0}{16\pi^2\epsilon_0}\frac{\max\limits_{\overline{r}',\overline{s}'\in S^1}(|\overline{J}|(\overline{r'},t-t_r)|\dot{\overline{J}}|(\overline{s'},t-t_r))}{(r-1)^3}.
\end{aligned}$$

It follows that, for $r>1$;

$$\begin{aligned}
&\max\left(\left|\int\limits_{S^3(r)}(\overline{E}_1\times\overline{B}_1).d\overline{S}(r)|,\int\limits_{S^3(r)}(\overline{E}_1\times\overline{B}_2).d\overline{S}(r)|,\int\limits_{S^3(r)}(\overline{E}_2\times\overline{B}_1).d\overline{S}(r)|,\int\limits_{S^3(r)}(\overline{E}_3\times\overline{B}_1).d\overline{S}(r)\right|\right)\\
&=\max\left(\left|\int\limits_{S^3(r)}(\overline{E}_1\times\overline{B}_1).\hat{\overline{n}}dS(r)|,|\int\limits_{S^3(r)}(\overline{E}_1\times\overline{B}_2).\hat{\overline{n}}dS(r)|,|\int\limits_{S^3(r)}(\overline{E}_2\times\overline{B}_1).\hat{\overline{n}}dS(r)|,|\int\limits_{S^3(r)}(\overline{E}_3\times\overline{B}_1).\hat{\overline{n}}dS(r)\right|\right)\\
&\leq\max(\int\limits_{S^3(r)}|\overline{E}_1\times\overline{B}_1)|dS(r)|,\int\limits_{S^3(r)}|(\overline{E}_1\times\overline{B}_2)|dS(r),\int\limits_{S^3(r)}|(\overline{E}_2\times\overline{B}_1)|dS(r),\int\limits_{S^3(r)}|(\overline{E}_3\times\overline{B}_1)|dS(r)|)\\
&\leq\int\limits_{S^3(r)}(\max(|\overline{E}_1\times\overline{B}_1|(\overline{r},t),|\overline{E}_1\times\overline{B}_2|(\overline{r},t),|\overline{E}_2\times\overline{B}_1|(\overline{r},t),|\overline{E}_3\times\overline{B}_1|(\overline{r},t)))dS(r)\\
&\leq 4\pi^2\frac{Area(\delta(S^3(r)))}{(r-1)^3}\frac{\mu_0}{16\pi^2\epsilon_0}\max_{\overline{r}',\overline{s}'\in S^1}\Big(|p|\,(\overline{r}',t-t_r)\,|\overline{J}(\overline{s}',t-t_r)|,|p|(\overline{r}',t-t_r)|\dot{\overline{J}}(\overline{s}',t-t_r)|,|\dot{p}(\overline{r}',t-t_r)|\\
&\quad\times|\overline{J}(\overline{s}',t-t_r)|,|\overline{J}(\overline{r}',t-t_r)||\dot{\overline{J}}((\overline{s}',t-t_r))|\Big)\\
&=4\pi^2\frac{4\pi r^2}{(r-1)^3}\frac{\mu_0}{16\pi^2\epsilon_0}\max_{\overline{r}',\overline{s}'\in S^1}(|p|(\overline{r}',t-t_r)|\overline{J}(\overline{s}',t-t_r)|,|p|(\overline{r}',t-t_r)|\dot{\overline{J}}(\overline{s}',t-t_r)|,|\dot{p}(\overline{r}',t-t_r)|\\
&\quad\times|\overline{J}(\overline{s}',t-t_r)|,|\overline{J}(\overline{r}',t-t_r)||\dot{\overline{J}}((\overline{s}',t-t_r))|)\\
&=C(r,t-t_r),
\end{aligned}$$

with $\lim\limits_{r\to\infty}C(r,t-t_r)=0$, for $t\in\mathcal{R}$, given the assumption that $lim_{t\to-\infty}\max\limits_{x\in S^1}(|p|,|\dot{p}|,|\overline{J}|,|\dot{\overline{J}})|_{x,t}\leq D$, with $D\in\mathcal{R}$, see Remark 1.

It follows that, for $t \in \mathcal{R}$,

$$\lim_{r\to\infty} P(r,t) = \lim_{r\to\infty} \int_{S^3(r)} (\overline{E} \times \overline{B})(\overline{r},t).d\overline{S}(r) = \lim_{r\to\infty} \int_{S^3(r)} (\overline{E}_2 \times \overline{B}_2 + \overline{E}_3 \times \overline{B}_2)(\overline{r},t).d\overline{S}(r),$$

as required. □

Lemma 9. *Let* $\rho(x,t) = \cos(mx)\cos(mt)$, $J(x,t) = \sin(mx)\sin(mt)$, *for* $m \in \mathcal{Z} > 0$, $x \in [-\pi,\pi)$, $t \in \mathcal{R}$, *with corresponding* $\rho(\theta,t)$ *and* $\overline{J}(\theta,t) = J(\theta)(-\sin(\theta),\cos(\theta),0)$ *with* $\theta \in [-\pi,\pi)$. *Let* $\overline{E}_m$ *and* $\overline{B}_m$ *be the causal fields determined by Jefimenko's equations, then, if m is even;*

$$\begin{aligned}
P(r,t) =& \beta\gamma m^2 \left[-\sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \right. \\
& \left. +2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right] \\
& \times \sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\le m,s,odd,s'\le m,s',odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)!(r^2+1)^{w+\frac{1}{2}}} \\
& \times (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'+1}}{(2w'+1)!(r^2+1)^{w'+\frac{1}{2}}} r^{2w+2w'+5} c_{w,w',s,s',m} + O\left(\frac{1}{r}\right).
\end{aligned}$$

and, if m is odd

$$\begin{aligned}
P(r,t) =& \beta\gamma m^2 \left[-\cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \right. \\
& \left. -2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right]
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\le m,s,odd,s'\le m,s',odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} (-1)^{\frac{s'-1}{2}} \\
& \times C_{s'}^m \frac{(-1)^{w'} m^{2w'}}{(2w')!(r^2+1)^{w'}} r^{2w+2w'+3} d_{w,w',s,s',m} + O\left(\frac{1}{r}\right).
\end{aligned}$$

Proof. Since, we have that

$$\overline{E}_2(\overline{r},t) = \frac{1}{4\pi\epsilon_0}\int [\frac{\dot{\rho}(\overline{r'},t_r)}{c\mathfrak{r}}\hat{\mathfrak{r}}]d\tau'$$

where $\bar{r} = (x, y, z)$, $\overline{r'} = (\cos(\theta, \sin(\theta), 0)$, $\bar{r} - \overline{r'} = (x - \cos(\theta), y - \sin(\theta), z)$, $\mathfrak{r} = |\bar{r} - \overline{r'}| = (x^2 + y^2 + z^2 + 1 - 2x\cos(\theta) - 2y\sin(\theta))^{\frac{1}{2}} = (r^2 + 1 - 2x\cos(\theta) - 2y\sin(\theta))^{\frac{1}{2}}$, $\hat{\mathfrak{r}} = \frac{(\bar{r} - \overline{r'})}{\mathfrak{r}} = \frac{(x - \cos(\theta), y - \sin(\theta), z)}{(x^2 + y^2 + z^2 + 1 - 2x\cos(\theta) - 2y\sin(\theta))^{\frac{1}{2}}}$, $\dot{\rho}(\theta, t) = -m\cos(m\theta)\sin(mt)$,

$$\begin{aligned}
\overline{E}_2\left(\bar{r}, t\right) &= \frac{1}{4\pi\epsilon_0 c} \int\limits_{-\pi}^{\pi} \frac{-m\cos\left(m\theta\right)\sin\left(mt_r\right)}{\left(r^2 + 1 - 2x\cos\left(\theta\right) - 2y\sin\left(\theta\right)\right)} \left(x - \cos\left(\theta\right), y - \sin\left(\theta\right), z\right) d\theta \\
&= \frac{1}{4\pi\epsilon_0 c\left(r^2 + 1\right)} \int\limits_{-\pi}^{\pi} \frac{-m\cos\left(m\theta\right)\sin\left(m\left(t - \frac{\mathfrak{r}}{c}\right)\right)}{\left(1 - \frac{2x\cos(\theta) - 2y\sin(\theta)}{(r^2+1)}\right)} \left(x - \cos\left(\theta\right), y - \sin\left(\theta\right), z\right) d\theta \\
&= \frac{1}{4\pi\epsilon_0 c\left(r^2 + 1\right)} \int\limits_{-\pi}^{\pi} -m\cos\left(m\theta\right)\sin\left(m\left(t - \frac{\mathfrak{r}}{c}\right)\right) \left(x - \cos\left(\theta\right), y - \sin\left(\theta\right), z\right) d\theta + O\left(\frac{1}{r^2}\right) \\
&= \frac{1}{4\pi\epsilon_0 c\left(r^2 + 1\right)} \int\limits_{-\pi}^{\pi} -m\cos\left(m\theta\right)\left[\sin\left(mt\right)\cos\left(m\frac{\mathfrak{r}}{c}\right) - \cos\left(mt\right)\sin\left(m\frac{\mathfrak{r}}{c}\right)\right] \left(x, y, z\right) d\theta + O\left(\frac{1}{r^2}\right) \\
&= \frac{-m\sin\left(mt\right)}{4\pi\epsilon_0 c\left(r^2 + 1\right)} \int\limits_{-\pi}^{\pi} \cos\left(m\theta\right)\cos\left(m\frac{\mathfrak{r}}{c}\right)\left(x, y, z\right) d\theta + \frac{m\cos\left(mt\right)}{4\pi\epsilon_0 c\left(r^2 + 1\right)} \int\limits_{-\pi}^{\pi} \cos\left(m\theta\right)\sin\left(m\frac{\mathfrak{r}}{c}\right)\left(x, y, z\right) d\theta \\
&\quad + O\left(\frac{1}{r^2}\right)
\end{aligned}$$

Observe that, using Taylor expansions;

$$\begin{aligned}
\cos\left(m\frac{\mathfrak{r}}{c}\right) &= \cos\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\left(1 + \frac{x\cos\left(\theta\right) + y\sin\left(\theta\right)}{r^2 + 1}\right)\right) + O\left(\frac{1}{r}\right) \\
&= \cos\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\right)\cos\left(\frac{mx\cos\left(\theta\right) + my\sin\left(\theta\right)}{\left(r^2 + 1\right)^{\frac{1}{2}}}\right) \\
&\quad - \sin\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\right)\sin\left(\frac{mx\cos\left(\theta\right) + my\sin\left(\theta\right)}{\left(r^2 + 1\right)^{\frac{1}{2}}}\right) + O\left(\frac{1}{r}\right) \\
\sin\left(m\frac{\mathfrak{r}}{c}\right) &= \sin\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\left(1 + \frac{x\cos\left(\theta\right) + y\sin\left(\theta\right)}{r^2 + 1}\right)\right) + O\left(\frac{1}{r}\right) \\
&= \sin\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\right)\cos\left(\frac{mx\cos\left(\theta\right) + my\sin\left(\theta\right)}{\left(r^2 + 1\right)^{\frac{1}{2}}}\right) \\
&\quad + \cos\left(m\frac{\left(r^2 + 1\right)^{\frac{1}{2}}}{c}\right)\sin\left(\frac{mx\cos\left(\theta\right) + my\sin\left(\theta\right)}{\left(r^2 + 1\right)^{\frac{1}{2}}}\right) + O\left(\frac{1}{r}\right),
\end{aligned}$$

so that;

$$\begin{aligned}\overline{E}_2\left(\overline{r},t\right) =&\frac{-m\sin\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\cos\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(x,y,z\right)d\theta\\ &+\frac{m\sin\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\cos\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(x,y,z\right)d\theta\\ &+\frac{m\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\cos\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(x,y,z\right)d\theta\\ &+\frac{m\cos\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\cos\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(x,y,z\right)d\theta+O\left(\frac{1}{r^2}\right).\end{aligned}\tag{15}$$

We have that

$$\cos\left(m\theta\right)=Re\left(\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)^m\right)=\sum_{s\leq m,s,even}\left(-1\right)^{\frac{s}{2}}C_s^m\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right),$$

$$\begin{aligned}\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)&=\sum_{w=0}^{\infty}\frac{\left(-1\right)^w m^{2w}\left(x\cos\left(\theta\right)+y\sin\left(\theta\right)\right)^{2w}}{(2w)!\left(r^2+1\right)^w}\\ &=\sum_{w=0}^{\infty}\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\sum_{v=0}^{2w}C_v^{2w}x^{2w-v}\cos^{2w-v}\left(\theta\right)y^v\sin^v\left(\theta\right),\end{aligned}$$

$$\begin{aligned}\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)&=\sum_{w=0}^{\infty}\frac{\left(-1\right)^w m^{2w+1}\left(x\cos\left(\theta\right)+y\sin\left(\theta\right)\right)^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\\ &=\sum_{w=0}^{\infty}\frac{\left(-1\right)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\sum_{v=0}^{2w+1}C_v^{2w+1}x^{2w+1-v}\cos^{2w+1-v}\left(\theta\right)y^v\sin^v\left(\theta\right),\end{aligned}$$

so that;

$$\begin{aligned}\overline{E}_2\left(\overline{r},t\right)=&\frac{-m\sin\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}\left(-1\right)^{\frac{s}{2}}C_s^m\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\sum_{v=0}^{2w}C_v^{2w}x^{2w-v}y^v\left(x,y,z\right)\\ &\times\int\limits_{-\pi}^{\pi}\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w-v}\left(\theta\right)\sin^v\left(\theta\right)d\theta+\frac{m\sin\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\\ &\times\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}\left(-1\right)^{\frac{s}{2}}C_s^m\frac{\left(-1\right)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\sum_{v=0}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^v\left(x,y,z\right)\\ &\times\int\limits_{-\pi}^{\pi}\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w+1-v}\left(\theta\right)\sin^v\left(\theta\right)d\theta+\frac{m\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\\ &\times\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}\left(-1\right)^{\frac{s}{2}}C_s^m\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\sum_{v=0}^{2w}C_v^{2w}x^{2w-v}y^v\left(x,y,z\right)\end{aligned}$$

$$\begin{aligned}
&\times \int_{-\pi}^{\pi} \cos^{m-s}(\theta)\sin^{s}(\theta)\cos^{2w-v}(\theta)\sin^{v}(\theta)\,d\theta + \frac{m\cos(mt)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\\
&\times \sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\sum_{v=0}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^v(x,y,z)\\
&\times \int_{-\pi}^{\pi} \cos^{m-s}(\theta)\sin^{s}(\theta)\cos^{2w+1-v}(\theta)\sin^{v}(\theta)\,d\theta + O\left(\frac{1}{r^2}\right).
\end{aligned}$$

We recall the result that;

$$\begin{aligned}
&I_\alpha^\beta = \int_{-\pi}^{\pi}\cos^\alpha(\theta)\sin^\beta(\theta)d\theta = \frac{\pi\alpha!\beta!}{2^{\alpha+\beta-1}(\frac{\alpha}{2})!(\frac{\beta}{2})!(\frac{\alpha+\beta}{2})!}, \quad \text{for } \alpha\geq 0, \beta\geq 0, \alpha \text{ and } \beta \text{ even},\\
&I_\alpha^\beta = 0, \quad \text{otherwise.}
\end{aligned}$$

Applying the result in this case, we obtain; if m is even, then;

$$\begin{aligned}
\overline{E}_2(\bar{r},t) =& \frac{-m\sin(mt)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w}}{(2w)!\left(r^2+1\right)^{w}}\\
&\times\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^v(x,y,z)\,I_{2w+m-s-v}^{s+v}\\
&+\frac{m\cos(mt)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w}}{(2w)!\left(r^2+1\right)^{w}}\\
&\times\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^v(x,y,z)\,I_{2w+m-s-v}^{s+v}+O\left(\frac{1}{r^2}\right),
\end{aligned}\tag{16}$$

if m is odd, then;

$$\begin{aligned}
\overline{E}_2(\bar{r},t) =& \frac{m\sin(mt)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\\
&\times\sum_{v=0,v,even}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^v(x,y,z)\,I_{2w+1+m-s-v}^{s+v}\\
&+\frac{m\cos(mt)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\\
&\times\sum_{v=0,v,even}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^v(x,y,z)\,I_{2w+1+m-s-v}^{s+v}+O\left(\frac{1}{r^2}\right).
\end{aligned}\tag{17}$$

We have that

$$\begin{aligned}
\overline{E}_3(\bar{r},t) &= \frac{-1}{4\pi\epsilon_0}\int\left[\frac{\dot{\overline{J}}\left(\overline{r'},t_r\right)}{c^2\mathfrak{r}}\right]d\tau'\dot{\overline{J}}(\theta,t)\\
&= m\sin(m\theta)\cos(mt)\left(-\sin(\theta),\cos(\theta),0\right),
\end{aligned}$$

$$
\begin{aligned}
\overline{E}_3\left(\overline{r},t\right) =& \frac{-1}{4\pi\epsilon_0 c^2}\int\limits_{-\pi}^{\pi}\frac{m\sin\left(m\theta\right)\cos\left(mt_r\right)}{\left(r^2+1-2x\cos\left(\theta\right)-2y\sin\left(\theta\right)\right)^{\frac{1}{2}}}\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
=& \frac{-1}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\frac{m\sin\left(m\theta\right)\cos\left(m\left(t-\frac{\mathfrak{r}}{c}\right)\right)}{\left(1-\frac{2x\cos(\theta)-2y\sin(\theta)}{(r^2+1)}\right)^{\frac{1}{2}}}\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
=& \frac{-1}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}m\sin\left(m\theta\right)\cos\left(m\left(t-\frac{\mathfrak{r}}{c}\right)\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta+O\left(\frac{1}{r^2}\right)\\
=& \frac{-1}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}m\sin\left(m\theta\right)\left[\cos\left(mt\right)\cos\left(m\frac{\mathfrak{r}}{c}\right)+\sin\left(mt\right)\sin\left(m\frac{\mathfrak{r}}{c}\right)\right]\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&+O\left(\frac{1}{r^2}\right)\\
=& \frac{-m\cos\left(mt\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(m\frac{\mathfrak{r}}{c}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&-\frac{m\sin\left(mt\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(m\frac{\mathfrak{r}}{c}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta+O\left(\frac{1}{r^2}\right)\\
=& \frac{-m\cos\left(mt\right)\cos\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&+\frac{m\cos\left(mt\right)\sin\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&-\frac{m\sin\left(mt\right)\sin\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&-\frac{m\sin\left(mt\right)\cos\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)d\theta\\
&+O\left(\frac{1}{r^2}\right).
\end{aligned}
\tag{18}
$$

We have that

$$
\sin\left(m\theta\right)=Im\left(\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)^m\right)=\sum_{s\leq m,sodd}\left(-1\right)^{\frac{s-1}{2}}C_s^m\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right),
$$

so that

$$
\begin{aligned}
\overline{E}_3\left(\overline{r},t\right) =& \frac{-m\cos\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}\left(-1\right)^{\frac{s-1}{2}}C_s^m\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\\
&\times\sum_{v=0}^{2w}C_v^{2w}x^{2w-v}y^v\int\limits_{-\pi}^{\pi}\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w-v}\left(\theta\right)\sin^v\left(\theta\right)d\theta\\
&+\frac{m\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2\left(r^2+1\right)^{\frac{1}{2}}}\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}\left(-1\right)^{\frac{s-1}{2}}C_s^m\frac{\left(-1\right)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{v=0}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v \int_{-\pi}^{\pi} \left(-\sin(\theta), \cos(\theta), 0\right) \cos^{m-s}(\theta) \sin^s(\theta) \cos^{2w+1-v}(\theta) \sin^v(\theta)\, d\theta \\
&- \frac{m \sin(mt) \sin\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)! \left(r^2+1\right)^w} \\
&\times \sum_{v=0}^{2w} C_v^{2w} x^{2w-v} y^v \int_{-\pi}^{\pi} \left(-\sin(\theta), \cos(\theta), 0\right) \cos^{m-s}(\theta) \sin^s(\theta) \cos^{2w-v}(\theta) \sin^v(\theta)\, d\theta \\
&- \frac{m \sin(mt) \cos\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)! \left(r^2+1\right)^{w+\frac{1}{2}}} \\
&\times \sum_{v=0}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v \int_{-\pi}^{\pi} \left(-\sin(\theta), \cos(\theta), 0\right) \cos^{m-s}(\theta) \sin^s(\theta) \cos^{2w+1-v}(\theta) \sin^v(\theta)\, d\theta + O\left(\frac{1}{r^2}\right).
\end{aligned}
$$

It follows that; for m even

$$
\begin{aligned}
\overline{E}_3(\overline{r}, t) = & + \frac{m \cos(mt) \sin\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)! \left(r^2+1\right)^{w+\frac{1}{2}}} \\
& \times \left(-\sum_{v=0, v, even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v I_{2w+1+m-v-s}^{v+s+1}, \sum_{v=0, v, odd}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v I_{2w+2+m-v-s}^{v+s}, 0\right) \\
& - \frac{m \sin(mt) \cos\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)! \left(r^2+1\right)^{w+\frac{1}{2}}} \\
& \times \left(-\sum_{v=0, v, even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v I_{2w+1+m-v-s}^{v+s+1}, \sum_{v=0, v, odd}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v I_{2w+2+m-v-s}^{v+s}, 0\right) \\
& + O\left(\frac{1}{r^2}\right), \qquad (19)
\end{aligned}
$$

and for m odd;

$$
\begin{aligned}
\overline{E}_3(\overline{r}, t) = & \frac{-m \cos(mt) \cos\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)! \left(r^2+1\right)^{w}} \\
& \times \left(-\sum_{v=0, v, even}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+m-v-s}^{v+s+1}, \sum_{v=0, v, odd}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+1+m-v-s}^{v+s}, 0\right) \\
& - \frac{m \sin(mt) \sin\left(m \frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi\epsilon_0 c^2 \left(r^2+1\right)^{\frac{1}{2}}} \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)! \left(r^2+1\right)^{w}} \\
& \times \left(-\sum_{v=0, v, even}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+m-v-s}^{v+s+1}, \sum_{v=0, v, odd}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+1+m-v-s}^{v+s}, 0\right) \\
& + O\left(\frac{1}{r^2}\right). \qquad (20)
\end{aligned}
$$

We have that;

$$\begin{aligned}\overline{B}_2\left(\overline{r},t\right)&=\frac{\mu_0}{4\pi}\int\left[\frac{\dot{\overline{J}}\left(\overline{r'},t_r\right)}{c\mathfrak{r}}\right]\times\hat{\mathfrak{r}}d\tau'\left(-\sin\left(\theta\right),\cos\left(\theta\right),0\right)\times\left(x-\cos\left(\theta\right),y-\sin\left(\theta\right),z\right)\\&=\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)\left(y-\sin\left(\theta\right)-\cos\left(\theta\right)\left(x-\cos\left(\theta\right)\right)\right),\end{aligned}$$

$$\begin{aligned}\overline{B}_2\left(\overline{r},t\right)&=\frac{\mu_0}{4\pi c}\int\limits_{-\pi}^{\pi}\frac{m\sin\left(m\theta\right)\cos\left(mt_r\right)}{\left(r^2+1-2x\cos\left(\theta\right)-2y\sin\left(\theta\right)\right)}\\&\quad\times\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)\left(y-\sin\left(\theta\right)-\cos\left(\theta\right)\left(x-\cos\left(\theta\right)\right)\right)\right)d\theta\\&=\frac{\mu_0}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\frac{m\sin\left(m\theta\right)\cos\left(m\left(t-\frac{\mathfrak{r}}{c}\right)\right)}{\left(1-\frac{2x\cos(\theta)-2y\sin(\theta)}{(r^2+1)}\right)}\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta+O\left(\frac{1}{r^2}\right)\\&=\frac{\mu_0}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}m\sin\left(m\theta\right)\cos\left(m\left(t-\frac{\mathfrak{r}}{c}\right)\right)\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta+O\left(\frac{1}{r^2}\right)\\&=\frac{\mu_0m\cos\left(mt\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(m\frac{\mathfrak{r}}{c}\right)\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta\\&\quad+\frac{\mu_0m\sin\left(mt\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(m\frac{\mathfrak{r}}{c}\right)\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta+O\left(\frac{1}{r^2}\right)\\&=\frac{\mu_0m\cos\left(mt\right)\cos\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\\&\quad\times\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta\\&\quad-\frac{\mu_0m\cos\left(mt\right)\sin\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\\&\quad\times\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta\\&\quad+\frac{\mu_0m\sin\left(mt\right)\sin\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\cos\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\\&\quad\times\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta\\&\quad+\frac{\mu_0m\sin\left(mt\right)\cos\left(m\left(r^2+1\right)^{\frac{1}{2}}\right)}{4\pi c\left(r^2+1\right)}\int\limits_{-\pi}^{\pi}\sin\left(m\theta\right)\sin\left(\frac{mx\cos\left(\theta\right)+my\sin\left(\theta\right)}{\left(r^2+1\right)^{\frac{1}{2}}}\right)\\&\quad\times\left(\cos\left(\theta\right)z,\sin\left(\theta\right)z,-\sin\left(\theta\right)y-\cos\left(\theta\right)x\right)d\theta+O\left(\frac{1}{r^2}\right)\\&=\frac{\mu_0m\cos\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}\left(-1\right)^{\frac{s-1}{2}}C_s^m\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\\&\quad\times\sum_{v=0}^{2w}C_v^{2w}x^{2w-v}y^v\int\limits_{-\pi}^{\pi}\left(z\cos\left(\theta\right),z\sin\left(\theta\right),-y\sin\left(\theta\right)-x\cos\left(\theta\right)\right)\\&\quad\times\cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w-v}\left(\theta\right)\sin^v\left(\theta\right)d\theta\\&\quad-\frac{\mu_0m\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)}\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}\left(-1\right)^{\frac{s-1}{2}}C_s^m\frac{\left(-1\right)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{v=0}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v \int_{-\pi}^{\pi} \left(z\cos\left(\theta\right), z\sin\left(\theta\right), -y\sin\left(\theta\right) - x\cos\left(\theta\right)\right) \\
&\times \cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w+1-v}\left(\theta\right)\sin^v\left(\theta\right) d\theta \\
&+ \frac{\mu_0 m \sin\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)} \sum_{w=0}^{\infty} \sum_{s\leq m,s,odd} \left(-1\right)^{\frac{s-1}{2}} C_s^m \frac{\left(-1\right)^w m^{2w}}{\left(2w\right)!\left(r^2+1\right)^w} \\
&\times \sum_{v=0}^{2w} C_v^{2w} x^{2w-v} y^v \int_{-\pi}^{\pi} \left(z\cos\left(\theta\right), z\sin\left(\theta\right), -y\sin\left(\theta\right) - x\cos\left(\theta\right)\right) \\
&\times \cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w-v}\left(\theta\right)\sin^v\left(\theta\right) d\theta \\
&+ \frac{\mu_0 m \sin\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)} \sum_{w=0}^{\infty} \sum_{s\leq m,s,odd} \left(-1\right)^{\frac{s-1}{2}} C_s^m \frac{\left(-1\right)^w m^{2w+1}}{\left(2w+1\right)!\left(r^2+1\right)^{w+\frac{1}{2}}} \\
&\times \sum_{v=0}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v \int_{-\pi}^{\pi} \left(z\cos\left(\theta\right), z\sin\left(\theta\right), -y\sin\left(\theta\right) - x\cos\left(\theta\right)\right) \\
&\times \cos^{m-s}\left(\theta\right)\sin^s\left(\theta\right)\cos^{2w+1-v}\left(\theta\right)\sin^v\left(\theta\right) d\theta + O\left(\frac{1}{r^2}\right). \qquad (21)
\end{aligned}
$$

It follows that; for m even;

$$
\begin{aligned}
\overline{B}_2\left(\overline{r},t\right) = &- \frac{\mu_0 m \cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)} \sum_{w=0}^{\infty} \sum_{s\leq m,s,odd} \left(-1\right)^{\frac{s-1}{2}} C_s^m \frac{\left(-1\right)^w m^{2w+1}}{\left(2w+1\right)!\left(r^2+1\right)^{w+\frac{1}{2}}} \\
&\times \left(\sum_{v=0,v,odd}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v z I_{2w+2+m-v-s}^{s+v}, \sum_{v=0,v,even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v z I_{2w+1+m-v-s}^{s+v+1} \right. \\
&\left. - \sum_{v=0,v,even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^{v+1} I_{2w+1+m-v-s}^{s+v+1} - \sum_{v=0,v,odd}^{2w+1} C_v^{2w+1} x^{2w+2-v} y^v I_{2w+2+m-v-s}^{s+v} \right) \\
&+ \frac{\mu_0 \sin\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)} \sum_{w=0}^{\infty} \sum_{s\leq m,s,odd} \left(-1\right)^{\frac{s-1}{2}} C_s^m \frac{\left(-1\right)^w m^{2w+1}}{\left(2w+1\right)!\left(r^2+1\right)^{w+\frac{1}{2}}} \\
&\times \left(\sum_{v=0,v,odd}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v z I_{2w+2+m-v-s}^{s+v}, \sum_{v=0,v,even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^v z I_{2w+1+m-v-s}^{s+v+1} \right. \\
&\left. - \sum_{v=0,v,even}^{2w+1} C_v^{2w+1} x^{2w+1-v} y^{v+1} I_{2w+1+m-v-s}^{s+v+1} - \sum_{v=0,v,odd}^{2w+1} C_v^{2w+1} x^{2w+2-v} y^v I_{2w+2+m-v-s}^{s+v} \right) + O\left(\frac{1}{r^2}\right), \qquad (22)
\end{aligned}
$$

and for m odd;

$$
\begin{aligned}
\overline{B}_2\left(\overline{r},t\right) = &\frac{\mu_0 m \cos\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)}{4\pi c\left(r^2+1\right)} \sum_{w=0}^{\infty} \sum_{s\leq m,s,odd} \left(-1\right)^{\frac{s-1}{2}} C_s^m \frac{\left(-1\right)^w m^{2w}}{\left(2w\right)!\left(r^2+1\right)^w} \\
&\times \left(\sum_{v=0,v,odd}^{2w} C_v^{2w} x^{2w-v} y^v z I_{2w+1+m-v-s}^{s+v}, \sum_{v=0,v,even}^{2w} C_v^{2w} x^{2w-v} y^v z I_{2w+m-v-s}^{s+v+1} \right. \\
&\left. - \sum_{v=0,v,even}^{2w} C_v^{2w} x^{2w-v} y^{v+1} I_{2w+m-v-s}^{s+v+1} - \sum_{v=0,v,odd}^{2w} C_v^{2w} x^{2w+1-v} y^v I_{2w+1+m-v-s}^{s+v} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mu_0 \sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)}{4\pi c\,(r^2+1)}\sum_{w=0}^{\infty}\sum_{s\le m,s,odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w)!\,(r^2+1)^w}\\
&\times\left(\sum_{v=0,v,odd}^{2w}C_v^{2w}x^{2w-v}y^v z I_{2w+1+m-v-s}^{s+v},\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^v z I_{2w+m-v-s}^{s+v+1}\right.\\
&\left.-\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^{v+1}I_{2w+m-v-s}^{s+v+1}-\sum_{v=0,v,odd}^{2w}C_v^{2w}x^{2w+1-v}y^v I_{2w+1+m-v-s}^{s+v}\right)+O\left(\frac{1}{r^2}\right).
\end{aligned}
\tag{23}
$$

We now compute the Poynting vectors $\overline{E}_2\times\overline{B}_2$ and $\overline{E}_3\times\overline{B}_2$ in the cases when m is even and m is odd. If m is even, by (15) and (16) we have that;

$$
\overline{E}_{2,e}=\overline{E}_{2,e}^1+\overline{E}_{2,e}^2=-\alpha m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma+\alpha m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma,
$$

where $\alpha=\frac{1}{4\pi\epsilon_0 c(r^2+1)}$ and $\Gamma=\int_{-\pi}^{\pi}\cos(m\theta)\cos\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(x,y,z)d\theta$. If m is even, by (21) and (22);

$$
\overline{B}_{2,e}=\overline{B}_{2,e}^1+\overline{B}_{2,e}^2=-\beta m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'+\beta m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'+O\left(\frac{1}{r^2}\right),
$$

where $\beta=\frac{\mu_0}{4\pi c(r^2+1)}$ and $\Gamma'=\int_{-\pi}^{\pi}\sin(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(\cos(\theta)z,\sin(\theta)z,-\sin(\theta)y-\cos(\theta)x)\,d\theta$. It follows that;

$$
\begin{aligned}
\overline{E}_{2,e}\times\overline{B}_{2,e}=&\overline{E}_{2,e}^1\times\overline{B}_{2,e}^1+\overline{E}_{2,e}^2\times\overline{B}_{2,e}^1+\overline{E}_{2,e}^1\times\overline{B}_{2,e}^2+\overline{E}_{2,e}^2\times\overline{B}_{2,e}^2+O\left(\frac{1}{r^3}\right)\\
=&\alpha\beta m^2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma\times\Gamma'\\
&-\alpha\beta m^2\cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma\times\Gamma'-\alpha\beta m^2\sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma\times\Gamma'\\
&+\alpha\beta m^2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma\times\Gamma'+O\left(\frac{1}{r^3}\right)\\
=&\alpha\beta m^2[-\sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\\
&+2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)]\Gamma\times\Gamma'+O\left(\frac{1}{r^3}\right).
\end{aligned}
$$

Similarly, by (18) and (19);

$$
\overline{E}_{3,e}=\overline{E}_{3,e}^1+\overline{E}_{3,e}^2=\gamma m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''-\gamma m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''+O\left(\frac{1}{r^2}\right),
$$

where $\gamma=\frac{1}{4\pi\epsilon_0 c^2(r^2+1)^{\frac{1}{2}}}$ and $\Gamma''=\int_{-\pi}^{\pi}\sin(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(-\sin(\theta),\cos(\theta),0)\,d\theta$. If m is even, it follows that;

$$
\overline{E}_{3,e}\times\overline{B}_{2,e}=\overline{E}_{3,e}^1\times\overline{B}_{2,e}^1+\overline{E}_{3,e}^2\times\overline{B}_{2,e}^1+\overline{E}_{3,e}^1\times\overline{B}_{2,e}^2+\overline{E}_{3,e}^2\times\overline{B}_{2,e}^2+O\left(\frac{1}{r^3}\right)
$$

$$\begin{aligned}
&= -\beta\gamma m^2 \cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''\times\Gamma' \\
&\quad + \beta\gamma m^2 \sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''\times\Gamma' \\
&\quad + \beta\gamma m^2 \sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''\times\Gamma' \\
&\quad - \beta\gamma m^2 \sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''\times\Gamma' + O\left(\frac{1}{r^3}\right) \\
&= \beta\gamma m^2[-\sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \\
&\quad + 2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)]\Gamma''\times\Gamma' + O\left(\frac{1}{r^3}\right),
\end{aligned}$$

If m is odd, by (15) and (17) we have that;

$$\overline{E}_{2,o} = \overline{E}^1_{2,o} + \overline{E}^2_{2,o} + O\left(\frac{1}{r^2}\right) = \alpha m \sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''' + \alpha m \cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma''',$$

where $\Gamma''' = \int\limits_{-\pi}^{\pi}\cos(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(x,y,z)d\theta$. If m is odd, by (21) and (23);

$$\overline{B}_{2,o} = \overline{B}^1_{2,o} + \overline{B}^2_{2,o} + O\left(\frac{1}{r^2}\right) = \beta m \cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''' + \beta m \sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''' + O\left(\frac{1}{r^2}\right),$$

where $\Gamma'''' = \int\limits_{-\pi}^{\pi}\sin(m\theta)\cos\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(\cos(\theta)z, \sin(\theta)z, -\sin(\theta)y - \cos(\theta)x)d\theta$. It follows that;

$$\begin{aligned}
\overline{E}_{2,o}\times\overline{B}_{2,o} &= \overline{E}^1_{2,o}\times\overline{B}^1_{2,o} + \overline{E}^2_{2,o}\times\overline{B}^1_{2,o} + \overline{E}^1_{2,o}\times\overline{B}^2_{2,o} + \overline{E}^2_{2,o}\times\overline{B}^2_{2,o} + O\left(\frac{1}{r^3}\right) \\
&= \alpha\beta m^2 \sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''\times\Gamma'''' \\
&\quad + \alpha\beta m^2 \cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''\times\Gamma'''' \\
&\quad + \alpha\beta m^2 \sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''\times\Gamma'''' \\
&\quad + \alpha\beta m^2 \sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Gamma'''\times\Gamma'''' + O\left(\frac{1}{r^3}\right) \\
&= \alpha\beta m^2[\cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) + \sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \\
&\quad + 2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)]\Gamma'''\times\Gamma'''' + O\left(\frac{1}{r^3}\right).
\end{aligned}$$

Similarly, by (18) and (20);

$$\overline{E}_{3,o}=\overline{E}^1_{3,o}+\overline{E}^2_{3,o}=-\gamma m\cos\left(mt\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''-\gamma m\sin\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''+O\left(\frac{1}{r^2}\right),$$

where $\Gamma'''''=\int\limits_{-\pi}^{\pi}\sin(m\theta)\cos\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(-\sin(\theta),\cos(\theta),0)d\theta$. If m is odd, it follows that;

$$\begin{aligned}\overline{E}_{3,o}\times\overline{B}_{2,o}=&\overline{E}^1_{3,o}\times\overline{B}^1_{2,o}+\overline{E}^2_{3,o}\times\overline{B}^1_{2,o}+\overline{E}^1_{3,o}\times\overline{B}^2_{2,o}+\overline{E}^2_{3,o}\times\overline{B}^2_{2,o}+O\left(\frac{1}{r^3}\right)\\
=&-\beta\gamma m^2\cos^2\left(mt\right)\cos^2\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''\times\Gamma''''\\
&-\beta\gamma m^2\sin\left(mt\right)\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''\times\Gamma''''\\
&-\beta\gamma m^2\sin\left(mt\right)\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''\times\Gamma''''\\
&-\beta\gamma m^2\sin^2\left(mt\right)\sin^2\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\Gamma'''''\times\Gamma''''+O\left(\frac{1}{r^3}\right)\\
=&\beta\gamma m^2[-\cos^2\left(mt\right)\cos^2\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)-\sin^2\left(mt\right)\sin^2\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\\
&-2\sin\left(mt\right)\cos\left(mt\right)\sin\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{\left(r^2+1\right)^{\frac{1}{2}}}{c}\right)]\Gamma'''''\times\Gamma''''+O\left(\frac{1}{r^3}\right).\end{aligned}$$

We compute $\Gamma\times\Gamma'$. By (16) and (22), we have that;

$$\Gamma=\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}\left(-1\right)^{\frac{s}{2}}C^m_s\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\sum_{v=0,v,even}^{2w}C^{2w}_v x^{2w-v}y^v\left(x,y,z\right)I_{2w+m-s-v,s+v},$$

$$\begin{aligned}\Gamma'=&\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}\left(-1\right)^{\frac{s-1}{2}}C^m_s\frac{\left(-1\right)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}}\\
&\times\left(\sum_{v=0,v,odd}^{2w+1}C^{2w+1}_v x^{2w+1-v}y^v zI^{s+v}_{2w+2+m-v-s},\sum_{v=0,v,even}^{2w+1}C^{2w+1}_v x^{2w+1-v}y^v zI^{s+v+1}_{2w+1+m-v-s}\right.\\
&\left.-\sum_{v=0,v,even}^{2w+1}C^{2w+1}_v x^{2w+1-v}y^{v+1}I^{s+v+1}_{2w+1+m-v-s}-\sum_{v=0,v,odd}^{2w+1}C^{2w+1}_v x^{2w+2-v}y^v I^{s+v}_{2w+2+m-v-s}\right),\end{aligned}$$

so that;

$$\begin{aligned}\Gamma\times\Gamma'=&\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\leq m,s,even,s'\leq m,s',odd}\left(-1\right)^{\frac{s}{2}}C^m_s\frac{\left(-1\right)^w m^{2w}}{(2w)!\left(r^2+1\right)^w}\left(-1\right)^{\frac{s'-1}{2}}C^m_{s'}\frac{\left(-1\right)^{w'}m^{2w'+1}}{(2w'+1)!\left(r^2+1\right)^{w'+\frac{1}{2}}}\\
&\left(-\sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1}C^{2w}_v C^{2w'+1}_{v'}x^{2w+2w'+1-v-v'}y^{v+v'+2}I^{s+v}_{2w+m-s-v}I^{s'+v'+1}_{2w'+1+m-v'-s'}\right.\\
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1}C^{2w}_v C^{2w'+1}_{v'}x^{2w+2w'+2-v-v'}y^{v+v'+1}I^{s+v}_{2w+m-s-v}I^{s'+v'}_{2w'+2-m-v'-s'}\\
&-\sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1}C^{2w}_v C^{2w'+1}_{v'}x^{2w+2w'+1-v-v'}y^{v+v'}z^2I^{s+v}_{2w+m-s-v}I^{s'+v'+1}_{2w'+1-m-v'-s'},\end{aligned}$$

$$
\begin{aligned}
&+\sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'+1}I_{2w+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'+1}\\
&+\sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w}C_{v'}^{2w'+1}x^{2w+2w'+3-v-v'}y^{v+v'}I_{2w+m-s-v}^{s+v}I_{2w'+2+m-v'-s'}^{s'+v'}\\
&+\sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w}C_{v'}^{2w'+1}x^{2w+2w'+1-v-v'}y^{v+v'}z^2I_{2w+m-s-v}^{s+v}I_{2w'+2+m-v'-s'}^{s'+v'},\\
&+\sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'}zI_{2w+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w}C_{v'}^{2w'+1}x^{2w+2w'+1-v-v'}y^{v+v'+1}zI_{2w+m-s-v}^{s+v}I_{2w'+2+m-v'-s'}^{s'+v'}\Bigg). \qquad (24)
\end{aligned}
$$

By (19), we have that;

$$
\begin{aligned}
\Gamma'' = &\sum_{w=0}^{\infty}\sum_{s\le m,s,odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}}\\
&\times\left(-\sum_{v=0,v,even}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^vI_{2w+1+m-v-s}^{v+s+1},\sum_{v=0,v,odd}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^vI_{2w+2+m-v-s}^{v+s},0\right).
\end{aligned}
$$

It follows that;

$$
\begin{aligned}
\Gamma''\times\Gamma' \quad = \quad &\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\le m,s,odd,s'\le m,s',odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}}(-1)^{\frac{s'-1}{2}}C_{s'}^m\frac{(-1)^{w'} m^{2w'+1}}{(2w'+1)!\,(r^2+1)^{w'+\frac{1}{2}}}\\
&\Bigg(-\sum_{v=0,v,odd,v'=0,v',even}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'+1}I_{2w+2+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,odd,v'=0,v',odd}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+3-v-v'}y^{v+v'}I_{2w+2+m-s-v}^{s+v}I_{2w'+2-m-v'-s'}^{s'+v'},\\
&-\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'+1}I_{2w+1+m-s-v}^{s+v+1}I_{2w'+1+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+3-v-v'}y^{v+v'}I_{2w+1+m-s-v}^{s+v+1}I_{2w'+2+m-v'-s'}^{s'+v'},\\
&-\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'}zI_{2w+1+m-s-v}^{s+v+1}I_{2w'+1+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,odd,v'=0,v',odd}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1}x^{2w+2w'+2-v-v'}y^{v+v'}zI_{2w+1++m-s-v}^{s+v+1}I_{2w'+2+m-v'-s'}^{s'+v'}\Bigg). \qquad (25)
\end{aligned}
$$

We now compute the flux of the Poynting vectors over the sphere $S(r)$. We have that $x = r\sin(\phi)\cos(\theta)$, $y = r\sin(\phi)\sin(\theta)$, $z = r\cos(\phi)$ with coordinates $0 \le \phi < \pi$ and $-\pi \le \theta < \pi$. We have that;

$$
\begin{aligned}
r_\phi\times r_\theta &= (r\cos(\phi)\cos(\theta), r\cos(\phi)\sin(\theta), -r\sin(\phi))\times(-r\sin(\phi)\sin(\theta), r\sin(\phi)\cos(\theta), 0)\\
&= r^2(\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi),
\end{aligned}
$$

so that;

$$
d\overline{S} = \hat{\overline{n}}dS = r^2(\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi))d\phi d\theta,
$$

and

$$
\int_{S(r)}(\Gamma\times\Gamma')\boldsymbol{.}d\overline{S} = \int_0^{\pi}\int_{-\pi}^{\pi}(\Gamma\times\Gamma')|_{(r\sin(\phi)\cos(\theta),r\sin(\phi)\sin(\theta),r\cos(\phi))}\boldsymbol{.}r^2(\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)d\theta d\phi.
$$

Applying this to (24) gives;

$$
\begin{aligned}
\int\limits_{S(r)} \left(\Gamma \times \Gamma'\right) . d\overline{S} = {} & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m, s, even, s' \leq m, s', odd} (-1)^{\frac{s}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)! \left(r^2+1\right)^w} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'+1}}{(2w'+1)! \left(r^2+1\right)^{w'+\frac{1}{2}}} \\
& \times \left(- \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'+1} \right. \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'+2}(\theta)\, d\theta d\phi \\
& - \sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+2-m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& - \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+1-m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3} \cos^2(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& + \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'+2}(\theta)\, d\theta d\phi \\
& + \sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+2+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& + \sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+2+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^2(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& + \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^2(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& - \sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'+1} C_v^{2w} C_{v'}^{2w'+1} I_{2w+m-s-v}^{s+v} I_{2w'+2+m-v'-s'}^{s'+v'} \\
& \left. \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^2(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \right). \qquad (26)
\end{aligned}
$$

An inspection of (26) shows that the terms 1 and 4, 2 and 5, 3 and 7, and 6 and 8 cancel. This proves that;

$$\int\limits_{S(r)} \left(\Gamma \times \Gamma'\right) . d\overline{S} = 0.$$

Applying the same method to (25), we have that;

$$\begin{aligned}\int\limits_{S(r)} \left(\Gamma'' \times \Gamma'\right) . d\overline{S} = & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m, s, odd, s' \leq m, s', odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)!\left(r^2+1\right)^{w+\frac{1}{2}}} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'+1}}{(2w'+1)!\left(r^2+1\right)^{w'+\frac{1}{2}}} \\
& - \sum_{v=0, v, odd, v'=0, v', even}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+2+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, odd, v'=0, v', odd}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+2+m-s-v}^{s+v} I_{2w'+2-m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+4-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'+2}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', odd}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+2+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^2(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, odd, v'=0, v', odd}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+1++m-s-v}^{s+v+1} I_{2w'+2+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^2(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi. \end{aligned} \tag{27}$$

Using the notation I_α^β again, and letting $J_\gamma = \frac{2^{\gamma+1}(\frac{\gamma-1}{2})!^2}{\gamma!} = \int\limits_0^{\pi} \sin^\gamma(\phi) d\phi$ for γ odd, we obtain that;

$$\begin{aligned}\int\limits_{S(r)} \left(\Gamma'' \times \Gamma'\right) . d\overline{S} = & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m, s, odd, s' \leq m, s', odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w+1}}{(2w+1)!(r^2+1)^{w+\frac{1}{2}}} \\
& \times (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'+1}}{(2w'+1)!(r^2+1)^{w'+\frac{1}{2}}} r^{2w+2w'+5} \\
& \times \Bigg(- \sum_{v=0, v, odd, v'=0, v', even}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+2+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'+1} I_{2w+2w'+3-v-v'}^{v+v'+1} J_{2w+2w'+5} \\
& - \sum_{v=0, v, odd, v'=0, v', odd}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+2+m-s-v}^{s+v} I_{2w'+2-m-v'-s'}^{s'+v'} I_{2w+2w'+4-v-v'}^{v+v'} J_{2w+2w'+5} \\
& - \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'+1} C_v^{2w+1} C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'+1} I_{2w+2w'+2-v-v'}^{v+v'+2} J_{2w+2w'+5} \end{aligned}$$

$$\begin{aligned}
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+2+m-v'-s'}^{s'+v'} I_{2w+2w'+3-v-v'}^{v+v'+1} J_{2w+2w'+5}\\
&-\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1} I_{2w+1+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'+1} I_{2w+2w'+2-v-v'}^{v+v'} (J_{2w+2w'+3}-J_{2w+2w'+5})\\
&-\sum_{v=0,v,odd,v'=0,v',odd}^{2w+1,2w'+1} C_v^{2w+1}C_{v'}^{2w'+1} I_{2w+1++m-s-v}^{s+v+1} I_{2w'+2+m-v'-s'}^{s'+v'} I_{2w+2w'+2-v-v'}^{v+v'} (J_{2w+2w'+3}-J_{2w+2w'+5})\Bigg)\\
=&\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\leq m,s,odd,s'\leq m,s',odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!(r^2+1)^{w+\frac{1}{2}}}\\
&\times(-1)^{\frac{s'-1}{2}}C_{s'}^m\frac{(-1)^{w'}m^{2w'+1}}{(2w'+1)!(r^2+1)^{w'+\frac{1}{2}}}r^{2w+2w'+5}c_{w,w',s,s',m},
\end{aligned}$$

where we have abbreviated the term in brackets to $c_{w,w',s,s',m}<0$. We compute $\Gamma'''\times\Gamma''''$. By (17) and (23);

$$\Gamma'''=\sum_{w=0}^{\infty}\sum_{s\leq m,s,even}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}}\sum_{v=0,v,even}^{2w+1}C_v^{2w+1}x^{2w+1-v}y^v\,(x,y,z)\,I_{2w+1+m-s-v}^{s+v}$$

and

$$\begin{aligned}
\Gamma''''=&\sum_{w=0}^{\infty}\sum_{s\leq m,s,odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w}}{(2w)!\,(r^2+1)^w}\\
&\times\Bigg(\sum_{v=0,v,odd}^{2w}C_v^{2w}x^{2w-v}y^v zI_{2w+1+m-v-s}^{s+v},\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^v zI_{2w+m-v-s}^{s+v+1}\\
&-\sum_{v=0,v,even}^{2w}C_v^{2w}x^{2w-v}y^{v+1}I_{2w+m-v-s}^{s+v+1}-\sum_{v=0,v,odd}^{2w}C_v^{2w}x^{2w+1-v}y^v I_{2w+1+m-v-s}^{s+v}\Bigg),
\end{aligned}$$

so that;

$$\begin{aligned}
\Gamma'''\times\Gamma''''=&\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\leq m,s,even,s'\leq m,s',odd}(-1)^{\frac{s}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}}(-1)^{\frac{s'-1}{2}}C_{s'}^m\frac{(-1)^{w'}m^{2w'}}{(2w')!\,(r^2+1)^{w'}}\\
&\times\Bigg(-\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+1-v-v'}y^{v+v'+2}I_{2w+1+m-s-v}^{s+v}I_{2w'+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+2-v-v'}y^{v+v'+1}I_{2w+1+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'}\\
&-\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+1-v-v'}y^{v+v'}z^2I_{2w+1+m-s-v}^{s+v}I_{2w'+m-v'-s'}^{s'+v'+1},\\
&+\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+2-v-v'}y^{v+v'+1}I_{2w+1+m-s-v}^{s+v}I_{2w'+m-v'-s'}^{s'+v'+1}\\
&+\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+3-v-v'}y^{v+v'}I_{2w+1+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'}\\
&+\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+1-v-v'}y^{v+v'}z^2I_{2w+1+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'},\\
&+\sum_{v=0,v,even,v'=0,v',even}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+2-v-v'}y^{v+v'}zI_{2w+1+m-s-v}^{s+v}I_{2w'+m-v'-s'}^{s'+v'+1}\\
&-\sum_{v=0,v,even,v'=0,v',odd}^{2w+1,2w'}C_v^{2w+1}C_{v'}^{2w'}x^{2w+2w'+1-v-v'}y^{v+v'+1}zI_{2w+1+m-s-v}^{s+v}I_{2w'+1+m-v'-s'}^{s'+v'}\Bigg).
\end{aligned}\tag{28}$$

Integrating (28), we obtain;

$$
\begin{aligned}
\int\limits_{S(r)} \left(\Gamma''' \times \Gamma''''\right) \boldsymbol{.}\, d\bar{S} = {} & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s\leq m, s, even, s' \leq m, s', odd} (-1)^{\frac{s}{2}} C_s^m \frac{(-1)^w \, m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} \, m^{2w'}}{(2w')!\,(r^2+1)^{w'}} \\
& \times \Bigg(- \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'+2}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', odd}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3} \cos^{2}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& + \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'+2}(\theta)\, d\theta d\phi \\
& + \sum_{v=0, v, even, v'=0, v', odd}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+5}(\phi) \cos^{2w+2w'+3-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& + \sum_{v=0, v, even, v'=0, v', odd}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^{2}(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \\
& + \sum_{v=0, v, even, v'=0, v', even}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^{2}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta)\, d\theta d\phi \\
& - \sum_{v=0, v, even, v'=0, v', odd}^{2w+1, 2w'} C_v^{2w+1} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \\
& \times \int\limits_0^{\pi} \int\limits_{-\pi}^{\pi} r^{2w+2w'+5} \sin^{2w+2w'+3}(\phi) \cos^{2}(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta)\, d\theta d\phi \Bigg).
\end{aligned}
\tag{29}
$$

An inspection of (29) shows that the terms 1 and 4, 2 and 5, 3 and 7, and 6 and 8 cancel again. This proves that;

$$
\int\limits_{S(r)} \left(\Gamma''' \times \Gamma''''\right) \boldsymbol{.}\, d\bar{S} = 0.
$$

By (20), we have that;

$$\begin{aligned}\Gamma''''' = & \sum_{w=0}^{\infty} \sum_{s \leq m, s, odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} \\ & \times \left(- \sum_{v=0, v, even}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+m-v-s}^{v+s+1}, \sum_{v=0, v, odd}^{2w} C_v^{2w} x^{2w-v} y^v I_{2w+1+m-v-s}^{v+s}, 0 \right).\end{aligned}$$

It follows that;

$$\begin{aligned}\Gamma''''' \times \Gamma'''' = & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m, s, odd, s' \leq m, s', odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'}}{(2w')!(r^2+1)^{w'}} \\ & \times \left(- \sum_{v=0, v, odd, v'=0, v', even}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'-v-v'} y^{v+v'+1} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \right. \\ & - \sum_{v=0, v, odd, v'=0, v', odd}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'+1-v-v'} y^{v+v'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1-m-v'-s'}^{s'+v'}, \\ & - \sum_{v=0, v, even, v'=0, v', even}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'-v-v'} y^{v+v'+1} I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} \\ & - \sum_{v=0, v, even, v'=0, v', odd}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'+1-v-v'} y^{v+v'} I_{2w+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'}, \\ & - \sum_{v=0, v, even, v'=0, v', even}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'-v-v'} y^{v+v'} z I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} \\ & \left. - \sum_{v=0, v, odd, v'=0, v', odd}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} x^{2w+2w'-v-v'} y^{v+v'} z I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \right). \end{aligned} \tag{30}$$

Integrating (30), we get that;

$$\begin{aligned}\int_{S(r)} (\Gamma''''' \times \Gamma'''') \cdot d\overline{S} = & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m, s, odd, s' \leq m, s', odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'}}{(2w')!(r^2+1)^{w'}} \\ & \times \left(- \sum_{v=0, v, odd, v'=0, v', even}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} \right. \\ & \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+3}(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta) d\theta d\phi \\ & - \sum_{v=0, v, odd, v'=0, v', odd}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1-m-v'-s'}^{s'+v'} \\ & \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+3}(\phi) \cos^{2w+2w'+2-v-v'}(\theta) \sin^{v+v'}(\theta) d\theta d\phi \\ & - \sum_{v=0, v, even, v'=0, v', even}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} \\ & \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+3}(\phi) \cos^{2w+2w'-v-v'}(\theta) \sin^{v+v'+2}(\theta) d\theta d\phi \\ & - \sum_{v=0, v, even, v'=0, v', odd}^{2w, 2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'} \\ & \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+3}(\phi) \cos^{2w+2w'+1-v-v'}(\theta) \sin^{v+v'+1}(\theta) d\theta d\phi \end{aligned}$$

$$
\begin{aligned}
& - \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} \\
& \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+1}(\phi) \cos^2(\phi) \cos^{2w+2w'-v-v'}(\theta) \sin^{v+v'}(\theta) d\theta d\phi \\
& - \sum_{v=0,v,odd,v'=0,v',odd}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} \Bigg) \\
& \times \int_0^{\pi} \int_{-\pi}^{\pi} r^{2w+2w'+3} \sin^{2w+2w'+1}(\phi) \cos^2(\phi) \cos^{2w+2w'-v-v'}(\theta) \sin^{v+v'}(\theta) d\theta d\phi \\
= & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m,s,odd,s' \leq m,s',odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} (-1)^{\frac{s'-1}{2}} C_{s'}^m \frac{(-1)^{w'} m^{2w'}}{(2w')!(r^2+1)^{w'}} r^{2w+2w'+3} \\
& \times \Bigg(- \sum_{v=0,v,odd,v'=0,v',even}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+m-v'-s'}^{s'+v'+1} I_{2w+2w'+1-v-v'}^{v+v'+1} J_{2w+2w'+3} \\
& - \sum_{v=0,v,odd,v'=0,v',odd}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1-m-v'-s'}^{s'+v'} I_{2w+2w'+2-v-v'}^{v+v'} J_{2w+2w'+3} \\
& - \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} I_{2w+2w'-v-v'}^{v+v'+2} J_{2w+2w'+3} \\
& - \sum_{v=0,v,even,v'=0,v',odd}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+1+m-v'-s'}^{s'+v'} I_{2w+2w'+1-v-v'}^{v+v'+1} J_{2w+2w'+3} \\
& - \sum_{v=0,v,even,v'=0,v',even}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+m-s-v}^{s+v+1} I_{2w'+m-v'-s'}^{s'+v'+1} I_{2w+2w'-v-v'}^{v+v'} (J_{2w+2w'+1} - J_{2w+2w'+3}) \\
& - \sum_{v=0,v,odd,v'=0,v',odd}^{2w,2w'} C_v^{2w} C_{v'}^{2w'} I_{2w+1+m-s-v}^{s+v} I_{2w'+1+m-v'-s'}^{s'+v'} I_{2w+2w'-v-v'}^{v+v'} (J_{2w+2w'+1} - J_{2w+2w'+3}) \Bigg) \\
= & \sum_{w=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{s \leq m,s,odd,s' \leq m,s',odd} (-1)^{\frac{s-1}{2}} C_s^m \frac{(-1)^w m^{2w}}{(2w)!(r^2+1)^w} (-1)^{\frac{s'-1}{2}} \\
& \times C_{s'}^m \frac{(-1)^{w'} m^{2w'}}{(2w')!(r^2+1)^{w'}} r^{2w+2w'+3} d_{w,w',s,s',m}, \qquad (31)
\end{aligned}
$$

where again we have denoted the term in brackets by $d_{w,w',s,s',m}$. We conclude that, if m is even;

$$
\begin{aligned}
P(r,t) = & \int_{S(r)} (E_{2,e} \times B_{2,e}) \cdot d\bar{S} + \int_{S(r)} (E_{3,e} \times B_{2,e}) \cdot d\bar{S} \\
= & \int_{S(r)} \Bigg(\Bigg[\alpha\beta m^2 \Bigg[-\sin^2(mt) \cos^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) - \cos^2(mt) \sin^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \\
& +2\sin(mt)\cos(mt)\sin\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \cos\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \Bigg] \Gamma \times \Gamma' \Bigg] + O\left(\frac{1}{r^3}\right) \Bigg) \cdot d\bar{S} \\
& + \int_{S(r)} \Bigg(\beta\gamma m^2 \Bigg[-\sin^2(mt) \cos^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) - \cos^2(mt) \sin^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \\
& +2\sin(mt)\cos(mt)\sin\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \cos\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) \Bigg] \Gamma'' \times \Gamma' + O\left(\frac{1}{r^3}\right) \Bigg) \cdot d\bar{S} \\
= & \beta\gamma m^2 \Bigg[-\sin^2(mt) \cos^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right) - \cos^2(mt) \sin^2\left(m \frac{(r^2+1)^{\frac{1}{2}}}{c} \right)
\end{aligned}
$$

$$+2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Bigg]$$
$$\times\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\leq m,s,odd,s'\leq m,s',odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w+1}}{(2w+1)!\,(r^2+1)^{w+\frac{1}{2}}}$$
$$\times(-1)^{\frac{s'-1}{2}}C_{s'}^m\frac{(-1)^{w'}m^{2w'+1}}{(2w'+1)!\,(r^2+1)^{w'+\frac{1}{2}}}r^{2w+2w'+5}c_{w,w',s,s',m}+O\left(\frac{1}{r}\right).$$

Similarly, if m is odd, we obtain that;

$$\begin{aligned}P(r,t)&=\int_{S(r)}(E_{2,o}\times B_{2,o})\cdot d\overline{S}+\int_{S(r)}(E_{3,o}\times B_{2,o})\cdot d\overline{S}\\&=\int_{S(r)}(E_{3,o}\times B_{2,o})\cdot d\overline{S}\\&=\beta\gamma m^2\Bigg[-\cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\\&\quad-2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Bigg]\\&\quad\times\sum_{w=0}^{\infty}\sum_{w'=0}^{\infty}\sum_{s\leq m,s,odd,s'\leq m,s',odd}(-1)^{\frac{s-1}{2}}C_s^m\frac{(-1)^w m^{2w}}{(2w)!\,(r^2+1)^w}(-1)^{\frac{s'-1}{2}}\\&\quad\times C_{s'}^m\frac{(-1)^{w'}m^{2w'}}{(2w')!\,(r^2+1)^{w'}}r^{2w+2w'+3}d_{w,w',s,s',m}+O\left(\frac{1}{r}\right).\end{aligned}$$

□

Remark 5. The previous result shows that a single standing wave radiates, oscillating with time t and radius r. We look for a cancellation by considering the other charge/current possibilities which satisfy the wave equation. This is the subject of the next theorem.

Definition 2. We let $\rho^1=\cos(mx)\cos(mt)$, $J^1=\sin(mx)\sin(mt)$, $\rho^2=\cos(mx)\sin(mt)$, $J^2=-\sin(mx)\cos(mt)$, $\rho^3=\sin(mx)\cos(mt)$, $J^3=-\cos(mx)\sin(mt)$ and $\rho^4=\sin(mx)\sin(mt)$, $J^4=\cos(mx)\cos(mt)$ so that the corresponding pairs $(\rho_i,\overline{J}_i)$, for $1\leq i\leq 4$, satisfy the continuity equation, and satisfy the prescription of Lemma 1. We let E_k^i and B_2^i, for $1\leq i\leq 4$, $2\leq k\leq 3$ be the corresponding causal fields.

Lemma 10. *For m even, we have that;* $\int_{S(r)}(E_2^i\times B_2^j)\cdot d\overline{S}=0$ *for* $1\leq i\leq j\leq 4$. *Moreover;*

$$\begin{aligned}E_3^1&=\left(\gamma m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\gamma m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Gamma'',\\B_2^1&=\left(-\beta m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\beta m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Gamma',\\E_3^2&=\left(\gamma m\sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\gamma m\cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Gamma'',\\B_2^2&=-\left(\beta m\sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\beta m\cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Gamma',\end{aligned}$$

$$\begin{aligned}
E_3^3 &= \left(-\gamma m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\gamma m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Delta'',\\
B_2^3 &= \left(\beta m\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\beta m\sin(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Delta',\\
E_3^4 &= \left(-\gamma m\sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\gamma m\cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Delta'',\\
B_2^4 &= \left(\beta m\sin(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\beta m\cos(mt)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\Delta',
\end{aligned}$$

where

$$\begin{aligned}
\Gamma' &= \int_{-\pi}^{\pi}\sin(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(\cos(\theta)z,\sin(\theta)z,-\sin(\theta)y-\cos(\theta)x)\,d\theta,\\
\Gamma'' &= \int_{-\pi}^{\pi}\sin(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(-\sin(\theta),\cos(\theta),0)\,d\theta,\\
\Delta' &= \int_{-\pi}^{\pi}\cos(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(\cos(\theta)z,\sin(\theta)z,-\sin(\theta)y-\cos(\theta)x)\,d\theta,\\
\Delta'' &= \int_{-\pi}^{\pi}\cos(m\theta)\sin\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(-\sin(\theta),\cos(\theta),0)\,d\theta.
\end{aligned}$$

Proof. It is easy to see, replacing summations over even indices, with odd indices, and vice versa, following the proof of the previous lemma, and assuming m is even, that $E_2^1 = c(r,m,t)\Gamma$, $E_2^2 = d(r,m,t)\Gamma$, $E_2^3 = e(r,m,t)\Delta$, $E_2^4 = f(r,m,t)\Delta$ where $\Gamma = \int_{-\pi}^{\pi}\cos(m\theta)\cos\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(x,y,z)d\theta$ and $\Delta = \int_{-\pi}^{\pi}\sin(m\theta)\cos\left(\frac{mx\cos(\theta)+my\sin(\theta)}{(r^2+1)^{\frac{1}{2}}}\right)(x,y,z)d\theta$ and $\{c,d,e,f\}$ are parameters not depending on θ, x, y or z. A similar argument works for Γ' and Δ' as in the statement of the lemma. It is therefore sufficient to check that $\int_{S(r)}(\Gamma\times\Gamma')\centerdot d\bar{S} = \int_{S(r)}(\Gamma\times\Delta')\centerdot d\bar{S} = \int_{S(r)}(\Delta\times\Gamma')\centerdot d\bar{S} = \int_{S(r)}(\Delta\times\Delta')\centerdot d\bar{S} = 0$

The first case was checked in the previous lemma. The remaining cases follow from the first case, by replacing even with odd summations in both v and s when replacing Γ by Δ, and, similarly, for the pair Γ' and Δ' in v' and s'. For the remainder of the lemma, carefully follow the calculation in the previous result, the details are left to the reader. □

Lemma 11. *For m even, we have that*

$$\begin{aligned}
E_3^1\times B_2^1 &= C_1\Gamma''\times\Gamma',\\
E_3^2\times B_2^2 &= C_2\Gamma''\times\Gamma',\\
E_3^2\times B_2^1 &= -C_3\Gamma''\times\Gamma',\\
E_3^1\times B_2^2 &= -C_3\Gamma''\times\Gamma',\\
E_3^3\times B_2^3 &= C_1\Delta''\times\Delta',\\
E_4^2\times B_2^3 &= -C_2\Delta''\times\Delta',\\
E_3^4\times B_2^3 &= -C_3\Delta''\times\Delta',\\
E_3^3\times B_2^4 &= -C_3\Delta''\times\Delta',\\
E_3^3\times B_2^1 &= -C_1\Delta''\times\Gamma',\\
E_3^4\times B_2^2 &= -C_2\Delta''\times\Gamma',\\
E_3^3\times B_2^2 &= C_3\Delta''\times\Gamma',
\end{aligned}$$

$$\begin{aligned}
E_3^4 \times B_2^1 &= C_3\Delta'' \times \Gamma', \\
E_3^1 \times B_2^3 &= -C_1\Gamma'' \times \Delta', \\
E_3^2 \times B_2^4 &= -C_2\Gamma'' \times \Delta', \\
E_3^1 \times B_2^4 &= C_3\Gamma'' \times \Delta', \\
E_3^2 \times B_2^3 &= C_3\Gamma'' \times \Delta'
\end{aligned}$$

where

$$\begin{aligned}
C_1 =&\beta\gamma m^2\left(-\cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right.\\
&\left.-\sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right),\\
C_2 =&\beta\gamma m^2\left(-\sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right.\\
&\left.-\cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right),\\
C_3 =&\beta\gamma m^2\left(\cos(mt)\sin(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\sin(mt)\cos(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right.\\
&\left.-\sin^2(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)+\cos^2(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right).
\end{aligned}$$

Proof. The proof is a simple calculation, using the result of the previous lemma.
□

Lemma 12. *There exists a family $(\rho, \overline{J})$ satisfying the continuity equation, with corresponding (ρ, J) satisfying the wave equation, such that for the solution $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfying Maxwell's equations, obtained from Jefimenko's equations, for any $r > 0$, $\int_t^{t+\frac{\pi}{m}} P(r,t)dt = O(\frac{1}{r})$, where $\int_t^{t+\frac{\pi}{m}} P(r,t)dt$ is the power radiated in a cycle, from a sphere $S(r)$ of radius r. In particularly, we have that*

$$lim_{r\rightarrow\infty} \int_t^{t+\frac{\pi}{m}} P(r,t)dt = 0$$

so the no radiation condition holds over a cycle. The family is obtained by setting any three of $\{a_1, a_2, a_3, a_4\} \subset \mathcal{R}$ to be equal, with the fourth having the reverse sign, and letting $\rho = a_1\rho_1 + a_2\rho_2 + a_3\rho_3 + a_4\rho_4$, $J = a_1J_1 + a_2J_2 + a_3J_3 + a_4J_4$.

Proof. We consider linear combinations $a_1\rho_1 + a_2\rho_2 + a_3\rho_3 + a_4\rho_4$ and $a_1J_1 + a_2J_2 + a_3J_3 + a_4J_4$, where $\{a_1, a_2, a_3, a_4\}$ are real scalars. Let $\{E_{m,comb}, B_{m,comb}\}$ denote the resulting fields obtained from Jefimenko's equations. We have, by linearity, that $E_{2,m,comb} = \sum_{i=1}^4 E_2^i$, $E_{3,m,comb} = \sum_{i=1}^4 E_3^i$, $B_{2,m,comb} = \sum_{i=1}^4 B_2^i$ and computing the power radiated through a sphere of radius r, using lemmas 10 and 11;

$$\begin{aligned}
P(r,t) &= \int_{S(r)} (E_{m,comb} \times B_{m,comb}) \cdot d\overline{S} + O\left(\frac{1}{r}\right) \\
&= \int_{S(r)} ((E_{2,m,comb} + E_{3,m,comb}) \times B_{2,m,comb}) \cdot d\overline{S} + O\left(\frac{1}{r}\right) \\
&= \int_{S(r)} (E_{2,m,comb} \times B_{2,m,comb}) \cdot d\overline{S} + \int_{S(r)} (E_{3,m,comb} \times B_{2,m,comb}) \cdot d\overline{S} + O\left(\frac{1}{r}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j=1}^{4} a_i a_j \int\limits_{S(r)} \left(E_2^i \times B_2^j\right) \boldsymbol{.}\, d\bar{S} + \sum_{i=1}^{4} a_i a_j \int\limits_{S(r)} \left(E_3^i \times B_2^j\right) \boldsymbol{.}\, d\bar{S} + O\left(\frac{1}{r}\right) \\
&= \sum_{i,j=1}^{4} a_i a_j \int\limits_{S(r)} \left(E_3^i \times B_2^j\right) \boldsymbol{.}\, d\bar{S} + O\left(\frac{1}{r}\right) \\
&= \left(a_1^2 C_1 + a_2^2 C_2 - 2a_1 a_2 C_3\right) \int\limits_{S(r)} \left(\Gamma'' \times \Gamma'\right) \boldsymbol{.}\, d\bar{S} + \left(a_3^2 C_1 + a_4^2 C_2 - 2a_3 a_4 C_3\right) \int\limits_{S(r)} \left(\Delta'' \times \Delta'\right) \boldsymbol{.}\, d\bar{S} \\
&\quad + \left(-a_1 a_3 C_1 - a_2 a_4 C_2 + \left(a_2 a_3 + a_1 a_4\right) C_3\right) \int\limits_{S(r)} \left(\Delta'' \times \Gamma'\right) \boldsymbol{.}\, d\bar{S} \\
&\quad + \left(-a_1 a_3 C_1 - a_2 a_4 C_2 + \left(a_1 a_4 + a_2 a_3\right) C_3\right) \int\limits_{S(r)} \left(\Gamma'' \times \Delta'\right) \boldsymbol{.}\, d\bar{S} + O\left(\frac{1}{r}\right)
\end{aligned}$$

We note the identities

$$\begin{aligned}
C_1 + C_2 &= \beta\gamma m^2 \left(-\cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) + 2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
&\quad \left. - \sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) + \beta\gamma m^2 \left(-\sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
&\quad \left. -2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) \\
&= \beta\gamma m^2 \left(-\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) \\
&= -\beta\gamma m^2
\end{aligned}$$

and;

$$\begin{aligned}
C_1 - C_2 &= \beta\gamma m^2 \left(-\cos^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) + 2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
&\quad \left. - \sin^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) - \beta\gamma m^2 \left(-\sin^2(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
&\quad \left. - 2\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos^2(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) \\
&= -\beta\gamma m^2 \left(\cos(2mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
&\quad \left. +4\sin(mt)\cos(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \cos(2mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) \\
&= \beta\gamma m^2 \left(\cos(2mt)\left(\cos\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) - \sin(2mt)\sin\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right. \\
\int\limits_{t}^{t+\frac{\pi}{m}} C_3\,dt &= \int\limits_{t}^{t+\frac{\pi}{m}} \left(\beta\gamma m^2 \left(\cos(mt)\sin(mt)\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right.\right. \\
&\quad - \sin(mt)\cos(mt)\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right) - \sin^2(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\cos^2(mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\Bigg)\Bigg)dt\\
=\;&\int_t^{t+\frac{\pi}{m}}\left(\beta\gamma m^2\left(\frac{\sin(2mt)}{2}\sin^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)-\frac{\sin(2mt)}{2}\cos^2\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right.\right.\\
&\left.\left.+\cos(2mt)\sin\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\cos\left(m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)\right)dt\\
=\;&0
\end{aligned}
$$

so we can obtain a convenient simplification by requiring that;

(i) $a_1^2 = a_2^2$,
(ii) $a_3^2 = a_4^2$,
(iii) $a_1a_2 = -a_3a_4$,
(iv) $a_1a_3 = -a_2a_4$,
(v) $a_2a_3 = -a_1a_4$.

The last 2 conditions give that $a_1 = \frac{-a_2a_4}{a_3}$ $(a_3 \neq 0)$, $a_2a_3 = -\frac{-a_2a_4}{a_3}a_4$, $a_3^2 = a_4^2$ $(a_3 \neq 0)$, and $a_3 = \frac{-a_1a_4}{a_2}$ $(a_2 \neq 0)$, $a_1 - \frac{-a_1a_4}{a_2} = -a_2a_4$, $a_1^2 = a_2^2$ $(a_4 \neq 0)$, which are conditions (i) and (ii). Conversely, if $a_1 = a_2$ and $a_3 = -a_4$ or $a_1 = -a_2$ and $a_3 = a_4$, then conditions (i),(ii),(iv),(v) are satisfied. Substituting into condition (iii), we then require that; $a_1^2 = a_3^2$, which we can achieve if; $a_1 = a_3$ or $a_1 = -a_3$. We then obtain that;

$$
\begin{aligned}
P(r,t) = &-a_1^2\beta\gamma m^2\int_{S(r)}\left(\Gamma''\times\Gamma'\right)\cdot d\bar{S} - a_3^2\beta\gamma m^2\int_{S(r)}\left(\Delta''\times\Delta'\right)\cdot d\bar{S} - 2a_1a_2C_3\int_{S(r)}\left(\Gamma''\times\Gamma'-\Delta''\times\Delta'\right)\cdot d\bar{S}\\
&- a_1a_3\beta\gamma m^2\cos(2mt)\left(\cos\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)-\sin(2mt)\sin\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\int_{S(r)}\left(\Delta''\times\Gamma'\right)\cdot d\bar{S}\\
&- a_1a_3\beta\gamma m^2\cos(2mt)\left(\cos\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right)-\sin(2mt)\sin\left(2m\frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\int_{S(r)}\left(\Gamma''\times\Delta'\right)\cdot d\bar{S}\\
&+O\left(\frac{1}{r}\right).
\end{aligned}
$$

If we integrate through a period of $\frac{\pi}{m}$, we obtain that;

$$
\begin{aligned}
\int_t^{t+\frac{\pi}{m}} P(r,t)\,dt &= -a_1^2\beta\gamma m^2\frac{\pi}{m}\int_{S(r)}\left(\Gamma''\times\Gamma'+\Delta''\times\Delta'\right)\cdot d\bar{S}+O\left(\frac{1}{r}\right)\\
&= -\pi a_1^2\beta\gamma m\int_{S(r)}\left(\Gamma''\times\Gamma'+\Delta''\times\Delta'\right)\cdot d\bar{S}+O\left(\frac{1}{r}\right),
\end{aligned}
$$

as

$$
\int_t^{t+\frac{\pi}{m}}\cos(2mt)\,dt = \int_t^{t+\frac{\pi}{m}}\sin(2mt)\,dt = \int_t^{t+\frac{\pi}{m}}C_3dt = 0.
$$

By Poynting's Theorem, we have that;

$$
\frac{dW}{dt} = -\frac{d}{dt}\int_{B(r)}\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)dB(r)-\frac{1}{\mu_0}P(r,t).
$$

Integrating this expression over a period of $\frac{\pi}{m}$ and using periodicity, from Jefimenko's equations that $\overline{E}\left(\overline{r},t+\frac{\pi}{m}\right)=-\overline{E}\left(\overline{r},t\right)$, and $\overline{B}\left(\overline{r},t+\frac{\pi}{m}\right)=-\overline{B}\left(\overline{r},t\right)$, we obtain that;

$$\begin{aligned}W\left(t+\frac{\pi}{m}\right)-W\left(t\right)&=-\int_{B(r)}\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)dB\left(r\right)|_{t+\frac{\pi}{m}}+\int_{B(r)}\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)dB\left(r\right)|_{t}-\frac{1}{\mu_0}\int_{t}^{t+\frac{\pi}{m}}P\left(r,t\right)\\&=\frac{1}{\mu_0}\pi a_1^2\beta\gamma m\int_{S(r)}\left(\Gamma''\times\Gamma'+\Delta''\times\Delta'\right)\centerdot d\overline{S}+O\left(\frac{1}{r}\right).\end{aligned}$$

It follows that, for $r>0$;

$$\int_{S(r)}\left(\Gamma''\times\Gamma'+\Delta''\times\Delta'\right)\centerdot d\overline{S}=\int_{B(r)}\left(\bigtriangledown\centerdot\left(\Gamma''\times\Gamma'+\Delta''\times\Delta'\right)\right)dB\left(r\right)=\frac{16\pi c^3c\left(t,m\right)\epsilon_0\left(r^2+1\right)^{\frac{3}{2}}}{a_1^2m}+O\left(r^2\right),$$

where $c(t,m)=W(t+\frac{\pi}{m})-W(t)$, the difference in mechanical energy of the charge/current distribution confined to a vanishing annulus containing S^1, is independent of $r>1$, as contained in $B(r)$, ([3]). Letting $f(x,y,z)=(\bigtriangledown\centerdot(\Gamma''\times\Gamma'+\Delta''\times\Delta'))$, we obtain that;

$$\int_{B(r)}fdB(r)=d(t,m)\left(r^2+1\right)^{\frac{3}{2}}+O(r^2),$$

where $d(t,m)$ is independent of r. Dividing by r^3, we obtain that;

$$\lim_{r\rightarrow\infty}\frac{1}{r^3}\int_{B(r)}fdB(r)=d(t,m)+O\left(\frac{1}{r}\right).$$

Taking a power series expansion of f in the variables $\{x,y,z\}$, and integrating term by term, we can see that f must be constant. Using the volume of the ball $B(r)$ as $\frac{4\pi r^3}{3}$, we must have that $f=d(t,m)=c(t,m)=0$. It follows, letting $r\rightarrow\infty$, that the mechanical energy of the matter wave doesn't vary over a cycle and moreover that the power radiated $\int_{t}^{t+m}P(r,t)dt$ over a ball $B(r)$ and a cycle is $O(\frac{1}{r})$, for any $r>0$, and $\lim_{r\rightarrow\infty}\int_{t}^{t+\frac{\pi}{m}}P(r,t)dt=0$. $\square$

Lemma 13. *The results of Lemma 1 and Lemma 12 hold for the wave equation with velocity c;*

$$\frac{\partial\Psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial\Psi}{\partial t^2}=0,\tag{32}$$

with the time cycle $\frac{\pi}{m}$ replaced by $\frac{\pi}{mc}$.

Proof. The first result is clear. If we denote a solution Ψ_{new} to (32) by $\Psi(x,ct)$, and let $J_{new}=cJ(x,ct)$, with corresponding $\{\rho_{new},\overline{J}_{new}\}$, then in the Jefimenko equations, we get that;

$$\begin{aligned}\overline{E}_{2,new}(x,t)&=c\overline{E}_2\left[\rho_{new},\frac{\overline{J}_{new}}{c}\right],\\\overline{E}_{3,new}&=c^2\overline{E}_2\left[\rho_{new},\frac{\overline{J}_{new}}{c}\right],\\\overline{B}_{2,new}&=c^2\overline{B}_2\left[\rho_{new},\frac{\overline{J}_{new}}{c}\right].\end{aligned}$$

---

[3] This idea relies on an argument in [2], that the work done on a charge q is $\overline{F}\centerdot d\overline{l}=q(\overline{E}+\overline{v}\times\overline{B})\centerdot\overline{v}dt=q\overline{E}\centerdot\overline{v}=\overline{E}\centerdot\overline{J}dt$, and can be summed over any $B(r)$ with $r>1$, as $\overline{J}$ is vanishing outside S^1. The idea that that $\overline{J}$ can be represented as $\rho\overline{v}$ is pursued in [13].

Again, we obtain that

$$\int\limits_{S(r)} \left(\overline{E}_{2,new} \times \overline{B}_{2,new}\right) \centerdot d\overline{S} = c^3 \int\limits_{S(r)} \left(\overline{E}_2 \times \overline{B}_2\right) \left[\rho_{new}, \frac{\overline{J}_{new}}{c}\right] \centerdot d\overline{S} = 0,$$

and;

$$\int\limits_{S(r)} \left(\overline{E}_{3,new} \times \overline{B}_{2,new}\right) \centerdot d\overline{S} = c^4 \int\limits_{S(r)} \left(\overline{E}_3 \times \overline{B}_2\right) \left[\rho_{new}, \frac{\overline{J}_{new}}{c}\right] \centerdot d\overline{S},$$

so that, following Lemma 8;

$$P^{new}(r,t) + O\left(\frac{1}{r}\right) = c^4 P(r,t) \left[\rho_{new}, \frac{\overline{J}_{new}}{c}\right] + O\left(\frac{1}{r}\right).$$

We then obtain, for the linear combination in Lemma 12 that;

$$\begin{aligned} P^{new}(r,t) =& c^4 \Bigg[-a_1^2 \beta \gamma m^2 \int\limits_{S(r)} \left(\Gamma'' \times \Gamma'\right) \centerdot d\overline{S} - a_3^2 \beta \gamma m^2 \int\limits_{S(r)} \left(\Delta'' \times \Delta'\right) \centerdot d\overline{S} - 2a_1 a_2 C_3' \int\limits_{S(r)} \left(\Gamma'' \times \Gamma' - \Delta'' \times \Delta'\right) \centerdot d\overline{S} \\ & - a_1 a_3 \beta \gamma m^2 \cos(2mct) \left(\cos\left(2m \frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) - \sin(2mct) \sin\left(2m \frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \int\limits_{S(r)} \left(\Delta'' \times \Gamma'\right) \centerdot d\overline{S} \\ & - a_1 a_3 \beta \gamma m^2 \cos(2mct) \left(\cos\left(2m \frac{(r^2+1)^{\frac{1}{2}}}{c}\right)\right) - \sin(2mct) \sin\left(2m \frac{(r^2+1)^{\frac{1}{2}}}{c}\right) \int\limits_{S(r)} \left(\Gamma'' \times \Delta'\right) \centerdot d\overline{S} \Bigg] \\ & + O\left(\frac{1}{r}\right), \end{aligned}$$

where $C_3' = C_3(ct)$. If we integrate through a period of $\frac{\pi}{mc}$, we obtain that;

$$\begin{aligned} \int\limits_t^{t+\frac{\pi}{mc}} P^{new}(r,t)\, dt &= -a_1^2 \beta \gamma m^2 \frac{c^4 \pi}{mc} \int\limits_{S(r)} \left(\Gamma'' \times \Gamma' + \Delta'' \times \Delta'\right) \centerdot d\overline{S} + O\left(\frac{1}{r}\right) \\ &= -c^3 \pi a_1^2 \beta \gamma m \int\limits_{S(r)} \left(\Gamma'' \times \Gamma' + \Delta'' \times \Delta'\right) \centerdot d\overline{S} + O\left(\frac{1}{r}\right). \end{aligned}$$

Now we can repeat the argument to obtain the same result, that $\lim\limits_{r \to \infty} \int\limits_t^{t+\frac{\pi}{mc}} P^{new}(r,t) dt = 0$.

□

Lemma 14. *Let $\rho_0 = a$ and $J_0 = b$ be constants, with $\{a, b\} \subset \mathcal{R}$, then for the corresponding $(\rho_0, \overline{J}_0)$ as in the paper, we have that $\frac{\partial \rho_0}{\partial t} + \bigtriangledown . \centerdot \overline{J}_0 = 0$. If $\{\overline{E}_0, \overline{B}_0\}$ are the corresponding causal fields, they satisfy the no radiation condition, and, if $(\rho, \overline{J})$ is the solution given either in Lemmas 12 or 13, with $\{\overline{E}_1, \overline{B}_1\}$ the corresponding causal fields, then they also satisfy the no radiation condition over a cycle, with the corresponding $(\rho + \rho_0, \overline{J} + \overline{J}_0)$ satisfying the continuity equation.*

Proof. We clearly have that $\frac{\partial \rho_0}{\partial t} + \frac{\partial J_0}{\partial x} = 0$, so that when we extend J_0 to S^1 by $\overline{J}_0(1, \theta) = J_0(\theta)(-\sin(\theta), \cos(\theta), 0)$, for $-\pi \leq \theta < \pi$, we have for the further extension $(\rho_0, \overline{J}_0)$ on $Ann(1, \epsilon) \times (-\epsilon, \epsilon)$, using Lemma 6, that $\frac{\partial \rho_0}{\partial t} + \bigtriangledown \centerdot \overline{J}_0 = 0$. Using Lemma 8, we have that;

$$\lim_{r \to \infty} P(r) = \lim_{r \to \infty} \int\limits_{S(r)} (\overline{E}_{0,2} \times \overline{B}_{0,2} + \overline{E}_{0,3} \times \overline{B}_{0,2}) d\overline{S}(r),$$

but as $\dot{\rho_0} = 0$ and $\dot{\overline{J}_0} = \overline{0}$, we have that $\overline{E}_{0,2} = \overline{E}_{0,3} = \overline{B}_{0,2} = \overline{0}$, so that $\lim_{r\to\infty} P(r) = 0$ and $\{\overline{E}_0, \overline{B}_0\}$ satisfy the no radiating condition. Again, we have that $\frac{\partial(\rho+\rho_0)}{\partial t} + \frac{\partial(J+J_0)}{\partial x} = 0$, with the same remark on the extension, so that $\frac{\partial(\rho+\rho_0)}{\partial t} + \bigtriangledown \centerdot (\overline{J} + \overline{J}_0) = 0$. We have that $\overline{E}_1 = \overline{E} + \overline{E}_0$ and $\overline{B}_1 = \overline{B} + \overline{B}_0$, so that;

$$\begin{aligned}
\lim_{r\to\infty} P(r) &= \lim_{r\to\infty} \int_{S(r)} (\overline{E}_1 \times \overline{B}_1)\, d\overline{S}(r) \\
&= \lim_{r\to\infty} \int_{S(r)} ((\overline{E} + \overline{E}_0) \times (\overline{B} + \overline{B}_0))\, d\overline{S}(r) \\
&= \lim_{r\to\infty} \left(\int_{S(r)} (\overline{E} \times \overline{B})\, d\overline{S}(r) + \int_{S(r)} (\overline{E}_0 \times \overline{B}_0)\, d\overline{S}(r) + \int_{S(r)} (\overline{E} \times \overline{B}_0)\, d\overline{S}(r) + \int_{S(r)} (\overline{E}_0 \times \overline{B})\, d\overline{S}(r) \right) \\
&= \lim_{r\to\infty} \left(\int_{S(r)} (\overline{E} \times \overline{B}_0)\, d\overline{S}(r) + \int_{S(r)} (\overline{E}_0 \times \overline{B})\, d\overline{S}(r) \right) \\
&= \lim_{r\to\infty} \left(\int_{S(r)} (\overline{E}_2 \times \overline{B}_{0,2})\, d\overline{S}(r) + \int_{S(r)} (\overline{E}_3 \times \overline{B}_{0,2})\, d\overline{S}(r) \right) \\
&\quad + \lim_{r\to\infty} \left(\int_{S(r)} (\overline{E}_{0,2} \times \overline{B}_2)\, d\overline{S}(r) + \int_{S(r)} (\overline{E}_{0,3} \times \overline{B}_2)\, d\overline{S}(r) \right) \\
&= 0,
\end{aligned}$$

as required. □

Definition 3. If $(\rho, \overline{J})$ satisfy the continuity equation, let;

$$\begin{aligned}
T(\overline{x}, t) &= \left| \frac{\overline{J}(\overline{x}, t)}{\rho(\overline{x}, t)} \right|, \quad \text{if} \quad \rho(\overline{x}, t) \neq 0, \\
T(\overline{x}, t) &= \lim_{r\to 0} \frac{1}{vol(B(\overline{x}, r))} \int_{B(\overline{x}, r)} \left| \frac{\overline{J}(\overline{y}, t)}{\rho(\overline{y}, t)} \right| d\overline{y}, \quad \text{if} \quad \rho(\overline{x}, t) = 0.
\end{aligned}$$

We say that $(\rho, \overline{J})$ are in electromagnetic thermal equilibrium at time t, if $T(\overline{x}, t)$ is constant on $\mathcal{R}^3$ and in electromagnetic thermal equilibrium if $T(\overline{x}, t)$ is constant on $\mathcal{R}^3 \times \mathcal{R}_{>0}$.

Remark 6. The definition is motivated by Boltzmann's definition of temperature for ideal gases as $\frac{m|\overline{v}|^2}{3k}$, see [12], where $|\overline{v}^2|$ is the mean square velocity of the particles making up the configuration, m is the molecular mass and k is Boltzmann's constant, and by the formula $\overline{J} = \rho\overline{v}$, see [2] and [13]. A local definition of temperature is given in [12], when the particles are moving with different velocities. In the electromagnetic case, if $\rho > 0$, a natural definition of local average speed would be;

$$\begin{aligned}
V_1(\overline{x}, t) &= \lim_{r\to 0} \frac{\int_{B(\overline{x}, r)} \rho(\overline{y}, t) |\overline{v}(\overline{y}, t)| dB(\overline{y})}{\int_{B(\overline{x}, r)} \rho(\overline{y}, t) dB(\overline{y})} \\
&= \lim_{r\to 0} \frac{\int_{B(\overline{x}, r)} |\overline{J}(\overline{y}, t)| dB(\overline{y})}{\int_{B(\overline{x}, r)} \rho(\overline{y}, t) dB(\overline{y})} \\
&= \lim_{r\to 0} \frac{\frac{1}{vol(B(\overline{x}, r))} \int_{B(\overline{x}, r)} |\overline{J}(\overline{y}, t)| dB(\overline{y})}{\frac{1}{vol(B(\overline{x}, r))} \int_{B(\overline{x}, r)} \rho(\overline{y}, t) dB(\overline{y})} = \left| \frac{\overline{J}(\overline{x}, t)}{\rho(\overline{x}, t)} \right|,
\end{aligned}$$

and a similar calculation works for the local average squared velocity;

$$V_2(\overline{x},t)=\frac{|\overline{J}(\overline{x},t)|^2}{\rho^2(\overline{x},t)}.$$

We then establish electromagnetic thermal equilibrium when V_1 or V_2 is constant, and similarly for the generalization T.

Lemma 15. *If we set $a_1=-a_4$, $a_2=a_3$, then the system defined by $(\rho,\overline{J})$, for the linear combination $\rho=\sum_{i=1}^4\rho^i$, $J=\sum_{i=1}^4 J^i$, from Definition 2, is in electromagnetic thermal equilibrium. In particularly, the configurations from Lemma 12 are in electromagnetic thermal equilibrium for all $t>0$.*

Proof. We have that;

$$\begin{aligned}
T(\overline{x},t)|_{S^1}&=\frac{|\overline{J}(\overline{x},t)|}{|\rho(\overline{x},t)|}\\
&=a\left|\frac{a_1\sin(mx)\sin(mt)-a_2\sin(mx)\cos(mt)-a_2\cos(mx)\sin(mt)-a_1\cos(mx)\cos(mt)}{a_1\cos(mx)\cos(mt)+a_2\cos(mx)\sin(mt)+a_2\sin(mx)\cos(mt)-a_1\sin(mx)\sin(mt)}\right|\\
&=|-1|\\
&=1,
\end{aligned}$$

where $a=|(-\sin(\theta),\cos(\theta),0)|=1$, when $\rho(\overline{x},t)\neq 0$, so that, using Lemma 6, for any configuration $(\rho,\overline{J})$, satisfying the continuity equation and extending (ρ,J), we have $T(\overline{x},t)|_{S^1}=1$. $\square$

Definition 4. We say that a configuration $(\rho,\overline{J},\overline{E},\overline{B})$ is classically radiating in a cycle, if for sufficiently large $r\in\mathcal{R}$, there exists a sequence of times $\{t_i:i\in\mathcal{Z}\}$ such that, for $t\in(t_i,t_{i+1})$;

$$\int_{B(\overline{0},r)}div(\overline{E}_{\overline{v}}\times\overline{B}_{\overline{v}})dB=f_{i,i+1}(t)$$

for some continuous function $f_{i,i+1}$ on $[t_i,t_{i+1}]$, with $f_{i,i+1}(t_i)=f_{i,i+1}(t_{i+1})=0$ and $|\int_{t_i}^{t_{i+1}}f_{i,i+1}(t)dt|>\epsilon>0$, with $\int_{t_i}^{t_{i+1}}f_{i,i+1}(t)dt$ of the same sign.

Lemma 16. *If we set $a_1=-a_4$, $a_2=a_3$, then the system defined by $(\rho,\overline{J})$, for the linear combination $\rho=\sum_{i=1}^4\rho^i$, $J=\sum\limits_{i=1}^4 J^i$, from Definition 2, has the property that it is classically non-radiating in a cycle, see [9], in all inertial frames, with respect to the causal solution $(\overline{E},\overline{B})$, provided by Jefimenko's equations.*

Proof. Suppose that there exists a frame $S_{\overline{v}}$ with the property that the transformed fields $(\overline{E}_{\overline{v}},\overline{B}_{\overline{v}})$ is classically radiating in a cycle. We can therefore assume that there exists a sequence of times $\{t_i:i\in\mathcal{Z}\}$, such that for sufficiently large $r\in\mathcal{R}_{>0}$, we have that;

$$\int_{B(\overline{0},r)}div(\overline{E}_{\overline{v}}\times\overline{B}_{\overline{v}})dB=f_{i,i+1}(t),$$

for some continuous function $f_{i,i+1}$ on $[t_i,t_{i+1}]$, $i\in\mathcal{Z}$, with $f_{i,i+1}(t_i)=f_{i,i+1}(t_{i+1})=0$ and $|\int_{t_i}^{t_{i+1}}f_{i,i+1}dt|>\epsilon$, $i\in\mathcal{Z}$. By Poynting's Theorem, we have that;

$$\frac{d}{dt}\int_{B(\overline{0},r)}(u_{mech}+u_{em})dB=-\int_{B(\overline{0},r)}div(\overline{E}_{\overline{v}}\times\overline{B}_{\overline{v}})dB=-f_{i,i+1}(t),$$

in the period (t_i,t_{i+1}), so that, by the fundamental theorem of calculus;

$$\int_{B(\overline{0},r)}(u_{mech}+u_{em})dB|_{t_{i+1}}-\int_{B(\overline{0},r)}(u_{mech}+u_{em})dB|_{t_i}=-\int_{t_i}^{t_{i+1}}f_{i,i+1}(t)dt.$$

It follows that, if $\int_{t_i}^{t_{i+1}} f_{i,i+1}(t)dt > 0$, $\int_{B(\overline{0},r)}(u_{mech}+u_{em})dB$ would decrease. Without loss of generality we can assume this as we can consider a time reversal;

$$(-\rho_{\overline{v}}(\overline{x},t_0-t),\overline{J}_{\overline{v}}(\overline{x},t_0-t),-\overline{E}_{\overline{v}}(\overline{x},t_0-t),\overline{B}_{\overline{v}}(\overline{x},t_0-t)),$$

which satisfies Maxwell's equations with the flux reversed. Then as $\sum_{i\in\mathcal{Z}}\int_{t_i}^{t_{i+1}} f_{i,i+1}(t)dt = \infty$ and $\int_{B(\overline{0},r)}(u_{mech}+u_{em})dB$ is finite, we can assume that that there exists $s_r \in \mathcal{R}_{>0}$, with $u_{mech}|_{B(\overline{0},r)} = 0$, either for $t \geq s_r$, or for $t \leq -s_r$, $(*)$. As u_{mech} is finite, and the property $(*)$ can be made uniform in r, for r sufficiently large, we can assume no charge flows to infinity. Then, we can assume, if $u_{mech} = 0$ in $B(\overline{0},r)$, that $\rho|_{B(\overline{0},r)} \neq 0$, and as local kinetic energy of the charge is zero, that the local speed $|\overline{w}|$ is zero, so that $\frac{|\overline{J}_{\overline{v}}|}{|\rho_{\overline{v}}|} = |\overline{w}| \to 0$. In particularly the components $\frac{j_{i,\overline{v}}}{\rho_{\overline{v}}} \to 0$. By the transformation laws back to the base frame, assuming, without loss of generality, that $\overline{v} = (v,0,0)$, we have that;

$$\frac{|\overline{J}|}{|\rho|} = \frac{|\gamma_v(\overline{J}_{\overline{v},||}+\gamma_v\overline{v}\rho_{\overline{v}})+\overline{J}_{\overline{v},\perp}|}{|\gamma_v(\rho_{\overline{v}}+\frac{\gamma_v v J_{\overline{v},||}}{c^2})|} = \frac{|(\gamma_v j_{1,v}+\gamma_v v\rho_v, j_{2,v}, j_{3,v})|}{|\gamma_v\rho_v+\frac{\gamma_v v j_{1,v}}{c^2}|} = \frac{|(\gamma_v\frac{j_{1,v}}{\rho_v}+\gamma_v v, \frac{j_{2,v}}{\rho_v}, \frac{j_{3,v}}{\rho_v})|}{|\gamma_v+\frac{\gamma_v v j_{1,v}}{\rho_v c^2}|} \to \frac{|\gamma_v v|}{|\gamma_v|} = v.$$

However, this contradicts Lemma 15 unless $v = 1$. We can then obtain the result by a limiting argument to exclude this special case. ☐

Remark 7. If thermal equilibrium held in all transformed frames, then one can probably adapt the argument of [9], replacing the classically non-radiating condition with the classically non-radiating in a cycle condition, to show that the above configuration $(\rho,\overline{J})$ would satisfy a global wave equation in $\mathcal{R}^4$. As this is clearly not the case in the base frame, thermal equilibrium cannot hold in all transformed frames. The above argument suggests that if a particle radiates in a cycle in some inertial frame, then it must eventually travel with constant velocity $\overline{v}$ in the base frame, so that in the transformed frame $S_{\overline{v}}$, it is stationary, developing an observation made by Rutherford.

Conflicts of Interest: "The author declares no conflict of interest."

References

[1] Batchelor, G. K. (2000). *An Introduction to Fluid Dynamics* (2nd Edition). (CUP).

[2] Griffiths, D. (2008). *Introduction to Electrodynamics* (3rd ed.). Pearson.

[3] Larmor, J. (1897), LXIII. On the theory of the magnetic influence on spectra, and on the Radiation from Moving Ions. *Philosophical Magazine*, *5*(44), (271), 503-512.

[4] Paterson, A. P. (1983). *A First Course in Fluid Dynamics.* (CUP).

[5] Pippard, A. B. (2008). *The Elements of Classical Thermodynamics.* (CUP).

[6] de Piro, T. (2019). Schrodinger's Equation and Related Charge Density. Retrieved from http://www.curvalinea.net

[7] de Piro, T. (2014). Nonstandard Analysis and Physics (in progress). Retrieved from http://www.curvalinea.net

[8] de Piro, T. (2021). Some arguments for the wave equation in quantum theory. *Open Journal of Mathematical Sciences*, *5*(1), 314 – 336.

[9] de Piro, T. (2022). Some arguments for the wave equation in quantum theory 2. *Open Journal of Mathematical Sciences*, *6*(1), 205 – 247.

[10] de Piro, T. (2023). Some arguments for the wave equation in quantum theory 4. Retrieved from http://www.curvalinea.net

[11] de Piro, T. (2020). A nonstandard solution to the wave equation. Retrieved from http://www.curvalinea.net

[12] de Piro, T. (2018). Nonstandard methods for solving the heat equation. Retrieved from http://www.curvalinea.net

[13] de Piro, T. (2022). The continuity equation and particle motion (in progress). Retrieved from http://www.curvalinea.net