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Some arguments for the wave equation in quantum theory 4

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Abstract: We classify particle paths for systems in thermal equilibrium satisfying the usual relations and prove that the only solutions are given by straight line parallel paths with speed c.

Keywords: Wave equation; Continuity equation; Maxwell's equations; Jefimenko's equations; Radiation.

1. Introduction

In this short paper, we consider the notion of thermal equilibrium for charge and current $(\rho, \overline{J})$ satisfying the continuity equation, $\frac{\partial \rho}{\partial t} = -div(\overline{J})$, in conjunction with the set of relations;

(i). $\square^2(\rho) = 0$.
(ii). $\square^2(\overline{J}) = \overline{0}$.
(iii). $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$,

where $\square^2$ is the d'Alembertian operator $\bigtriangledown^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

The significance of the relations $(i) - (iii)$ is that they are invariant under the Lorentz transformation of frames defined by special relativity, and characterise systems in which there exists electromagnetic fields $(\overline{E}, \overline{B})$ in every frame such that $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfy Maxwell's equations, $\overline{B} = \overline{0}$ and $\square^2 \overline{E} = \overline{0}$. Such configurations have the characteristic that the Poynting vector $\overline{E} \times \overline{B} = \overline{0}$ and the flux $div(\overline{E} \times \overline{B}) = 0$, so that there is no energy loss of the signal at any distance, in any inertial frame. The fields $(\overline{E}, \overline{B})$ may not be the causal fields defined by Jefimenko's equations, or related by Lorentz transformations in different frames. These ideas are developed in detail in the papers [1] and [2].

In order to define thermal equilibrium in this paper, we need that the charge $\rho > 0$, in which case we require that $|\frac{\overline{J}}{\rho}| = d$, where $d \in \mathcal{R}_{\geq 0}$. The significance of $\frac{\overline{J}}{\rho}$ is that it defines the velocities of particles which make up the charge. More precisely, one can start with $(\rho, \overline{J})$ satisfying the continuity equation, and use ρ_0 to define the initial position of the particles which make up the charge. The particle paths are then defined by $\overline{x}(t+\epsilon) - \overline{x}(t) = \epsilon \frac{\overline{J}}{\rho'}|_{\overline{x},t}$, where ϵ is an infinitesimal, with $\overline{x}(0)$ given and ρ' defined step by step using the new particle configuration. It is the aim of the paper [3] to prove that ρ' is S-continuous as a nonstandard process, has a well defined standard part ${}^\circ\rho'$ and that $(\overline{J}, {}^\circ\rho')$ satisfies the continuity equation. It then follows, by taking the difference of the processes, that;

$$\frac{\partial(\rho - {}^\circ\rho')}{\partial t} = div(\overline{J}) - div(\overline{J}) = 0 \ \text{ and } \ \rho_0 = {}^\circ\rho'_0,$$

so that $\rho_t = {}^\circ\rho'_t$, for $t \geq 0$, and the processes coincide. Physicists, as in [4], use this relation $\overline{J} = \rho\overline{v}$ intuitively.

In Definition 1, we define the notion of a simple system in which all the particles travel with constant velocity. This notion is stronger than that of thermal equilibrium. In Lemma 1, we prove that simple systems have the important property that they are classically non-radiating in every inertial frame, but the property of thermal equilibrium is in general not preserved between frames. The result relies on the work of Larmor, see [5], who characterises radiation fields for moving particles, although it is still to be shown that the radiation field defined by Jefimenko's equations, see [4], coincides with the radiation field of the sum of its constituents. We isolate simple systems as parallel and divergent, the idea being that in any other system, straight line paths would intersect transversely and the current would not be well defined at the intersection point. We show that there are systems $(\rho, \overline{J})$ satisfying the continuity equation with $\rho > 0$ which are parallel but, using the main result of [2], there are none which are divergent. We did, however, find a divergent system in [3] and [6], which satisfies the additional relations, but is not simple or in thermal equilibrium. In Lemma 2, we exclude the possibility that $d \neq c$ for parallel systems, when the additional relations outlined above are in place and, in Lemma 3, we show that the condition of parallel with $d = c$ is non vacuous, that is parallel systems with $d = c$ and satisfying the additional relations exist. In the final Definition 2, we define trajectories and flow lines, and, in Lemma 4, we exclude the possibility of $(\rho, \overline{J})$ satisfying the continuity equation and the additional relations, with circular flow lines centred at the origin. We did manage to find a system in thermal equilibrium with these properties in [7] but it does not have the property that $\rho > 0$ and it does not satisfy the additional relations, though we present an argument that it is classically non-radiating in all inertial frames. If thermal equilibrium holds, we use an important result in model theory due to Wilkie, see [8], that real fields with Pfaffian functions are O-minimal, to show that the flowlines are unbounded. This allows us to prove that, if the additional relations are in place as well, that the system must be parallel, with $d = c$, by the above.

Definition 1. Suppose that $(\rho, \overline{J})$ satisfy the continuity equation, with $\rho > 0$, so that we can use the results of [3], in defining constituent particles and their velocities. We assume this throughout the paper. We say that the system is simple if the velocities of the individual particles are equal, with zero acceleration. We say the system is parallel, if the particles travel in parallel straight lines at constant velocity. We say the system is divergent, if all the particles travel in straight lines with constant velocity and the paths can only intersect at one point. We say the system is in thermal equilibrium if $|\frac{\overline{J}}{\rho}| = d$, for some $d \in \mathcal{R}_{>0}$.

Lemma 1. *A simple system has the property that it is classically non radiating in the sense of [2], in every inertial frame, and is in thermal equilibrium in the base frame. The parallel systems are given by the prescription that the charge ρ satisfies the transport equation; $\frac{\partial \rho}{\partial t} = -\overline{\lambda} \centerdot \bigtriangledown(\rho)$ for the velocity vector $\overline{\lambda}$, and the current $\overline{J}$ satisfies; $\overline{J} = \overline{\lambda}\rho$. In particular parallel systems exists. Without loss of generality, the divergent systems are given by the prescription that ρ satisfies the equation; $\frac{\partial \rho}{\partial t} = -\frac{d}{|\overline{x}|}(\bigtriangledown(\rho) \centerdot \overline{x} - 2\rho)$, $(\overline{x} \neq \overline{0})$, where $d \in \mathcal{R}$ and the current satisfies; $\overline{J} = d\rho\frac{\overline{x}}{|\overline{x}|}$. However, no non trivial divergent systems satisfying the continuity equation exist.*

Proof. As all the particles in a simple system have zero acceleration, using the calculation of the Lienard-Wiechert potentials for a single particle, and the fact that all the acceleration fields vanish, we have that; $lim_{r\rightarrow\infty}\int_{B(0,r)} div(\overline{E}_t \times \overline{B}_t)d\overline{x} = 0$, for the causal fields $(\overline{E}, \overline{B})$. When we transform between frames, the particles still move with constant but differing velocities and again all the acceleration fields vanish, so the result holds for the causal fields in all frames. Thermal equilibrium in the base frame follows from the fact that $|\frac{\overline{J}}{\rho}| = d$, as all the particles have the same constant speed. For the second claim, as the velocity $\overline{\lambda}$ is constant, and the paths are parallel, we must have that, $\overline{J} = \overline{\lambda}\rho$. In order to satisfy the continuity equation, we must have that;

$$\frac{\partial \rho}{\partial t} = -div(\overline{J}) = -div(\overline{\lambda}\rho) = -\overline{\lambda} \centerdot \bigtriangledown(\rho).$$

This is a transport equation with solution $\rho(\overline{x}, t) = g(\overline{x} - \overline{\lambda}t)$, where $g \in C^{\infty}(\mathcal{R}^3)$. For the third claim, in a divergent system, with the intersection point centred at the origin, we must have that; $\overline{J} = d\rho\frac{\overline{x}}{|\overline{x}|}$, (V). By the continuity equation, we obtain that;

$$\frac{\partial \rho}{\partial t} = -div(\overline{J}) = -div(d\rho\frac{\overline{x}}{|\overline{x}|}) = -d\bigtriangledown(\rho) \centerdot \frac{\overline{x}}{|\overline{x}|} - d\rho div(\frac{\overline{x}}{|\overline{x}|}) = -d\bigtriangledown(\rho) \centerdot \frac{\overline{x}}{|\overline{x}|} - \frac{2d\rho}{|\overline{x}} = -\frac{d}{|\overline{x}|}(\bigtriangledown(\rho) \centerdot \overline{x} - 2\rho)(A)$$

If $(\overline{E}, \overline{J}) = 0$ for the causal fields such that $(\rho, \overline{J}, \overline{E}, \overline{B})$ satisfies Maxwell's equations, then, for any volume $V \subset \mathcal{R}^3$, we would have that $\frac{dW}{dt} = 0$, for the total mechanical energy W of the charge distribution in V. In particular, as all the particles are travelling with equal speed, we must have that;

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V \rho d\overline{x} = \int_V \frac{\partial \rho}{\partial t} d\overline{x} = 0,$$

so that, as the volume V was arbitrary, $\frac{\partial \rho}{\partial t}$ = and ρ is time independent. From (A), we then obtain that $\bigtriangledown(\rho) \centerdot \overline{x} = 2\rho$. However;

if $\rho = \frac{1}{2}(\rho_x x + \rho_y y + \rho_z z)$, then $\rho_x = \frac{1}{2}(\rho_{xx}x + \rho_x + \rho_{xy}y + \rho_{xz}z)$ and $\bigtriangledown(\rho_x) \centerdot \overline{x} = \rho_x$. Similarly, it follows that; $\bigtriangledown(\rho_y) \centerdot \overline{x} = \rho_y$ and $\bigtriangledown(\rho_z) \centerdot \overline{x} = \rho_z$.

Repeating the argument, we obtain that; $\bigtriangledown(\rho_{xx}) \centerdot \overline{x} = \bigtriangledown(\rho_{xy}) \centerdot \overline{x} = \bigtriangledown(\rho_{xz}) \centerdot \overline{x} = \bigtriangledown(\rho_{yy}) \centerdot \overline{x}$

$\bigtriangledown(\rho_{yz}) \centerdot \overline{x} = \bigtriangledown(\rho_{zz}) \centerdot \overline{x} = 0$

It follows that:

$$\rho_{rxx} = \rho_{rxy} = \rho_{rxz} = \rho_{ryy} = \rho_{ryz} = \rho_{rzz} = 0, \rho_{rx} = C_1\rho_{ry} = C_2\rho_{rz} = C_3, \rho_r = C_1 x + C_2 y + C_3 z.$$

By the divergence theorem and the continuity equation, we have that;

$$\int_{B(\overline{0},r)} \frac{\partial \rho}{\partial t} d\overline{x} = -\int_{B(\overline{0},r)} div(\overline{J}) d\overline{x} = -\int_{S(\overline{0},r)} \overline{J} \centerdot d\overline{S} = -\int_{S(\overline{0},r)} d\rho \frac{\overline{x}}{|\overline{x}|} \centerdot \frac{\overline{x}}{|\overline{x}|} dS = -d\int_{S(\overline{0},r)} \rho dS = 0$$

so that $\int_{S(\overline{0},r)} \rho dS = 0$. It follows, by the Reynolds transport theorem, that;

$$\frac{d}{dr}\int_{B(\overline{0},r)} \rho d\overline{x} = \int_{B(\overline{0},r)} \frac{\partial \rho}{\partial r} d\overline{x} + \int_{S(\overline{0},r)} \rho dS = \int_{B(\overline{0},r)} \overline{C} \centerdot \overline{x} d\overline{x} = 0$$

and;

$$\int_{B(\overline{0},r)} \rho d\overline{x} = 0 \;\text{ for }\; r \in \mathcal{R}(E)$$

As $\rho \geq 0$, the condition (E) then implies that $\rho = 0$, so that $\overline{J} = \overline{0}$ as well. We can therefore assume that for the causal fields $(\overline{E}, \overline{B})$ in the based frame, that $(\overline{E}, \overline{J}) \neq 0$. As the set of causal fields in the frames $S_{\overline{v}}$, where $|\overline{v}| < c$ is definable, and the condition $(\overline{E}, \overline{J}) = 0$ defines a closed set, we can assume that generically in $S_{\overline{v}}$, $(\overline{E}_{\overline{v},causal}, \overline{J}_{\overline{v}}) \neq 0$. We can then apply the arguments in [2], together with the classically non-radiating property proved above, to conclude that there is a transfer of mechanical energy between two volumes $\{S, S_\kappa\}$. As this occurs over a finite time interval $(t_0 = \epsilon, t_0 + \epsilon)$ we can construct corresponding volumes $\{T, T_\kappa\}$ in the base frame. Generically, either the energy change in T is positive and the energy change in T_κ is negative ot the energy change in T is negative and the energy change in T_κ is positive, both of which contradict thermal equilibrium in the base frame. Alternatively, the energy change in $\{T, T_\kappa\}$ is of the same sign, in which case, as the process is reversible, we can assume that both energy changes are negative. In this case, we can assume that energy is transferred into the field, again a contradiction. By the main result of [2], we can conclude that $(\rho, \overline{J})$ must satisfy the wave equations $\square^2\rho = 0, \square^2\overline{J} = \overline{0}$ in the base frame, as well as the connecting relation $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$. We have, by the definition of $\overline{J}$, that; $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial \overline{J}}{\partial t} = \overline{0}$ iff $\bigtriangledown(\rho) + \frac{1}{c^2} d\frac{\overline{x}}{|\overline{x}|}\frac{\partial \rho}{\partial t} = \overline{0}$.

In particularly, $\bigtriangledown(\rho)$ is parallel to $\overline{x}$, so that ρ is constant on spheres $S(\overline{0}, r)$, for $r \in \mathcal{R}_{>0}$. We have that $\square^2(\rho) = 0$, so writing the Laplacian $\bigtriangledown^2$ in polar form, we obtain that; $\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\rho) - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = 0$ so that $r\rho(r,t)$ satisfies the 1-dimensional wave equation $\square_r^2(r\rho) = 0$, (K), with speed c. Similarly, $\square_r^2(r\frac{\partial \rho}{\partial t}) = 0$. From (A), we have that;

$\frac{\partial \rho}{\partial t} = -\frac{d}{|\overline{x}|}(\bigtriangledown(\rho) \centerdot \overline{x} - 2\rho)$ (A)

$= -d\frac{\partial \rho}{\partial r} - \frac{2d\rho}{r}$

so that; $r\frac{\partial \rho}{\partial t} = -dr\frac{\partial \rho}{\partial r} - 2d\rho$ and, as $\square_r^2(r\frac{\partial \rho}{\partial t}) = 0$, we have that; $\square_r^2(-dr\frac{\partial \rho}{\partial r} - 2d\rho) = 0$, (F). Also; $d\frac{\partial (r\rho)}{\partial r} = dr\frac{\partial \rho}{\partial r} + d\rho$ so that as $d\square_r^2(\frac{\partial (r\rho)}{\partial r}) = 0$, we have that; $\square_r^2(dr\frac{\partial \rho}{\partial r} + d\rho) = 0$ (G).

Combining $(F),(G)$ gives that $\Box_r^2(\rho)=0$. It follows from this and (K) that;

$\frac{\partial^2\rho}{\partial^2 r}=\frac{1}{c^2}\frac{\partial^2\rho}{\partial^2 t}$

$$\frac{\partial^2(r\rho)}{\partial r^2}=\frac{\partial(\rho+r\frac{\partial\rho}{\partial r})}{\partial r}=2\frac{\partial\rho}{\partial r}+r\frac{\partial^2\rho}{\partial^2 r}=\frac{1}{c^2}\frac{\partial^2\rho}{\partial^2 t}$$

so that; $(1-r)\frac{\partial^2\rho}{\partial^2 r}=2\frac{\partial\rho}{\partial r}$. Letting $g=\frac{\partial\rho}{\partial r}$, we have that; $(1-r)\frac{\partial g}{\partial r}=2g$, so that $g=A(t)e^{-2ln(1-r)}=A(t)(1-r)^{-2}=\frac{A(t)}{(1-r)^2}$ and $\rho=\frac{A(t)}{(1-r)}+B(t)$. It follows from $\Box_r^2(\rho)=0$, that; $\frac{1}{c^2}(\frac{A''(t)}{(1-r)}+B''(t))=\frac{2A(t)}{(1-r)^3}$, so that $A''(t)=B''(t)=A(t)=0$, and $\rho=B(t)=\alpha t+\beta$. It follows that $\overline{J}=d(\alpha t+\beta)\frac{\overline{x}}{|\overline{x}|}$ and clearly, we cannot have that $\Box^2\overline{J}=\overline{0}$. □

Lemma 2. *Suppose that $(\rho,\overline{J})$ satisfy the wave equations $\Box^2\rho=0$, $\Box^2\overline{J}=\overline{0}$, the continuity equation $\frac{\partial\rho}{\partial t}=-div(\overline{J})$ and the connecting relation $\bigtriangledown(\rho)+\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\overline{0}$, such that the system is parallel, then, if $|\overline{\lambda}|\neq c$, where λ is the velocity, $\{\rho,\overline{J}\}$ are time independent and harmonic. In particularly, if for $t\in\mathcal{R}$, $\rho_t\in S(\mathcal{R}^3)$, $\overline{J}_t\in S(\mathcal{R}^3)$, then $\rho=0,\overline{J}=\overline{0}$. In the remaining case, when $|\overline{\lambda}|=c$, we obtain the additional relation $\rho=\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\overline{J}$. With the same hypotheses as the first claim, if thermal equilibrium holds instead of parallel, we obtain that ρ is time independent and harmonic, and if $|\overline{\lambda}|=c$, $div(\overline{c})=0$.*

Proof. As the acceleration is zero, the velocity λ is constant, so that, $\overline{J}=\overline{\lambda}\rho$, (A). By the continuity equation and the connecting relation, we have that;

$$\frac{\partial\rho}{\partial t}=-div(\overline{J})=-div(\overline{\lambda}\rho)=-\overline{\lambda}\boldsymbol{\cdot}\bigtriangledown(\rho)=\overline{\lambda}\boldsymbol{\cdot}(-\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t})=-\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\frac{\partial\overline{J}}{\partial t}=\frac{\partial}{\partial t}(\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\overline{J})$$

so that, by the FTC;

$$\rho=\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\overline{J}+d(\overline{x})=\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\overline{\lambda}\rho+d(\overline{x})=\frac{|\lambda^2|}{c^2}\rho+d(\overline{x})$$

so that either ρ is time independent or $|\lambda|=c$ and $d(\overline{x})=0$. In the first case, as $\Box^2\rho=0$, we have that $\bigtriangledown^2(\rho)=0$ and ρ is harmonic. Then, as $\overline{J}=\overline{\lambda}\rho$, we have that, as $\Box^2\overline{J}=\overline{0}$, and the components of $\overline{J}$ are time independent, that the components of $\overline{J}$ are harmonic. In the second case, we obtain that $\rho=\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\overline{J}$, (B). The penultimate claim is clear.

For the final claim concerning thermal equilibrium, we have that, $\overline{J}=\overline{\lambda}\rho$, with $|\overline{\lambda}|=d$, so that $\frac{\partial\overline{\lambda}}{\partial t}\boldsymbol{\cdot}\overline{\lambda}=0$. As above, we have that;
$\frac{\partial\rho}{\partial t}=-div(\overline{J})=-div(\rho\overline{\lambda})=-\overline{\lambda}\boldsymbol{\cdot}\bigtriangledown(\rho)-\rho div(\overline{\lambda})(U)=-\overline{\lambda}\boldsymbol{\cdot}(-\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t})-\rho div(\overline{\lambda})=\frac{\overline{\lambda}}{c^2}\boldsymbol{\cdot}\frac{\partial\overline{J}}{\partial t}-\rho div(\overline{\lambda})=\frac{\partial}{\partial t}(\frac{\overline{\lambda}\boldsymbol{\cdot}\overline{J}}{c^2})-\frac{1}{c^2}\frac{\partial\overline{\lambda}}{\partial t}\boldsymbol{\cdot}\overline{J}-\rho div(\overline{\lambda})=\frac{\partial}{\partial t}(\frac{\overline{\lambda}\boldsymbol{\cdot}\overline{\lambda}\rho}{c^2})-\frac{\rho}{c^2}\frac{\partial\overline{\lambda}}{\partial t}\boldsymbol{\cdot}\overline{\lambda}-\rho div(\overline{\lambda})=\frac{\partial}{\partial t}\frac{d^2\rho}{c^2}-\rho div(\overline{\lambda})$.

If $d=c$, we obtain $\rho div(\lambda)=0$, so that with analyticity assumptions, $\rho\neq 0$, we obtain that $div(\lambda)=0$. If $d\neq c$, letting $\epsilon=1-\frac{d^2}{c^2}$, we have that; $\epsilon\frac{\partial\rho}{\partial t}=-\rho div(\overline{\lambda})$ iff $\epsilon(div(\overline{J}))=\rho div(\overline{\lambda})$.

Again, we have that;

$$div(\overline{J})=\bigtriangledown(\rho)\boldsymbol{\cdot}\overline{\lambda}+\rho div(\overline{\lambda})$$

which implies that; $\frac{\rho}{\epsilon}div(\overline{\lambda})=\bigtriangledown(\rho)\boldsymbol{\cdot}\overline{\lambda}+\rho div(\overline{\lambda})$, so that; $\rho div(\overline{\lambda})(\frac{1}{\epsilon}-1)=\bigtriangledown(\rho)\boldsymbol{\cdot}\overline{\lambda}$ and, from (U); $\frac{\partial\rho}{\partial t}=-\overline{\lambda}\boldsymbol{\cdot}\bigtriangledown(\rho)-\frac{\epsilon}{\epsilon-1}\overline{\lambda}\boldsymbol{\cdot}\bigtriangledown(\rho)=\frac{1-2\epsilon}{\epsilon-1}\overline{\lambda}\boldsymbol{\cdot}\bigtriangledown(\rho)=\frac{1-2\epsilon}{\epsilon-1}\overline{\lambda}\boldsymbol{\cdot}-\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\frac{1-2\epsilon}{\epsilon-1}\overline{\lambda}\boldsymbol{\cdot}-\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\overline{\lambda})=-\frac{1-2\epsilon}{\epsilon-1}\frac{1}{c^2}(d^2)\frac{\partial\rho}{\partial t}$, so that; $(1+\frac{1-2\epsilon}{\epsilon-1}\frac{1}{c^2}(d^2))\frac{\partial\rho}{\partial t}=0$ iff $2(1-\frac{d^2}{c^2})\frac{\partial\rho}{\partial t}=0$

so that ρ is time independent. The conclusion that ρ is harmonic follows again from $\Box^2(\rho)=0$. □

Lemma 3. *Parallel systems with $|\overline{\lambda}|=c$, which satisfy the additional connecting relation; $\bigtriangledown(\rho)+\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}=\overline{0}$ are characterised by; $\rho(\overline{x},t)=h(\frac{\lambda_1}{c}x+\frac{\lambda_2}{c}y+\frac{\lambda_3}{c}z-ct)$ and $\overline{J}=\overline{\lambda}\rho$, where $h\in C^\infty(\mathcal{R})$. In particular, we have that the addition relations $\Box^2\rho=0$, $\Box^2\overline{J}=\overline{0}$ is satisfied and there is a solution with this requirement.*

Proof. As the system is parallel, by the proof of Lemma 1, we have that; $\overline{J} = \overline{\lambda}\rho$ and $\frac{\partial\rho}{\partial t} = -\overline{\lambda} \centerdot \bigtriangledown(\rho)$ (A). The connecting relation implies that;

$$\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} = \bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial(\overline{\lambda}\rho)}{\partial t} = \bigtriangledown(\rho) + \frac{\overline{\lambda}}{c^2}\frac{\partial\rho}{\partial t} = \overline{0},$$

so that $\bigtriangledown(\rho) = -\frac{\overline{\lambda}}{c^2}\frac{\partial\rho}{\partial t}$ (B).

Conversely, if ρ satisfies $(A), (B)$, with $\overline{J} = \overline{\lambda}\rho$, then we obtain a parallel system with the additional connecting relation. By (A), we have that $\rho(\overline{x}, t) = g(\overline{x} - \overline{\lambda}t)$, (C). In the case that $\overline{\lambda} = (c, 0, 0)$, we have from $(B), (C)$, that we require; $(g_x, g_y, g_z) = -\frac{1}{c^2}(c, 0, 0)\frac{\partial\rho}{\partial t}$, so that, in particular, $g_y = g_z = 0$, $g(\overline{x}) = h(x)$ and then; $\rho(\overline{x}, t) = g(\overline{x} - \overline{\lambda}t) = h(x - ct)$. Observe that if $\bigtriangledown(\rho) = \overline{\lambda}\delta(\overline{x}, t)$, then, as $|\lambda| = c$;

$$\frac{\partial\rho}{\partial t} = -\overline{\lambda} \centerdot \bigtriangledown(\rho) = -\overline{\lambda} \centerdot \lambda\delta(\overline{x}, t) = -c^2\delta(\overline{x}, t)$$

and $\delta(\overline{x}, t) = -\frac{1}{c^2}\frac{\partial\rho}{\partial t}$, so that; $\bigtriangledown(\rho) = -\frac{\overline{\lambda}}{c^2}\frac{\partial\rho}{\partial t}$, which is (B), $(*)$.

We have that $\rho_x = g_x$, $\rho_{xx} = g_{xx}$, $\rho_y = g_y = 0$, $\rho_{yy} = g_{yy} = 0$, $\rho_z = g_z = 0$, $\rho_{zz} = g_{zz} = 0$, $\rho_t = h_x(-c)$, $\rho_{tt} = h_{xx}c^2$, so that $\bigtriangledown^2(\rho) - \frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} = h_{xx} - h_{xx} = 0$ and $\square^2(\rho) = 0$, so that $\square^2(\overline{J}) = \overline{0}$. In the case of an arbitrary $\overline{\lambda}$ with $|\overline{\lambda}| = c$, choose an orthogonal matrix U with $U(1, 0, 0) = \frac{\overline{\lambda}}{c}$. We have found $\rho(\overline{x}, t)$ such that; $\bigtriangledown(\rho)(\overline{x}, t) = -\frac{1}{c^2}(1, 0, 0)m(\overline{x}.t)$, so that, applying U to both sides, we have that;

$$U(\bigtriangledown(\rho))(\overline{x}, t) = \bigtriangledown(\rho)(U^{-1}(\overline{x}), t) = -\frac{1}{c^2}\frac{\overline{\lambda}}{c}m(\overline{x}.t)$$

We have that;

$$\rho(U^{-1}(\overline{x}), t) = g(U^{-1}(\overline{x}) - \overline{\lambda}t) = h(U^{-1}(\overline{x})_1 - ct) = h(\frac{\lambda_1}{c}x + \frac{\lambda_2}{c}y + \frac{\lambda_3}{c}z - ct).$$

Observe that if $g(\overline{x}) = h(\frac{\lambda_1}{c}x + \frac{\lambda_2}{c}y + \frac{\lambda_3}{c}z)$, then; $g(\overline{x} - \overline{\lambda}t) = h(\frac{\lambda_1}{c}x + \frac{\lambda_2}{c}y + \frac{\lambda_3}{c}z - \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{c}t) = h(\frac{\lambda_1}{c}x + \frac{\lambda_2}{c}y + \frac{\lambda_3}{c}z - ct)$, so that (A) is still satisfied, and, by the above remark $(*)$, (B) is satisfied as well, as $\bigtriangledown(\rho \circ U^{-1})$ is parallel to λ. Again $\rho \circ U^{-1}$ satisfies the wave equation $\square^2(\rho \circ U^{-1}) = \bigtriangledown^2(\rho \circ U^{-1}) - \frac{1}{c^2}\frac{\partial^2\rho\circ U^{-1}}{\partial t^2} = 0$. Defining $\overline{J} = \overline{\lambda}\rho \circ U^{-1}$, we have that $\square^2(\overline{J}) = \overline{0}$ as well. $\square$

Definition 2. Given $(\rho, \overline{J})$ satisfying the continuity equation with $\rho > 0$, we define a trajectory $\overline{\gamma}$ to be an integral curve for the velocity field $\frac{\overline{J}}{\rho}$. We define a flow line $\overline{\gamma} : (0, \infty) \rightarrow \mathcal{R}^3$ to be a solution of the differential equation; $\overline{\gamma}'(s) = \frac{\overline{J}}{\rho}(\gamma(s), t_0 + s)$. We define a system to be circular if all the flow lines are circular orbits centred at the origin. We define a system to be closed if all the flow lines define closed curves. We define a system to be open if none of the flow lines are closed or bounded. We define thermal equilibrium by the condition $\frac{(\overline{J}, \overline{J})}{\rho^2} = d^2$, where $d \in \mathcal{R}_{>0}$.

Lemma 4. *Suppose that $(\rho, \overline{J})$ satisfy the wave equations $\square^2\rho = 0$, $\square^2\overline{J} = \overline{0}$, the continuity equation $\frac{\partial\rho}{\partial t} = -div(\overline{J})$ and the connecting relation $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} = \overline{0}$. Then, if the system is circular, ρ is time independent and harmonic, and, for $t \in \mathcal{R}$, the components of $\overline{J}_t$ are harmonic. If the system is in thermal equilibrium, then every closed trajectory intersects the locus $\frac{\partial\rho}{\partial t}|_t = 0$, for all $t \in \mathcal{R}$. If the system is in thermal equilibrium and ρ is not time independent, $\rho(\overline{\gamma}(s), s)$ is constant along the flowline. Moreover, every flowline is open and if $\rho_t - z \in S(\mathcal{R}^3)$ for $t \in \mathcal{R}$ some $z \in \mathcal{R}$, the system must be parallel.*

Proof. For the first claim, we have that $\frac{\overline{J}}{\rho} \centerdot \overline{x} = 0$, in particular $\overline{J} \centerdot \overline{x} = 0$. As $\Box^2(\overline{J}) = \overline{0}$, and the continuity equations holds, we have that;

$$\begin{aligned}
\Box^2(\overline{J} \centerdot \overline{x}) &= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})(j_1 x + j_2 y + j_3 z) \\
&= \frac{\partial^2}{\partial x^2}(j_1 x) + \frac{\partial^2}{\partial y^2}(j_2 y) + \frac{\partial^2}{\partial z^2}(j_3 z) + ((\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_1)x + ((\frac{\partial^2}{\partial y^x} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_2)y \\
&\quad + ((\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_3)z \\
&= 2\frac{\partial j_1}{\partial x} + 2\frac{\partial j_2}{\partial y} + 2\frac{\partial j_3}{\partial z} + (\frac{\partial^2}{\partial x^2}j_1)x + (\frac{\partial^2}{\partial y^2}j_2)y + (\frac{\partial^2}{\partial z^2}j_3)z + ((\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_1)x \\
&\quad + ((\frac{\partial^2}{\partial y^x} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_2)y + ((\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})j_3)z \\
&= 2div(\overline{J}) + (\Box^2 j_1)x + (\Box^2 j_2)y + (\Box^2 j_3)z = 2div(\overline{J}) = -2\frac{\partial\rho}{\partial t} = 0,
\end{aligned}$$

so that ρ is time independent and, as $\Box^2(\rho) = 0$, that ρ is harmonic. As $\bigtriangledown(\rho)$ is time independent, by the connecting relation $\bigtriangledown(\rho) + \frac{1}{c^2}\frac{\partial\overline{J}}{\partial t} = \overline{0}$, we have that $\frac{\partial\overline{J}}{\partial t}$ is time independent and $\frac{\partial^2\overline{J}}{\partial t^2} = \overline{0}$, so that, as $\Box^2\overline{J} = \overline{0}$, the the components of $\overline{J}_t$ are harmonic for $t \in \mathcal{R}$. For the second claim, we have that, if $\overline{\gamma}$ is a trajectory;

$$\begin{aligned}
\frac{d\rho}{ds}(\overline{\gamma}(s), t) &= \bigtriangledown(\rho)|_{\overline{\gamma}(s),t} \centerdot \overline{\gamma}'(s) = -\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}|_{\overline{\gamma}(s),t} \centerdot \frac{\overline{J}}{\rho}|_{\overline{\gamma}(s),t} \\
&= -\frac{1}{c^2\rho}|_{\overline{\gamma}(s),t}(\frac{\partial\overline{J}}{\partial t} \centerdot \overline{J})|_{\overline{\gamma}(s),t} = -\frac{1}{2c^2\rho}|_{\overline{\gamma}(s),t}\frac{\partial}{\partial t}(\overline{J} \centerdot \overline{J})|_{\overline{\gamma}(s),t} \\
&= -\frac{1}{2c^2\rho}|_{\overline{\gamma}(s),t}\frac{\partial}{\partial t}d^2\rho^2|_{\overline{\gamma}(s),t} = -\frac{d^2}{c^2\rho}|_{\overline{\gamma}(s),t}\rho|_{\overline{\gamma}(s),t}\frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),t} \\
&= -\frac{d^2}{c^2}\frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),t}.
\end{aligned}$$

If a closed trajectory $\overline{\gamma}$ is disjoint from the locus of $\frac{\partial\rho}{\partial t}_{t_0}$, with $\overline{\gamma}(a) = \overline{\gamma}(b)$, for some $t_0 \in \mathcal{R}$, then either $\frac{\partial\rho}{\partial t}|_W > 0$ or $\frac{\partial\rho}{\partial t}|_W < 0$. We have that;

$$\int_W \bigtriangledown(\rho) \centerdot d\overline{\gamma} = \int_a^b \frac{d\rho}{ds}(\overline{\gamma}(s), t)ds = -\frac{d^2}{c^2}\int_a^b \frac{\partial\rho}{\partial t}(\gamma(s), t)ds = \overline{\gamma}(b) - \overline{\gamma}(a) = 0$$

which is a contradiction. For the next claim, we calculate, if $\overline{\gamma}$ is a flowline;

$$\begin{aligned}
\frac{d\rho}{ds}(\overline{\gamma}(s), s) &= \bigtriangledown(\rho)|_{\overline{\gamma}(s),s} \centerdot \overline{\gamma}'(s) + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} = -\frac{1}{c^2}\frac{\partial\overline{J}}{\partial t}|_{\overline{\gamma}(s),s} \centerdot \frac{\overline{J}}{\rho}|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} \\
&= -\frac{1}{c^2\rho}|_{\overline{\gamma}(s),s}(\frac{\partial\overline{J}}{\partial t} \centerdot \overline{J})|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} = -\frac{1}{2c^2\rho}|_{\overline{\gamma}(s),s}\frac{\partial}{\partial t}(\overline{J} \centerdot \overline{J})|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} \\
&= -\frac{1}{2c^2\rho}|_{\overline{\gamma}(s),s}\frac{\partial}{\partial t}d^2\rho^2|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} = -\frac{d^2}{c^2\rho}|_{\overline{\gamma}(s),s}\rho|_{\overline{\gamma}(s),s}\frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} \\
&= -\frac{d^2}{c^2}\frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} + \frac{\partial\rho}{\partial t}|_{\overline{\gamma}(s),s} = 0,
\end{aligned}$$

as $d = c$, by Lemma 2. If $\overline{\gamma}$ is a flowline, then it cannot intersect the locus of $\overline{J}_t = \overline{0}$ at any time t, as we have thermal equilibrium $\overline{c}$, with $|\overline{c}| = c$. It follows that the flowline cannot have any equilibrium points in its closure, and if it is bounded, must be a closed loop or spirals into a closed loop. The first case can be excluded, by O-minimality of the real closed field with Pfaffian functions, using the fact that for a point $\overline{x}$ on the loop $Range(\overline{\gamma})$, we can define;

$$\{t \in \mathcal{R}_{>0} : \overline{\gamma}(t) = \overline{x}\}$$

which cannot be a finite union of points and intervals, as the set is unbounded and discrete, see [8] and [9]. More specifically, we can approximate $\frac{\bar{J}}{\rho}$ by a polynomial vector field $\overline{W}$, using the Stone-Weierstrass approximation theorem, on an open ball $B(\bar{0}, r_0)$ containing the loop, in such a way that a flowline $\overline{\gamma}_1$ for $\overline{W}$ is still a bounded closed loop. This follows as the cycle maps $\theta_{t_0,t}$ for $\frac{\bar{J}}{\rho}$, defined by;

$$\theta_{t_0,t}(\overline{x}) = \overline{\gamma}(t_0 + t)$$

where $\overline{\gamma}(t_0) = \overline{x}$ are invertible, by reversing the flow, letting;

$$\frac{\bar{J}}{\rho}^{rev}(\overline{x}, s) = -\frac{\bar{J}}{\rho}(\overline{x}, t_0 + t - s), 0 \leq s \leq t$$

and using the cycle map $\theta_{0,t}^{rev}$ for the corresponding flowline. It follows, by continuity, that the cycle maps $\theta_{t_0,t}$ are proper and the deviation map $\phi_{t_0,t}(\overline{x}) = \theta_{t_0,t}(\overline{x}) - \overline{x}$ is proper. As $\overline{\gamma}$ forms a closed loop, we have that $\bar{0} \in \phi_{t_0,t}(B^\circ(\bar{0}, r_0))$ and not a boundary point. By continuity, when we construct $\overline{W}$, we can still obtain a closed loop. In the second case, we could find a limit point $\overline{x} \in Range(\overline{\gamma})$, which is not an equilibrium point and which, wlog, is not on the trajectory, $(*)$. This follows as if it lies on the trajectory twice, we would form a closed loop. By the argument (X) below, we can exclude the case that $\overline{x}$ is an endpoint of $\overline{\gamma}$. It follows that it must be part of a compact ω-limit set with no fixed points. By the Poincare-Bendixson Theorem, applied to planar projections of the system in $\mathcal{R}^3$, we would obtain a periodic orbit again, and we can repeat the argument to obtain a contradiction. We can, therefore assume that the system is open.

If the system is not parallel or divergent, then there exists a ball $B(\overline{x}, r) = \bigcup_{w \in I}\{\overline{\gamma}_w(t) : 0 \leq t \leq t_(w)\}$, where I is a closed interval and $\{\overline{\gamma}_w : w \in I\}$ is a set of flowlines, with the properties that;

(i). $\overline{\gamma}_w(0)$ and $\overline{\gamma}_w(t_w)$ are the unique points of $\overline{\gamma}_w([0, t_w])$ lying on $\delta B(\overline{x}, r)$.

(ii). We have that $-\overline{c}(\overline{\gamma}_w(0)) \cdot \hat{\bar{n}} > \overline{c}(\overline{\gamma}_w(t_w)) \cdot \hat{\bar{n}}$

The property (i) follows from the fact that the flowlines are unbounded and the existence of flowlines through a given point, the last property being a consequence of Peano's existence theorem. The property (ii) follows from the asymmetry of a path which is not straight and continuity. By the above, we have that ρ is constant along the flowline in the sense that $\rho(\overline{\gamma}_w(s), s) = f(w)$ for some smooth function f, (S). By the continuity equation and divergence theorem, we have that;

$$\int_{B(\overline{x},r)} \frac{\partial \rho}{\partial t} dB = -\int_{\delta B(\overline{x},r)} \bar{J} \cdot \hat{\bar{n}} dS = -\int_{\delta B(\overline{x},r)} \rho \overline{c} \cdot \hat{\bar{n}} dS (T)$$

so that by $(S), (T)$ and the properties $(i), (ii)$, the charge ρ would monotonically increase or decrease inside the ball $B(\overline{x}, r)$. We can then either use the fact that $\rho \geq 0$ or the fact that $\square^2(\rho) = 0$, together with Kirchoff's formula for ρ, with initial conditions in $S(\mathcal{R}^3)$, so that $|\rho_t| \leq \frac{M}{t}$, where M is independent of t, to get a contradiction. The addition of a constant doesn't effect the argument.

We show that every trajectory $\overline{\gamma} : [0, \infty) \to \mathcal{R}$ doesn't have an endpoint, (X). Suppose that a trajectory has an endpoint $\overline{x}$. Then, by thermal equilibrium, we may suppose that $\frac{\bar{J}}{\rho}(\overline{x}) = \overline{v} \neq \bar{0}$. Without loss of generality, assume that $v_2 = v_3 = 0$. By continuity, we may suppose that $|\frac{j_1}{\rho}| \geq |\frac{v_1}{2}|$, for $\overline{x} \in B(\overline{x}, r)$, with $r < \frac{|v_1|}{2}$, and, there exists $t_0 \in \mathcal{R}_{>0}$, with $|\overline{\gamma}(t) - \overline{x}| < r < \frac{|v_1|}{2}$ (A), for $t \geq t_0$. By the intermediate value theorem, we have that;

$$\frac{|(\gamma_{\overline{v}}(t_0 + s))_1 - (\gamma_{\overline{v}}(t_0))_1| = |s(\gamma'_{\overline{v}}(t_0 + s_0))_1| \geq s v_1}{2}$$

with $s_0 \in (0, s)$, so that if $s > 3$, we obtain a contradiction with (A), as;

$|(\gamma_{\overline{v}}(t_0 + s)) - (\gamma_{\overline{v}}(t_0))| \geq |(\gamma_{\overline{v}}(t_0 + s))_1 - (\gamma_{\overline{v}}(t_0))_1| > v_1$

and, by (A);

$|(\gamma_{\overline{v}}(t_0 + s)) - (\gamma_{\overline{v}}(t_0))| \leq |(\gamma_{\overline{v}}(t_0 + s)) - \overline{x}| + |(\gamma_{\overline{v}}(t_0)) - \overline{x}| \leq 2\frac{v_1}{2} = v_1$ (X)

which is a contradiction.

□

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