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Some new results on soft quasilinear spaces

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Abstract: In this article, we focus on developing new results regarding normed quasilinear spaces. We provide a definition for soft homogenized quasilinear spaces and obtain some related results. Furthermore, we explore the floor of soft normed quasilinear spaces. Using some soft linearity and soft quasilinearity methods, we derive new results and examples. Finally, we also obtain some new consequences that we believe will facilitate the development of quasilinear functional analysis in a soft inner product quasilinear space.

Keywords: Soft set; Soft quasilinear space; Soft normed quasilinear space; Homogenized quasilinear spaces.

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1. Introduction

In this article, we focus on developing new results regarding normed quasilinear spaces. Aseev introduced quasilinear spaces, normed quasilinear spaces, and quasilinear operators in [1]. In [2–4], proper quasilinear spaces were defined, and many results were obtained on normed quasilinear spaces. In [5,6], authors worked on bounded quasilinear interval-valued functions and analyzed the Hahn Banach extension theorem for interval-valued functions. To develop quasilinear functional analysis, quasilinear inner product spaces, which are generalizations of inner product spaces, were defined in [7–10]. In [11], a special class of fuzzy number sequences was shown to be a Hilbert quasilinear space. In [12], authors provided easy examples of approximate estimations of deterministic autocorrelation of some semi non-deterministic signals or signals with inexact data in [12]. In [13], a new mathematical method for processing such a non-deterministic signal was presented by using interval-valued functions, which is called its model interval signal. Moreover, in [14], a new continuous-time epidemic model including nonlinear delay differential equations was analyzed by using parameters and functions selected from a class of intervals whose algebraic basis is based on quasilinear spaces.

In 1999, Molodtsov introduced soft set theory in [15] and showed various applications of this theory on economics, engineering, medical science, etc. In [16], several operations on soft sets were presented. Das and Samanta introduced the notions of soft element in [17] and soft real number in [18]. They worked on soft linear spaces, soft normed linear spaces, soft linear operators, soft inner product spaces, and their properties in [19–22]. In [23,24], soft normed space was introduced from a new point of view, and soft inner product space and soft Hilbert space on soft linear spaces were introduced, respectively.

Based on these studies on soft linear spaces and quasilinear spaces, in [25], the notions of soft quasilinear spaces and soft normed quasilinear spaces were introduced. Afterwards, in [26], definitions of soft inner product quasilinear spaces and soft Hilbert quasilinear spaces were given, and some properties of soft inner product quasilinear spaces were studied.

In this paper, we define the concept of soft homogenized quasilinear space and obtain new results related to this new concept. We also provide properties of these spaces. Furthermore, we present new theorems and examples regarding the floor of soft quasilinear spaces and soft quasilinear inner product spaces. Our results contribute significantly to the development of quasilinear functional analysis.

2. Preliminaries

In this section, we introduce some notions related to soft set theory, as well as basic concepts such as soft quasilinear spaces, soft normed quasilinear spaces, and soft inner product quasilinear spaces.

Let Q be a universe and P be a set of parameters. $P(Q)$ denotes the power set of Q, and B denotes a non-empty subset of P.

Definition 1. [15] A pair (G,P) is called a soft set over Q, where G is a mapping defined by $G : P \to P(Q)$.

Definition 2. [20] A soft set (G,P) over Q is said to be an absolute soft set represented by $\widetilde{Q}$, if for every $\lambda \in P$, $G(\lambda) = Q$. A soft set (G,P) over Q is said to be a null soft set represented by Φ, if for every $\lambda \in P$, $G(\lambda) = \emptyset$.

Definition 3. [18] Let Q be a non-empty set and P be a non-empty parameter set. Then a function $q : P \to Q$ is said to be a soft element of Q. A soft element q of Q is said to belong to a soft set G of Q, which is denoted by $q \in Q$, if $q(\lambda) \in G(\lambda)$, $\lambda \in P$. So, for a soft set G of Q with respect to the index set P, we have $G(\lambda) = \{q(\lambda), \lambda \widetilde{\in} P\}$. A soft set (G,P) for which $G(\lambda)$ is a singleton set, $\forall \lambda \in P$ can be determined with a soft element by simply determining the singleton set with the element that it contains $\forall \lambda \in P$.

The set of all soft sets (G,P) over Q is described by $S\left(\widetilde{Q}\right)$ for which $G(\lambda) \neq \emptyset$, for all $\lambda \in P$, and the collection of all soft elements of (G,P) over Q is denoted by $SE\left(\widetilde{Q}\right)$.

Now, let us give a definition that is meaningful in soft quasilinear spaces but not in soft linear spaces.

Definition 4. [25] Let Q be a quasilinear space and P be a parameter set. Let G be a soft set over (Q,P). G is said to be a soft quasilinear space of Q if $Q(\lambda)$ is a quasilinear subspace of Q for every $\lambda \in P$.

We use the notation $\widetilde{q}, \widetilde{w}, \widetilde{z}$ to indicate soft quasi vectors of a soft quasilinear space and $\widetilde{a}, \widetilde{b}, \widetilde{c}$ to specify soft real numbers. If a soft quasi element $\widetilde{q}$ has an inverse i.e. $\widetilde{q} - \widetilde{q} = \widetilde{\theta}$ such that $\widetilde{q}(\lambda) - \widetilde{q}(\lambda) = \widetilde{\theta}(\lambda)$ for every $\lambda \in P$ then it is called regular. If a soft quasi element $\widetilde{q}$ has no inverse, then it is called singular. Also, $\widetilde{Q}_r$ express for the set of all soft regular elements in $\widetilde{Q}$ and $\widetilde{Q}_s$ imply the sets of all soft singular elements in $\widetilde{Q}$.

Definition 5. [25] Let $\widetilde{Q}$ be the absolute soft quasilinear space i.e. $\widetilde{Q}(\lambda) = Q$ for every $\lambda \in P$. Then a mapping $\|.\| : SE\left(\widetilde{Q}\right) \longrightarrow \mathbb{R}(P)$ is said to be soft norm on the soft quasilinear space $\widetilde{Q}$, if $\|.\|$ satisfies the following conditions:

i) $\|\widetilde{q}\| \widetilde{\geq} 0$ if $\widetilde{q} \neq \overline{\theta}$ for every $\widetilde{q} \in \widetilde{Q}$,
ii) $\|\widetilde{q} + \widetilde{w}\| \widetilde{\leq} \|\widetilde{q}\| + \|\widetilde{w}\|$ for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$,
iii) $\|\widetilde{\alpha} \cdot \widetilde{q}\| = |\widetilde{\alpha}| \cdot \|\widetilde{q}\|$ for every $\widetilde{q} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$,
iv) if $\widetilde{q} \widetilde{\preceq} \widetilde{w}$, then $\|\widetilde{q}\| \widetilde{\leq} \|\widetilde{w}\|$ for every $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$,
v) if for any $\varepsilon \widetilde{>} 0$ there exists an element $\widetilde{q}_\varepsilon \in \widetilde{Q}$ such that, $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_\varepsilon$ and $\|\widetilde{q}_\varepsilon\| \widetilde{\leq} \varepsilon$ then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$ for any soft elements $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$.

A soft quasilinear space $\widetilde{Q}$ with a soft norm$\|.\|$ on $\widetilde{Q}$ is called soft normed quasilinear space and is indicated by $\left(\widetilde{Q}, \|.\|\right)$ or $\left(\widetilde{Q}, \|.\|, P\right)$.

Theorem 1. [26] *If a soft norm $\|.\|$ on soft normed quasilinear space $\widetilde{Q}$ satisfied the condition " $\xi \in Q$, and $\lambda \in P$, $\{\|\widetilde{q}\|(\lambda) = \xi\}$ is a singleton set.". If for every $\lambda \in P$, $\|.\|_\lambda : Q \to \mathbb{R}^+$ be a mapping such that for every $\xi \in Q$, $\|\xi\|_\lambda = \|\widetilde{q}\|(\lambda)$, where $\widetilde{q} \in \widetilde{Q}$ such that $\widetilde{q}(\lambda) = \xi$. Then for every $\lambda \in P$, $\|.\|_\lambda$ is a norm on quasilinear space Q.*

Let $\widetilde{Q}$ be a soft normed quasilinear space. Then, soft Hausdorff or soft norm metric on $\widetilde{Q}$ is defined by

$$h_Q(\widetilde{q}, \widetilde{w}) = \inf\left\{\widetilde{r} \widetilde{\geq} 0 : \widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_1^r, \widetilde{w} \widetilde{\preceq} \widetilde{q} + \widetilde{q}_2^r, \|\widetilde{q}_i^r\| \widetilde{\leq} \widetilde{r}\right\}.$$

Definition 6. [26] Let $\widetilde{Q}$ be a soft quasilinear space, $\widetilde{W} \subseteq \widetilde{Q}$ and $\widetilde{q} \in \widetilde{W}$. The set

$$F_{\widetilde{q}}^{\widetilde{W}} = \{\widetilde{m} \in \widetilde{W}_r : \widetilde{m} \widetilde{\preceq} \widetilde{q}\},$$

is called floor in $\widetilde{W}$ of $\widetilde{q}$. If $\widetilde{W} = \widetilde{Q}$ then we will say only floor of $\widetilde{q}$ and written shortly $F_{\widetilde{q}}$ instead of $F_{\widetilde{q}}^{\widetilde{Q}}$.

Definition 7. [26] Let $\widetilde{Q}$ be a soft quasilinear space, $\widetilde{Q}$ is called a solid floored soft quasilinear space whenever

$$\widetilde{q} = \sup\{\widetilde{m} \in \widetilde{W}_r : \widetilde{m} \widetilde{\preceq} \widetilde{q}\}$$

for every $\widetilde{q} \in \widetilde{Q}$. Otherwise, $\widetilde{Q}$ is called a non-solid floored soft quasilinear space.

Theorem 2. [26] *Absolute soft quasilinear space* $\left(\widetilde{\Omega_C(\mathbb{R})}\right)$ *is a solid floored.*

Definition 8. [26] Let $\widetilde{Q}$ be a soft quasilinear space. Consolidation of floor of $\widetilde{Q}$ is the smallest solid floored soft quasilinear space $\widehat{\left(\widetilde{Q}\right)}$ containing $\widetilde{Q}$, namely, if there exists different solid floored soft quasilinear space $\widetilde{W}$ including $\widetilde{Q}$, then $\widehat{\left(\widetilde{Q}\right)} \widetilde{\subseteq} \widetilde{W}$.

Definition 9. [26] Let $\widetilde{Q}$ be the absolute soft quasilinear space i.e. $\widetilde{Q}(\lambda) = Q, \forall \lambda \in P$. Then a mapping

$$\langle . \rangle : SE\left(\widetilde{Q}\right) \times SE\left(\widetilde{Q}\right) \to \Omega\left(\mathbb{R}\right)(P)$$

is said to be a soft quasi inner product on the soft quasilinear space $\widetilde{Q}$, if $\langle . \rangle$ satisfies the following conditions:

i) $\langle \widetilde{q}, \widetilde{w} \rangle \in (\Omega(\mathbb{R}))_r \equiv \widetilde{\mathbb{R}}$ if $\widetilde{q}, \widetilde{w} \in \widetilde{Q}_r$,
ii) $\langle \widetilde{q} + \widetilde{w}, \widetilde{z} \rangle \widetilde{\subseteq} \langle \widetilde{q}, \widetilde{z} \rangle + \langle \widetilde{w}, \widetilde{z} \rangle$ for all $\widetilde{q}, \widetilde{w}, \widetilde{z} \in \widetilde{Q}$,
iii) $\langle \widetilde{\alpha} \cdot \widetilde{q}, \widetilde{w} \rangle = \widetilde{\alpha} \cdot \langle \widetilde{q}, \widetilde{w} \rangle$ for all $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$ and for every soft scalar $\widetilde{\alpha}$,
iv) $\langle \widetilde{q}, \widetilde{w} \rangle = \langle \widetilde{w}, \widetilde{q} \rangle$ for all $\widetilde{q}, \widetilde{w} \in \widetilde{Q}$,
v) $\langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\supseteq} \bar{0}$ if $\widetilde{q} \in \widetilde{Q}_r$ and $\langle \widetilde{q}, \widetilde{q} \rangle = \{\bar{0}\} \Leftrightarrow \widetilde{q} = \{\theta\}$,
vi) $\|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = \sup\left\{ \|\langle x, y \rangle\| : x \in F_{\widetilde{q}}^{\widehat{(\widetilde{Q})}}, y \in F_{\widetilde{w}}^{\widehat{(\widetilde{Q})}} \right\}$,
vii) $\langle \widetilde{q}, \widetilde{w} \rangle \widetilde{\subseteq} \langle \widetilde{z}, \widetilde{v} \rangle$ if $\widetilde{q} \widetilde{\preceq} \widetilde{z}$ and $\widetilde{w} \widetilde{\preceq} \widetilde{v}$ for all $\widetilde{q}, \widetilde{w}, \widetilde{z}, \widetilde{v} \in \widetilde{Q}$,
viii) $\forall \widetilde{\epsilon} \widetilde{\geq} \bar{0}$, $\exists \widetilde{q}_{\widetilde{\epsilon}} \in \widetilde{Q}$ such that $\widetilde{q} \widetilde{\preceq} \widetilde{w} + \widetilde{q}_{\widetilde{\epsilon}}$ and $\langle \widetilde{q}_{\widetilde{\epsilon}}, \widetilde{q}_{\widetilde{\epsilon}} \rangle \widetilde{\subseteq} S_{\widetilde{\epsilon}}(\theta)$ then $\widetilde{q} \widetilde{\preceq} \widetilde{w}$.

A soft quasilinear space $\widetilde{Q}$ with a soft quasi inner product $\langle . \rangle$ on $\widetilde{Q}$ is called a soft quasilinear inner product space and denoted by $\left(\widetilde{Q}, \langle . \rangle, P\right)$.

Remark 1. If $\widetilde{Q}$ is a soft linear space, then above conditions are determined by conditions of the real soft inner product spaces. Moreover, a regular subspace $\widetilde{Q}_r$ of a soft quasilinear inner product space $\widetilde{Q}$ is a soft (linear) inner product space with the same inner product.

Definition 10. [26] A soft quasi vector $\widetilde{q}$ of soft quasilinear inner product space $\widetilde{Q}$ is said to be orthogonal to soft quasi element $\widetilde{w} \in \widetilde{Q}$ if

$$\|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = \bar{0}.$$

It is also denoted by $\widetilde{q} \perp \widetilde{w}$. Let $\widetilde{M}$ be a non-null soft quasi subset of soft quasilinear inner product space $\widetilde{Q}$ such that $\widetilde{M}(\lambda) \neq \emptyset$ for every $\lambda \in P$. If a soft quasi vector $\widetilde{q}$ of soft quasilinear inner product space $\widetilde{Q}$ orthogonal to every soft quasi vectors of $\widetilde{M}$, then we say that $\widetilde{q}$ is orthogonal to $\widetilde{M}$ and we write $\widetilde{q} \perp \widetilde{M}$. A non-null orthonormal soft quasi subset $\widetilde{M}$ of soft quasilinear inner product space $\widetilde{Q}$ such that $\widetilde{M}(\lambda) \neq \emptyset$ for every $\lambda \in P$ is a orthogonal soft quasi subset in $\widetilde{Q}$ whose soft quasi vectors have norm $\bar{1}$; that is, for all $\widetilde{q}, \widetilde{w} \in \widetilde{M}$

$$\|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = \begin{cases} \bar{0}, & \widetilde{q} = \widetilde{w} \\ \bar{1}, & \widetilde{q} \neq \widetilde{w} \end{cases}.$$

Definition 11. The set of all soft elements of $\widetilde{Q}$ orthogonal to $\widetilde{W}$, denoted by $\widetilde{W}^{\perp}$, is called the orthogonal complement of $\widetilde{W}$ and is indicated by

$$\widetilde{W}^{\perp} = \left\{ \widetilde{q} \in \widetilde{Q} : \|\langle \widetilde{q}, \widetilde{w} \rangle\|_{\Omega(\mathbb{R})} = 0, \widetilde{w} \in \widetilde{W} \right\}.$$

3. Main results

Definition 12. Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{\alpha}\widetilde{\beta} \widetilde{\geq} \bar{0}$ for every soft scalars $\widetilde{\alpha}, \widetilde{\beta}$. If

$$\left(\widetilde{\alpha}+\widetilde{\beta}\right)\cdot\widetilde{q}=\widetilde{\alpha}\cdot\widetilde{q}+\widetilde{\beta}\cdot\widetilde{q}$$

for every $\widetilde{q}\in\widetilde{Q}$, then we say that $\widetilde{Q}$ is a soft homogenized quasilinear space.

Theorem 3. *Let $\widetilde{Q}$ be a soft quasilinear space. $\widetilde{\Omega_C(Q)}$ is a soft homogenized quasilinear space but $\widetilde{\Omega(Q)}$ is not a soft homogenized quasilinear space.*

Proof. First, we will show that $\widetilde{\Omega_C(Q)}$ is a soft homogenized quasilinear space for a soft quasilinear space $\widetilde{Q}$. We have to show that $\left(\widetilde{\alpha}+\widetilde{\beta}\right)\cdot\widetilde{q}=\widetilde{\alpha}\cdot\widetilde{q}+\widetilde{\beta}\cdot\widetilde{q}$ for every $\widetilde{q}\in\widetilde{\Omega_C(Q)}$ and $\widetilde{\alpha}\widetilde{\beta}\widetilde{\geq}\widetilde{0}$ for $\widetilde{\Omega_C(Q)}$ to be soft homogenized quasilinear space. We obtain $\left(\widetilde{\alpha}+\widetilde{\beta}\right)\cdot\widetilde{q}\subseteq\widetilde{\alpha}\cdot\widetilde{q}+\widetilde{\beta}\cdot\widetilde{q}$ since $\widetilde{\Omega_C(Q)}$ is a quasilinear space. Let $c\in\left(\widetilde{\alpha}\cdot\widetilde{q}+\widetilde{\beta}\cdot\widetilde{q}\right)(\lambda)$ for a parameter λ. Then, we get

$$c=\widetilde{\alpha}\left(\lambda\right)a+\widetilde{\beta}\left(\lambda\right)b$$

for $a,b\in\widetilde{q}\left(\lambda\right)$. Here, we can write

$$c=(\widetilde{\alpha}+\widetilde{\beta})\left(\lambda\right)\cdot\left[\frac{\widetilde{\alpha}\left(\lambda\right)}{\widetilde{\alpha}\left(\lambda\right)+\widetilde{\beta}\left(\lambda\right)}a+\frac{\widetilde{\beta}\left(\lambda\right)}{\widetilde{\alpha}\left(\lambda\right)+\widetilde{\beta}\left(\lambda\right)}b\right]. \tag{1}$$

If we take $\widetilde{t}=\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}$ and $\widetilde{k}=\frac{\widetilde{\beta}}{\widetilde{\alpha}+\widetilde{\beta}}$, then we obtain

(i) $\widetilde{\alpha}\widetilde{\leq}\widetilde{\alpha}+\widetilde{\beta}\Rightarrow\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}\widetilde{\leq}\widetilde{1}$ and $\widetilde{0}\widetilde{\leq}\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}$ for $\widetilde{\alpha},\widetilde{\beta}\in\widetilde{\mathbb{R}}^{+}$.

(ii) $\widetilde{\alpha}+\widetilde{\beta}\widetilde{\leq}\widetilde{\alpha}\Rightarrow\widetilde{1}\widetilde{\geq}\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}$ and $\widetilde{0}\widetilde{\leq}\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}$ $\widetilde{\alpha},\widetilde{\beta}\in\widetilde{\mathbb{R}}^{-}$

for $\widetilde{\alpha}\widetilde{\beta}\widetilde{\geq}\widetilde{0}$. So, we have $\widetilde{0}\leq\widetilde{t}\leq\widetilde{1}$ from (i) and (ii).

Further, from definition of convexity on quasilinear spaces we find $\frac{\widetilde{\alpha}(\lambda)}{\widetilde{\alpha}(\lambda)+\widetilde{\beta}(\lambda)}a+\frac{\widetilde{\beta}(\lambda)}{\widetilde{\alpha}(\lambda)+\widetilde{\beta}(\lambda)}b\in\widetilde{q}\left(\lambda\right)$ since $\widetilde{t}+\widetilde{k}=\frac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}+\frac{\widetilde{\beta}}{\widetilde{\alpha}+\widetilde{\beta}}=\frac{\widetilde{\alpha}+\widetilde{\beta}}{\widetilde{\alpha}+\widetilde{\beta}}=\widetilde{1}$. Thus, for a $k\in\widetilde{q}\left(\lambda\right)$ we obtain

$$c=(\widetilde{\alpha}+\widetilde{\beta})\left(\lambda\right)k\in\widetilde{q}\left(\lambda\right)$$

from (1). Therefore, we get $c\in\left(\left(\widetilde{\alpha}+\widetilde{\beta}\right)\cdot\widetilde{q}\right)(\lambda)$ for arbitrary soft parameter λ. □

Example 1. $\widetilde{\Omega(\mathbb{R})}$ is not a homogenized soft quasilinear space. Let

$$\begin{array}{ccl}\widetilde{A} & : & M\longrightarrow\Omega(\mathbb{R})\\ \lambda & \longrightarrow & \widetilde{A}(\lambda)=\{1,2\}.\end{array}$$

From here, we obtain $\left(\widetilde{2}\cdot\widetilde{A}\right)(\lambda)=\{2,4\}$. Otherside, we get $\left(\widetilde{A}+\widetilde{A}\right)(\lambda)=\widetilde{A}(\lambda)+\widetilde{A}(\lambda)=\{2,3,4\}$. Also, $\widetilde{\alpha}\widetilde{\beta}\widetilde{\geq}\widetilde{0}$ we obtain $\widetilde{2}\cdot\widetilde{A}\neq\widetilde{A}+\widetilde{A}$ for every parameter $\widetilde{\alpha}\left(\lambda\right)=\alpha$.

Example 2. Let $\widetilde{Q}=\widetilde{\Omega(\mathbb{R}^2)}$ and $\widetilde{A}$ be a soft quasi vector on $\widetilde{\Omega(\mathbb{R}^2)}$. Let $\widetilde{A}(\lambda)=\{A_1,A_2\}$ such that $A_1=\{(0,t):0\leq t\leq 1\}$ and $A_2=\{(t,0):0\leq t\leq 1\}$ for a λ parameter. Then, similar to above example if we take $\widetilde{\alpha}\left(\lambda\right)=\alpha$ for every λ parameter, then we obtain

$$\left(\widetilde{2}\cdot\widetilde{A}\right)(\lambda)=\{2A_1,2A_2\}.$$

Otherside, we have

$$\begin{array}{rcl}\left(\widetilde{A}+\widetilde{A}\right)(\lambda) & = & \{A_1,A_2\}+\{A_1,A_2\}\\ & = & \{(0,2t):0\leq t\leq 1\}\cup\{(t,t):0\leq t\leq 1\}\cup\{(2t,0):0\leq t\leq 1\}.\end{array}$$

This gives $\widetilde{\Omega(\mathbb{R}^2)}$ is not a homogenized soft quasilinear space for $\widetilde{\alpha}(\lambda)=1$ and $\widetilde{\beta}(\lambda)=1$ soft scalars.

Theorem 4. *Let $\widetilde{Q}$ be a soft homogenized quasilinear space. If $\widetilde{A}\in\widetilde{Q_d}$, then there exist a $\widetilde{B}\in\widetilde{Q}$ such that $\widetilde{A}=\widetilde{B}-\widetilde{B}$.*

Proof. Let $\widetilde{A}$ be a symmetric soft quasi vector of homogenized soft quasilinear space $\widetilde{Q}$ i.e. $\widetilde{A}=-\widetilde{A}$. Sametime, we obtain $\widetilde{A}+\widetilde{A}=\widetilde{A}-\widetilde{A}$ since $\widetilde{A}=\widetilde{A}$. For every λ parameter we obtain $\left(\widetilde{A}+\widetilde{A}\right)(\lambda)=\left(\widetilde{A}-\widetilde{A}\right)(\lambda)$. Since X is a homogenized quasilinear space, we find $\widetilde{A}(\lambda)+\widetilde{A}(\lambda)=(\widetilde{2A})(\lambda)$. This gives $\widetilde{A}(\lambda)=\frac{\widetilde{A}}{2}(\lambda)-\frac{\widetilde{A}}{2}(\lambda)$. So, we obtain $\widetilde{A}=\frac{\widetilde{A}}{2}-\frac{\widetilde{A}}{2}$. □

Now, let us give our results regarding floor in a soft quasilinear space.

Theorem 5. *Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{U}\subseteq\widetilde{Q}$. Then the following conditions satisfies:*

a) $\left\{\widetilde{0}\right\}\in F^{\perp}_{\widetilde{U}}$,
b) *If* $\widetilde{U}\subseteq\widetilde{V}$, *then* $F_{\widetilde{U}}\subseteq F_{\widetilde{V}}$ *and if* $\widetilde{U}^{\perp}\subseteq\widetilde{V}^{\perp}$, *then* $F^{\perp}_{\widetilde{U}}\subseteq F^{\perp}_{\widetilde{V}}$,
c) $F_{\{\widetilde{0}\}}=\left\{\widetilde{0}\right\}$.

Proof. Let $\widetilde{q}$ is a arbitrary soft quasi vector in $\widetilde{U}$. $\left\{\widetilde{0}\right\}\in F^{\perp}_{\widetilde{U}}$ since $\left\|\left\langle\widetilde{q},\left\{\widetilde{0}\right\}\right\rangle_{\widetilde{\Omega(\mathbb{R})}}\right\|=\left\|\left\langle\widetilde{q}(\lambda),\left\{\widetilde{0}\right\}(\lambda)\right\rangle_{\Omega(\mathbb{R})}\right\|=\left\|\langle\widetilde{q}(\lambda),0\rangle_{\Omega(\mathbb{R})}\right\|=\left\{\widetilde{0}\right\}$ for a parameter λ.

Let $\widetilde{U}\subseteq\widetilde{V}$. We must show that $\widetilde{q}\in F_{\widetilde{V}}$ for every $\widetilde{q}\in F_{\widetilde{U}}$. If $\widetilde{q}\in F_{\widetilde{U}}$, then there exist a $\widetilde{u}\in\widetilde{U}$ such that $\widetilde{q}\in F_{\widetilde{u}}$. Since $\widetilde{U}\subseteq\widetilde{V}$, we find $\widetilde{u}\in\widetilde{V}$. Thus, we obtain $F_{\widetilde{u}}\widetilde{\subseteq}\cup_{\widetilde{u}\in\widetilde{U}}F_{\widetilde{v}}$. This gives $\widetilde{q}\in F_{\widetilde{V}}$. if $\widetilde{U}^{\perp}\subseteq\widetilde{V}^{\perp}$, then $F^{\perp}_{\widetilde{U}}\subseteq F^{\perp}_{\widetilde{V}}$ can be showed similarly.

Clearly,

$$\begin{aligned}F_{\{\widetilde{0}\}} &= \left\{\widetilde{q}\in\widetilde{Q_r}:\widetilde{q}\widetilde{\leq}\left\{\widetilde{0}\right\}\right\}\\ &= \left\{\widetilde{q}(\lambda)\in Q_r:\widetilde{q}(\lambda)\leq\left\{\widetilde{0}\right\}(\lambda)\right\}\\ &= \{\widetilde{q}(\lambda)\in Q_r:\widetilde{q}(\lambda)\leq 0\}\\ &= \left\{\widetilde{0}\right\}\end{aligned}$$

for a parameter λ. □

Definition 13. Let $\widetilde{Q}$ be a soft quasilinear space. $\widetilde{M}\subseteq\widetilde{Q}$ is a convex if and only if $\widetilde{\alpha}\widetilde{q}+(1-\widetilde{\alpha})\widetilde{w}\in F_{\widetilde{M}}$ for every $\widetilde{q},\widetilde{w}\in F_{\widetilde{M}}$ and soft scalar $\widetilde{\alpha}$.

Theorem 6. *Let $\widetilde{Q}$ be a soft quasilinear space and $\widetilde{M}$ is a subspace of $\widetilde{Q}$. Then $F_{\widetilde{M}}$ is a convex subspace of soft quasilinear space $\widetilde{Q}$ if and only if F_M is a convex subspace of Q.*

Proof. Let $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in F_{\widetilde{M}}$ for $\forall\widetilde{a},\widetilde{b}\in F_{\widetilde{M}}$ such that $\widetilde{a}(\lambda)=a\in F_M$,$\widetilde{b}(\lambda)=b\in F_M$,$\widetilde{x}(\lambda)=x\in M$,$\widetilde{y}(\lambda)=y\in M$ and soft scalar $\widetilde{\alpha}(\lambda)=\alpha$ for every parameter λ. From Definition 6, there exists $\widetilde{x},\widetilde{y}\in\widetilde{M}$ such that $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\widetilde{\leq}\widetilde{\alpha}\cdot\widetilde{x}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{y}$. Then for every parameter λ, we get $(\widetilde{\alpha}\cdot\widetilde{a})(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}(\lambda)\leq(\widetilde{\alpha}\cdot\widetilde{x})(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{y}(\lambda)$. This gives

$$\widetilde{\alpha}(\lambda)\cdot\widetilde{a}(\lambda)+(\widetilde{1}-\widetilde{\alpha})(\lambda)\cdot\widetilde{b}(\lambda)\leq\widetilde{\alpha}(\lambda)\cdot\widetilde{x}(\lambda)+(\widetilde{1}-\widetilde{\alpha})(\lambda)\cdot\widetilde{y}(\lambda).$$

So, we obtain $\alpha\cdot a+(1-\alpha)\cdot b\leq\alpha\cdot x+(1-\alpha)\cdot y\in M$. Otherside, there exists a $\widetilde{c}\in\widetilde{M}$ such that $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}+\widetilde{c}=\widetilde{\theta}$ because of $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in\widetilde{M}_r$. So, we obtain $(\widetilde{\alpha}\cdot\widetilde{a})(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}(\lambda)+\widetilde{c}(\lambda)=\theta$ for every parameter λ. This gives $\alpha\cdot a+(1-\alpha)\cdot b\in M_r$. Therefore, we get $\alpha\cdot a+(1-\alpha)\cdot b\in F_M$.

Let $\alpha\cdot a+(1-\alpha)\cdot b\in F_M$ for $\forall a,b\in F_M$ such that $\widetilde{a}(\lambda)=a\in F_M$,$\widetilde{b}(\lambda)=b\in F_M$, $\widetilde{x}(\lambda)=x\in M$,$\widetilde{y}(\lambda)=y\in M$ and soft scalar $\widetilde{\alpha}(\lambda)=\alpha$ for every parameter λ. From Definition 6, there exists $x,y\in M$ such that $\alpha\cdot a+(1-\alpha)\cdot b\leq\alpha\cdot x+(1-\alpha)\cdot y$. Then for every parameter λ, we get

$$(\widetilde{\alpha}\cdot\widetilde{a})(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}(\lambda)=\alpha\cdot a+(1-\alpha)\cdot b\widetilde{\leq}\alpha\cdot x+(1-\alpha)\cdot y=(\widetilde{\alpha}\cdot\widetilde{x})(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{y}(\lambda).$$

This gives $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\widetilde{\leq}\widetilde{\alpha}\cdot\widetilde{x}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{y}$. Otherside, there exists a $d\in M$ such that $\alpha\cdot a+(1-\alpha)\cdot b+d=\theta$ because of $\alpha\cdot a+(1-\alpha)\cdot b\in M_r$. So, we obtain $(\widetilde{\alpha}\cdot\widetilde{a}(\lambda)+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}(\lambda)+\widetilde{d}(\lambda)=\theta$ for every parameter λ. This gives $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in\widetilde{M}_r$. Therefore, we get $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in F_{\widetilde{M}}$. □

Theorem 7. *Let $\widetilde{Q}$ be a soft homogenized quasilinear space. $F_{\widetilde{q}}$ is a convex for a $\widetilde{q}\in\widetilde{Q}$.*

Proof. We assume that $\widetilde{Q}$ be a soft homogenized quasilinear space. We have to show that $\widetilde{\lambda}\cdot\widetilde{w}+(1-\widetilde{\lambda})\cdot\widetilde{w'}\in F_{\widetilde{q}}$ for $\widetilde{w},\widetilde{w'}\in F_{\widetilde{q}}$. For a $\widetilde{q}\in\widetilde{Q}$, we have $F_{\widetilde{q}}=\{\widetilde{w}\in\widetilde{Q}_r:\widetilde{w}\widetilde{\preceq}\widetilde{q}\}$. From here, we obtain $\widetilde{w}\widetilde{\preceq}\widetilde{q}$ and $\widetilde{w'}\widetilde{\preceq}\widetilde{q}$ for every $\widetilde{w},\widetilde{w'}\in F_{\widetilde{q}}$. Since $\widetilde{Q}$ be a soft quasilinear space, we find $\widetilde{\lambda}\cdot\widetilde{w}\widetilde{\preceq}\widetilde{\lambda}\cdot\widetilde{q}$ and $(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{w'}\widetilde{\preceq}(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{q}$ for every $\widetilde{0}\widetilde{\preceq}\widetilde{\lambda}\widetilde{\preceq}\widetilde{1}$ and $\widetilde{\lambda}\cdot\widetilde{w}+(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{w'}\widetilde{\preceq}\widetilde{\lambda}\cdot\widetilde{q}+(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{q}$. Since $\widetilde{Q}$ is a homogenized soft quasilinear space, we obtain $\widetilde{\lambda}\cdot\widetilde{q}+(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{q}=(\widetilde{\lambda}+\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{q}=\widetilde{q}$. This gives $\widetilde{\lambda}\cdot\widetilde{w}+(\widetilde{1}-\widetilde{\lambda})\cdot\widetilde{w'}\in F_{\widetilde{q}}$. □

Remark 2. A floor of a soft quasi vector of a soft quasilinear space $\widetilde{Q}$ is a convex if and only if soft quasilinear space $\widetilde{Q}$ is homogenized. In the above theorem, if $\widetilde{Q}$ would not be homogenized, then $F_{\widetilde{q}}$ had not been convex for a $\widetilde{q}\in\widetilde{Q}$ because of the equality $\widetilde{\alpha}\cdot\widetilde{q}+\widetilde{\beta}\cdot\widetilde{q}=(\widetilde{\alpha}+\widetilde{\beta})\cdot\widetilde{q}$ may not satisfy for all soft scalars $\widetilde{\alpha},\widetilde{\beta}$.

Theorem 8. *Let $\widetilde{Q}$ be a soft Hilbert quasilinear space and $\widetilde{K}$ is a soft convex subspace of $\widetilde{Q}$. Then the set of floor of $\widetilde{K}$ i.e. $F_{\widetilde{K}}$ is a convex complete soft subspace of $\widetilde{Q}$.*

Proof. First, we will show that $F_{\widetilde{K}}$ is a convex soft subspace of $\widetilde{Q}$. We have $\widetilde{\alpha}\cdot\widetilde{q}+(\widetilde{1}-\widetilde{\alpha})\widetilde{w}\in\widetilde{K}$ for every $\widetilde{q},\widetilde{w}\in\widetilde{K}$ and $\widetilde{0}\widetilde{\preceq}\widetilde{\alpha}\widetilde{\preceq}\widetilde{1}$ since $\widetilde{K}$ is a soft convex set. Let $\widetilde{a},\widetilde{b}\in F_{\widetilde{K}}$. Then there exists a $\widetilde{q},\widetilde{w}\in\widetilde{K}$ such that $\widetilde{\alpha}\widetilde{\preceq}\widetilde{q}$ and $\widetilde{b}\widetilde{\preceq}\widetilde{w}$. Otherwise, we find $\widetilde{\alpha}\cdot\widetilde{a}\widetilde{\leq}\widetilde{\alpha}\cdot\widetilde{q}$ and $(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\widetilde{\leq}(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{w}$ for $\widetilde{0}\widetilde{\preceq}\widetilde{\alpha}\widetilde{\preceq}\widetilde{1}$. We obtain $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\widetilde{\preceq}\widetilde{\alpha}\cdot\widetilde{q}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{w}$ since $\widetilde{Q}$ is a soft quasilinear space. Further, we find $\widetilde{\alpha}\cdot\widetilde{q}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{w}\in\widetilde{K}$ because of $\widetilde{K}$ is a soft convex subspace of $\widetilde{Q}$. Also, we obtain $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in\widetilde{K_r}$. This gives $\widetilde{\alpha}\cdot\widetilde{a}+(\widetilde{1}-\widetilde{\alpha})\cdot\widetilde{b}\in F_{\widetilde{K}}$. Now, we will show that $F_{\widetilde{K}}$ is a complete. Let $\widetilde{a}_n\in F_{\widetilde{K}}$ and $\widetilde{a}_n\to\widetilde{a}\in\widetilde{Q}$ for $n\to\infty$. If $\widetilde{a}_n\in F_{\widetilde{K}}$, then there exists $\widetilde{q}_n\in\widetilde{K}$ for every $n\in\mathbb{N}$ such that

$$\widetilde{\alpha_n}\widetilde{\preceq}\widetilde{q}_n. \tag{2}$$

On the other hand, for every $\widetilde{\varepsilon}\widetilde{\geq}\widetilde{0}$ there exists a $n_0\in\mathbb{N}$ such that for every $n>n_0$, we get

$$\widetilde{a}_n\widetilde{\leq}\widetilde{a}+\widetilde{a}_{1n}^{\varepsilon},\ \widetilde{a}\widetilde{\leq}\widetilde{a}_n+\widetilde{a}_{2n}^{\varepsilon}\text{ and }\|\widetilde{a}_{in}^{\varepsilon}\|\widetilde{\leq}\widetilde{\varepsilon}. \tag{3}$$

From (2) and (3), we have $\widetilde{a}\widetilde{\leq}\widetilde{q}_n+\widetilde{a}_{2n}^{\varepsilon}$ and $\|\widetilde{a}_{2n}^{\varepsilon}\|\widetilde{\leq}\widetilde{\varepsilon}$ for every $n\in\mathbb{N}$. Additionally, we find $\widetilde{a}\widetilde{\leq}\widetilde{q}_n$ for every $n\in\mathbb{N}$ from Definition 5. Now, we show that $\widetilde{a}\in\widetilde{K_r}$. Since $\widetilde{Q}$ is a soft Hilbert quasilinear space and [25], we find $-\widetilde{a}_n\to-\widetilde{a}$ and $\widetilde{a}_n-\widetilde{a}_n\to\widetilde{a}-\widetilde{a}$ for $n\to\infty$. Thus, for every $\widetilde{\varepsilon}\widetilde{\geq}\widetilde{0}$ there exists a $n_0\in\mathbb{N}$ such that for every $n>n_0$, we get

$$\widetilde{a}_n-\widetilde{a}_n\widetilde{\leq}\widetilde{a}-\widetilde{a}+\widetilde{a}_{1n}^{\varepsilon},\ \widetilde{a}-\widetilde{a}\widetilde{\leq}\widetilde{a}_n-\widetilde{a}_n+\widetilde{a}_{2n}^{\varepsilon}\text{ and }\|\widetilde{a}_{in}^{\varepsilon}\|\widetilde{\leq}\widetilde{\varepsilon}.$$

From here, we have

$$\widetilde{0}\widetilde{\leq}\widetilde{a}-\widetilde{a}+\widetilde{a}_{1n}^{\varepsilon},\ \widetilde{a}-\widetilde{a}\widetilde{\leq}\widetilde{0}+\widetilde{a}_{2n}^{\varepsilon}\text{ and }\|\widetilde{a}_{in}^{\varepsilon}\|\widetilde{\leq}\widetilde{\varepsilon}$$

and

$$\widetilde{0}\widetilde{\leq}\widetilde{a}-\widetilde{a},\ \widetilde{a}-\widetilde{a}\widetilde{\leq}\widetilde{0}$$

since $\widetilde{a}_n\in F_{\widetilde{K}}$ and $\widetilde{Q}$ is a soft Hilbert quasilinear space. From [25] Definition 11., we find $\widetilde{0}=\widetilde{a}-\widetilde{a}$. This gives $\widetilde{a}\in\widetilde{K_r}$. Therefore, we say that $F_{\widetilde{K}}$ is complete. □

Remark 3. Here we note that the floor of subspace of a Hilbert quasilinear space is always complete whether the this subspace is complete or not.

Theorem 9. *Let $\widetilde{Q}$ be a soft quasilinear inner product space. Then $F_{\widetilde{w}}$ is a closed and bounded for every $\widetilde{w}\in\widetilde{Q}$.*

Proof. Let $(\widetilde{q}_n)\in F_{\widetilde{w}}$ and $\widetilde{q}_n\to\widetilde{q}\in\widetilde{Q}$ for every $n\to\infty$. So, we know that for every $\widetilde{\varepsilon}\widetilde{\succ}\widetilde{0}$ there exist a $n_0\in\mathbb{N}$ such that for every $n\geq n_0$, we have

$$\widetilde{q}_n\widetilde{\leq}\widetilde{q}+\widetilde{q}_{1n}^{\varepsilon},\ \widetilde{q}\widetilde{\leq}\widetilde{q}_n+\widetilde{q}_{2n}^{\varepsilon},\ \|\widetilde{q}_{in}^{\varepsilon}\|\widetilde{\leq}\widetilde{\varepsilon}^{1/2}.$$

Further, for every $n \in N$ we find $\widetilde{q}_n \widetilde{\leq} \widetilde{w}$ since $\widetilde{q}_n \in F_{\widetilde{w}}$. Since $\widetilde{Q}$ be a soft inner product quasilinear space, we get

$$\widetilde{q} \widetilde{\leq} \widetilde{q}_n \widetilde{\leq} \widetilde{w} \text{ for } \widetilde{q} \widetilde{\leq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n} \text{ and } \|\widetilde{q}^{\varepsilon}_{2n}\|^2 = \|\langle q^{\varepsilon}_{2n}, q^{\varepsilon}_{2n}\rangle\| \widetilde{\leq} \widetilde{\varepsilon}.$$

Now, we will show that $\widetilde{q} \in \widetilde{Q}_r$. If $\widetilde{q}_n \to \widetilde{q}$, then we have $-\widetilde{q}_n \to -\widetilde{q}$. So, for every $\widetilde{\varepsilon} \widetilde{\succ} \widetilde{0}$ there exist a $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have

$$\widetilde{q}_n \widetilde{\leq} \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n}\,, \widetilde{q} \widetilde{\leq} \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n}\,, \|\widetilde{q}^{\varepsilon}_{in}\| \widetilde{\leq} \frac{\varepsilon^{1/2}}{2}$$

$$-\widetilde{q}_n \widetilde{\leq} -\widetilde{q} + \widetilde{u}^{\varepsilon}_{1n}\,, -\widetilde{q} \widetilde{\leq} -\widetilde{q}_n + \widetilde{u}^{\varepsilon}_{2n}\,, \|\widetilde{u}^{\varepsilon}_{in}\| \widetilde{\leq} \frac{\varepsilon^{1/2}}{2}.$$

Moreover, since $\widetilde{q}_n \in \widetilde{Q}_r$, we obtain $\widetilde{q}_n - \widetilde{q}_n = \widetilde{0}$. Since $\widetilde{Q}$ is a soft quasilinear inner product space, we find

$$\widetilde{q}_n - \widetilde{q}_n \widetilde{\leq} \widetilde{q} - \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n} + \widetilde{u}^{\varepsilon}_{1n} \text{ and } \widetilde{q} - \widetilde{q} \widetilde{\leq} \widetilde{q}_n - \widetilde{q}_n + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{u}^{\varepsilon}_{2n}\,, \|\widetilde{q}^{\varepsilon}_{in} + \widetilde{u}^{\varepsilon}_{in}\| \widetilde{\leq} \varepsilon^{1/2}.$$

From Definition 9, we get

$$\widetilde{0} \widetilde{\leq} \widetilde{q} - \widetilde{q} + \widetilde{q}^{\varepsilon}_{1n} + \widetilde{u}^{\varepsilon}_{1n} \text{ and } \|\widetilde{q}^{\varepsilon}_{in} + \widetilde{u}^{\varepsilon}_{in}\|^2 = \|\langle \widetilde{q}^{\varepsilon}_{1n} + \widetilde{u}^{\varepsilon}_{1n}, \widetilde{q}^{\varepsilon}_{1n} + \widetilde{u}^{\varepsilon}_{1n}\rangle\|_{\Omega(\mathbb{R})} \widetilde{\leq} \varepsilon \Rightarrow \widetilde{0} \widetilde{\leq} \widetilde{q} - \widetilde{q}$$

and

$$\widetilde{q} - \widetilde{q} \widetilde{\leq} \widetilde{0} + \widetilde{q}^{\varepsilon}_{2n} + \widetilde{u}^{\varepsilon}_{2n} \text{ and } \|\widetilde{q}^{\varepsilon}_{2n} + \widetilde{u}^{\varepsilon}_{2n}\|^2 = \|\langle \widetilde{q}^{\varepsilon}_{2n} + \widetilde{u}^{\varepsilon}_{2n}, \widetilde{q}^{\varepsilon}_{2n} + \widetilde{u}^{\varepsilon}_{2n}\rangle\|_{\Omega(\mathbb{R})} \widetilde{\leq} \varepsilon \Rightarrow \widetilde{q} - \widetilde{q} \widetilde{\leq} \widetilde{0}.$$

So, we obtain $\widetilde{q} - \widetilde{q} = \widetilde{0}$. This gives $\widetilde{q} \in \widetilde{Q}_r$. Thus we get that $F_{\widetilde{w}}$ is closed.

Moreover, we know that $\widetilde{w} \widetilde{\leq} \widetilde{q}$ for every $\widetilde{w} \in F_{\widetilde{q}}$ and $\widetilde{w} \in \widetilde{Q}_r$. Since $\widetilde{Q}$ is a soft quasilinear inner product space, we get $\|\widetilde{w}\| \leq \|\widetilde{q}\|$. This gives us that $F_{\widetilde{q}}$ is bounded. □

Theorem 10. *In a soft quasilinear inner product space, the floor of any soft quasi vector may not be a subspace. But, the orthogonal complement set of the floor of a soft quasi vector is always a subspace of space.*

Proof. Let $\widetilde{Q}$ be a soft quasilinear inner product space. For a $\widetilde{q} \in \widetilde{Q}$, we have $F_{\widetilde{q}} = \left\{\widetilde{a} \in \widetilde{Q}_r : \widetilde{a} \widetilde{\leq} \widetilde{q}\right\}$. If $\widetilde{a}, \widetilde{b} \in F_{\widetilde{q}}$, then $\widetilde{a} \widetilde{\leq} \widetilde{q}$ and $\widetilde{b} \widetilde{\leq} \widetilde{q}$. Since $\widetilde{Q}$ be a soft quasilinear space, we get

$$\widetilde{\alpha} \cdot \widetilde{a} \widetilde{\leq} \widetilde{\alpha} \cdot \widetilde{q} \text{ and } \widetilde{\beta} \cdot \widetilde{b} \widetilde{\leq} \widetilde{\beta} \cdot \widetilde{q}$$

for soft scalars $\widetilde{\alpha}, \widetilde{\beta}$. Since $\widetilde{Q}$ is a soft quasilinear space, we get $\widetilde{\alpha} \cdot \widetilde{a} + \widetilde{\beta} \cdot \widetilde{b} \widetilde{\leq} \widetilde{\alpha} \cdot \widetilde{q} + \widetilde{\beta} \cdot \widetilde{q}$. But, because of the inequality $\widetilde{\alpha} \cdot \widetilde{a} + \widetilde{\beta} \cdot \widetilde{b} \widetilde{\leq} \widetilde{q}$ may not be satisfy for every soft scalars $\widetilde{\alpha}, \widetilde{\beta}$, we obtain $\widetilde{\alpha} \cdot \widetilde{a} + \widetilde{\beta} \cdot \widetilde{b} \widetilde{\nleq} \widetilde{q}$. So, the floor of a soft vector of a quasilinear space $\widetilde{Q}$ may not be subspace.

Now, we will show that $F^{\perp}_{\widetilde{q}}$ is a subspace of a soft quasilinear space $\widetilde{Q}$. Let $\widetilde{c}, \widetilde{d} \in F^{\perp}_{\widetilde{q}}$ and $\widetilde{z} \in F_{\widetilde{q}}$. For soft scalars $\widetilde{\alpha}, \widetilde{\beta}$, we find

$$\left\|\left\langle \widetilde{z}, \widetilde{\alpha} \cdot \widetilde{c} + \widetilde{\beta} \cdot \widetilde{d}\right\rangle\right\| \widetilde{\leq} \|\langle \widetilde{z}, \widetilde{\alpha} \cdot \widetilde{c}\rangle\| + \left\|\left\langle \widetilde{z}, \widetilde{\beta} \cdot \widetilde{d}\right\rangle\right\| = \widetilde{\alpha}\, \|\langle \widetilde{z}, \widetilde{c}\rangle\| + \widetilde{\beta} \left\|\left\langle \widetilde{z}, \widetilde{d}\right\rangle\right\| = \widetilde{0}.$$

This gives $\widetilde{\alpha} \cdot \widetilde{c} + \widetilde{\beta} \cdot \widetilde{d} \in F^{\perp}_{\widetilde{q}}$. So, we obtain $F_{\widetilde{q}}$ is a subspace of quasilinear space $\widetilde{Q}$ for every $\widetilde{q} \in \widetilde{Q}$. □

Example 3. Let $\widetilde{Q} = \widetilde{\Omega_C(\mathbb{R})}$ and we take a soft quasi vector $\widetilde{q} \in \widetilde{Q}$ such that $\widetilde{q}(\lambda) = [1, 3]$ for every parameter λ. If $\widetilde{q}_1(\lambda) = \{1\}$ and $\widetilde{q}_2(\lambda) = \{3\}$ then we find $\widetilde{q}_1(\lambda) + \widetilde{q}_2(\lambda) = \{4\}$ for $\widetilde{q}_1, \widetilde{q}_2 \in F_{\widetilde{q}}$ and λ parameter. But $\{4\} \notin F_{\widetilde{q}(\lambda)}$. So, $F_{\widetilde{q}}$ is a not subspace of $\widetilde{Q}$.

4. Conclusions and Future Works

In our study, a special type of soft quasilinear spaces, which are homogenized soft quasilinear spaces have been introduced. Moreover, some related properties and examples of homogenized soft quasilinear spaces have been given. Lastly, related theorems including floor of soft quasilinear theory and many conclusions are researched. Some algebraic properties of soft normed quasilinear spaces such as basis, dimensions and properness will be studied in further investigations depending on the descriptions of homogenized soft quasilinear space given in this research.

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