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Sun's six conjectures on Apéry-like sums involving ordinary harmonic numbers

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Abstract: The main goal of this brief article is to provide an elementary proof of Sun's six conjectures on Apéry-like sums involving ordinary harmonic numbers.

Keywords: Riemann zeta function; Catalans constant; Apéry-like sums; Harmonic numbers; Polylogarithm function; Polygamma function; Logarithmic integrals; Clausen function.

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1. Introduction

Significant progress has been made on the calculations of the ε-expansion of multiloop Feynman diagram in the past quarter of a century. It turns out that special values of multiple polylogarithms have played indispensable roles in these computations.

Many experimental work emerged around the beginning of this century, for instance, see [1–4], in which a special class of series emerges. These infinite sums are often called *Apéry-type series* (or *Apéry-like sums*) because the simplest cases were used by Apéry in his celebrated proof of irrationality of $\zeta(3)$ in 1979. In particular, he showed that

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

In his new book [5], Zhi-Wei Sun listed 820 mathematical conjectures, including several concerning the Apéry-like sums involving ordinary and hyper-harmonic numbers, some of which had appeared in his previous paper [6].

The main aim of this breif article is develop an elementary aproach to prove the following six conjectures of Sun [6, Conjecture 3.1 and 3.2]. Our approach throughout is completely analytic and the proofs rely on simple yet interesting logarithmic integral evaluations.

Conjecture 1 ([6, Conjecture 3.1 and 3.2]). *The following identities hold*

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}\left(H_{2n}+\frac{2}{3n}\right) = \zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}\left(H_{2n}+2H_n\right) = \frac{5}{3}\zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{H_{2n}+17H_n}{n^2\binom{2n}{n}} = \frac{5\pi}{6\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right)-\psi^{(1)}\left(\frac{2}{3}\right)\right),$$

$$\sum_{n=1}^{\infty}\frac{2^n}{n^2\binom{2n}{n}}\left(2H_{2n}-3H_n+\frac{2}{n}\right) = \frac{7}{4}\zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{2^n}{n^2\binom{2n}{n}}\left(6H_{2n}-11H_n+\frac{8}{n}\right) = 2\pi G,$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2\binom{2n}{n}} \left(7H_n - 2H_{2n} - \frac{2}{n}\right) = \frac{\pi^2}{2} \log 2,$$

where $H_n = \sum_{k=1}^{n} k^{-1}$ denotes the classical harmonic number, $G = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-2}$ denotes the Catalans constant, $\psi^{(1)}(z) = \sum_{n=0}^{\infty} (z+n)^{-2}$ represents the trigamma function, and $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ represents the Riemann zeta function.

Our main result in this manuscript is as follows:

Theorem 2. *The following identities hold*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2\binom{2n}{n}} = -\frac{1}{9}\zeta(3) + \frac{\pi}{18\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right), \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{2^n H_n}{n^2\binom{2n}{n}} = \frac{7}{16}\zeta(3) + \frac{3}{4}\zeta(2)\log 2, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{2^n H_{2n}}{n^2\binom{2n}{n}} = \frac{119}{32}\zeta(3) + \frac{3}{8}\zeta(2)\log 2 - \pi G, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{H_{2n}}{n^2\binom{2n}{n}} = \frac{17}{9}\zeta(3) - \frac{\pi}{9\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right), \tag{4}$$

Remark 1. Note that similar Apéry-type series evaluations were provided by Chu in [7,8]. We provide a new and elementary approach to evaluate these type of sums.

Proof of Theorem 2 is provided in Section 2. Combining Theorem 2 with the remarkable identities proved by I. Zucker [9, Eqn. 2.9 and 2.10] in 1985, that is

$$\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} = \frac{\pi}{6\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right),$$

and

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3\binom{2n}{n}} = \frac{\pi^2}{8}\log 2 + \pi G - \frac{35}{16}\zeta(3),$$

completes the proof of Conjecture 1.

Beyond this specific case, we hope that the tools used in this study will be useful to the readers in their endeavour of evaluating more interesting Apéry-like series and integrals, and they will be an integral tool in proving remaining unsolved similar conjectures of Sun.

2. Proof of Theorem 2

2.1. Proof of identity 1

Start with the following series expansion:

$$\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{n\binom{2n}{n}} z^{2n-1}, \quad |z| < 1. \tag{5}$$

Some routine manipulations produce

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2\binom{2n}{n}} = -4\int_0^1 z \log\left(1 - 4\sin^2 z\right) dz.$$

Next, we replace $z \mapsto \arctan z$ to get

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2\binom{2n}{n}} = 4\left(\int_0^{\frac{1}{\sqrt{3}}} \frac{\arctan(z)\log\left(1+z^2\right)}{1+z^2}\,\mathrm{d}z - \int_0^{\frac{1}{\sqrt{3}}} \frac{\arctan(z)\log\left(1-3z^2\right)}{1+z^2}\,\mathrm{d}z\right)$$
$$= 4\left(\mathfrak{T}_1 - \mathfrak{T}_2\right). \tag{6}$$

The change of variable $z = \tan x$ produces

$$\mathfrak{T}_1 = -2\int_0^{\pi/6} x\log\left(\cos(x)\right)\mathrm{d}x$$
$$= \frac{11}{72}\zeta(3) + \frac{1}{6}\log(2)\,\zeta(2) - \frac{\pi}{36\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right), \tag{7}$$

where we have used the Fourier series expansion of $\log(\cos x)$. To evaluate $\mathfrak{T}_2$, let

$$\mathfrak{F}(\omega) = \int_0^{\omega} \frac{\arctan(z)\log\left(1-\omega^{-2}z^2\right)}{1+z^2}\,\mathrm{d}z, \quad \omega \in \mathbb{R},$$

and observe that $\mathfrak{T}_2 = \mathfrak{F}(1/\sqrt{3})$.

For the reader's convenience, we will use and restate computation of $\mathfrak{F}(\omega)$ from [10]:

$$\mathfrak{F}(\omega) = \frac{1}{8}\zeta(3) - \frac{1}{8}Cl_3(4\alpha) - \frac{1}{2}\alpha^2\log\left(\sin^2\alpha\right) - \alpha\,\mathrm{Cl}_2(2\alpha), \tag{8}$$

where $\alpha = \arctan\omega$, $Cl_n(\vartheta)$ represents the Clausen function of order n, defined by the Fourier series

$$Cl_{2m}(\vartheta) = \sum_{k=1}^{\infty} \frac{\sin(k\vartheta)}{k^{2m}}, \quad Cl_{2m+1}(\vartheta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2m+1}}.$$

The change of variable $\arctan z = \tau$ with $\omega = \tan\alpha$ produces

$$\begin{aligned}
\mathfrak{F}(\omega) &= \int_0^{\omega} \frac{\arctan(z)\log\left(1-\omega^{-2}z^2\right)}{1+z^2}\,\mathrm{d}z \\
&= \int_0^{\tan\alpha} \frac{\arctan(z)\log\left(1-z^2\cot^2(\alpha)\right)}{1+z^2}\,\mathrm{d}z \\
&= \int_0^{\alpha} \tau\log\left(1-\cot^2(\alpha)\tan^2(\tau)\right)\mathrm{d}\tau \\
&= \int_0^{\alpha} \tau\log\left(\frac{\sin^2(\alpha)\cos^2(\tau)-\cos^2(\alpha)\sin^2(\tau)}{\sin^2(\alpha)\cos^2(\tau)}\right)\mathrm{d}\tau \\
&= \int_0^{\alpha} \tau\log\left(\csc^2(\alpha)\sec^2(\tau)\sin(\alpha-\tau)\sin(\alpha+\tau)\right)\mathrm{d}\tau \\
&= -\int_0^{\alpha} \tau\left(\log\left(\sin^2\alpha\right)+\log\left(\cos^2\tau\right)-\log\left(\sin(\alpha-\tau)\right)-\log\left(\sin(\alpha+\tau)\right)\right)\mathrm{d}\tau \\
&= -\int_0^{\alpha} \tau\log\left(\sin^2\alpha\right)\mathrm{d}\tau - \int_0^{\alpha} \tau\left(2\log(\cos\tau)-\log(\sin(\alpha-\tau))-\log(\sin(\alpha+\tau))\right)\mathrm{d}\tau \\
&= -\frac{1}{2}\alpha^2\log\left(\sin^2\alpha\right) - \int_0^{\alpha}\int_0^{\tau} \left(2\log(\cos\tau)-\log(\sin(\alpha-\tau))-\log(\sin(\alpha+\tau))\right)\mathrm{d}\vartheta\,\mathrm{d}\tau \\
&= -\frac{1}{2}\alpha^2\log\left(\sin^2\alpha\right) - \int_0^{\alpha}\int_{\vartheta}^{\alpha} \left(2\log(\cos\tau)-\log(\sin(\alpha-\tau))-\log(\sin(\alpha+\tau))\right)\mathrm{d}\tau\,\mathrm{d}\vartheta \\
&= -\frac{1}{2}\alpha^2\log\left(\sin^2\alpha\right) - \mathfrak{T}_3.
\end{aligned}$$

Next, we find that

$$\begin{aligned}
\mathfrak{T}_3 &= \int_0^{\alpha}\int_{\vartheta}^{\alpha}\left(2\log\left(2\cos\tau\right)-\log\left(2\sin\left(\alpha-\tau\right)\right)-\log\left(2\sin\left(\alpha+\tau\right)\right)\right)\mathrm{d}\tau\,\mathrm{d}\vartheta\\
&= \int_0^{\alpha}\left[2\int_{\vartheta}^{\alpha}\log\left(2\cos\tau\right)\mathrm{d}\tau-\int_{\vartheta}^{\alpha}\log\left(2\sin\left(\alpha-\tau\right)\right)\mathrm{d}\tau-\int_{\vartheta}^{\alpha}\log\left(2\sin\left(\alpha+\tau\right)\right)\mathrm{d}\tau\right]\mathrm{d}\vartheta\\
&= \int_0^{\alpha}\left[2\int_{\pi/2-\alpha}^{\pi/2-\vartheta}\log\left(2\sin\tau\right)\mathrm{d}\tau-\int_{\vartheta}^{\alpha-\vartheta}\log\left(2\sin\tau\right)\mathrm{d}\tau-\int_{\alpha+\vartheta}^{2\alpha}\log\left(2\sin\tau\right)\mathrm{d}\tau\right]\mathrm{d}\vartheta\\
&= \frac{1}{2}\int_0^{\alpha}\left[2\int_{\pi-2\alpha}^{\pi-2\vartheta}\log\left(2\sin\frac{\tau}{2}\right)\mathrm{d}\tau-\int_{0}^{2\alpha-2\vartheta}\log\left(2\sin\frac{\tau}{2}\right)\mathrm{d}\tau-\int_{2\alpha+2\vartheta}^{4\alpha}\log\left(2\sin\frac{\tau}{2}\right)\mathrm{d}\tau\right]\mathrm{d}\vartheta\\
&= \frac{1}{2}\int_0^{\alpha}\left[\mathrm{Cl}_2\left(2\alpha-2\vartheta\right)+\mathrm{Cl}_2\left(4\alpha\right)-\mathrm{Cl}_2\left(2\alpha+2\vartheta\right)+2\,\mathrm{Cl}_2\left(\pi-2\alpha\right)-2\,\mathrm{Cl}_2\left(\pi-2\vartheta\right)\right]\mathrm{d}\vartheta\\
&= \alpha\,\mathrm{Cl}_2\left(2\alpha\right)+\frac{1}{2}\int_0^{\alpha}\mathrm{Cl}_2\left(2\alpha-2\vartheta\right)\mathrm{d}\vartheta-\frac{1}{2}\int_0^{\alpha}\mathrm{Cl}_2\left(2\alpha+2\vartheta\right)\mathrm{d}\vartheta-\int_0^{\alpha}\mathrm{Cl}_2\left(\pi-2\vartheta\right)\mathrm{d}\vartheta,
\end{aligned}$$

where we have used the integral representation of Clausen function of order 2, that is

$$\mathrm{Cl}_2\left(\vartheta\right)=-\int_0^{\vartheta}\log\left(\left|2\sin\left(\frac{\tau}{2}\right)\right|\right)\mathrm{d}\tau.$$

Replacing $\vartheta\mapsto\frac{\vartheta}{2}$ produces

$$\begin{aligned}
&\frac{1}{2}\int_0^{\alpha}\mathrm{Cl}_2\left(2\alpha-2\vartheta\right)\mathrm{d}\vartheta-\frac{1}{2}\int_0^{\alpha}\mathrm{Cl}_2\left(2\alpha+2\vartheta\right)\mathrm{d}\vartheta-\int_0^{\alpha}\mathrm{Cl}_2\left(\pi-2\vartheta\right)\mathrm{d}\vartheta\\
&= \frac{1}{4}\int_0^{2\alpha}\mathrm{Cl}_2\left(2\alpha-\vartheta\right)\mathrm{d}\vartheta-\frac{1}{4}\int_0^{2\alpha}\mathrm{Cl}_2\left(2\alpha+\vartheta\right)\mathrm{d}\vartheta-\frac{1}{2}\int_0^{2\alpha}\mathrm{Cl}_2\left(\pi-\vartheta\right)\mathrm{d}\vartheta\\
&= \frac{1}{4}\int_0^{2\alpha}\mathrm{Cl}_2\left(\vartheta\right)\mathrm{d}\vartheta-\frac{1}{4}\int_{2\alpha}^{4\alpha}\mathrm{Cl}_2\left(\vartheta\right)\mathrm{d}\vartheta-\frac{1}{2}\int_{\pi-2\alpha}^{\pi}\mathrm{Cl}_2\left(\vartheta\right)\mathrm{d}\vartheta\\
&= \frac{1}{4}\mathrm{Cl}_3\left(0\right)-\frac{1}{4}\mathrm{Cl}_3\left(2\alpha\right)+\frac{1}{4}\mathrm{Cl}_3\left(4\alpha\right)-\frac{1}{4}\mathrm{Cl}_3\left(2\alpha\right)+\frac{1}{2}\mathrm{Cl}_3\left(\pi\right)-\frac{1}{2}\mathrm{Cl}_3\left(\pi-2\alpha\right)\\
&= \frac{1}{4}\mathrm{Cl}_3\left(0\right)+\frac{1}{2}\mathrm{Cl}_3\left(\pi\right)+\frac{1}{4}\mathrm{Cl}_3\left(4\alpha\right)-\frac{1}{2}\mathrm{Cl}_3\left(2\alpha\right)-\frac{1}{2}\mathrm{Cl}_3\left(\pi-2\alpha\right)=\frac{1}{8}\mathrm{Cl}_3\left(4\alpha\right)-\frac{1}{8}\zeta(3).
\end{aligned}$$

Putting all things together produces identity 8. Plugging in $\omega=\frac{1}{\sqrt{3}}$ in identity 8 produces

$$\alpha=\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6},$$

and thus

$$\mathfrak{T}_2=\frac{1}{8}\zeta(3)+\frac{1}{6}\zeta(2)\log 2-\frac{\pi}{6}Cl_2\left(\frac{\pi}{3}\right)-\frac{1}{8}Cl_3\left(\frac{2\pi}{3}\right) \tag{9}$$

$$=\frac{13}{72}\zeta(3)+\frac{1}{6}\zeta(2)\log 2-\frac{\pi}{24\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right)-\psi^{(1)}\left(\frac{2}{3}\right)\right), \tag{10}$$

where we have simply used the fact that

$$Cl_2\left(\frac{\pi}{3}\right)=\frac{1}{4\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right)-\psi^{(1)}\left(\frac{2}{3}\right)\right).$$

Combining equations (6), (7), and (10) finally gives us the desired result.

2.2. Proof of identity 2

Using the series expansion 5, we find that

$$\sum_{n=1}^{\infty}\frac{2^n H_n}{n^2\binom{2n}{n}}=-4\int_0^{\frac{1}{\sqrt{2}}}\frac{\log\left(1-2z^2\right)\arcsin\left(z\right)}{\sqrt{1-z^2}}\mathrm{d}z.$$

The change of variable $z = \sin z$ produces

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin(z)\log(1-2z^2)}{\sqrt{1-z^2}}\,\mathrm{d}z = \int_0^{\pi/4} z\log\left(1-2\sin^2(z)\right)\mathrm{d}z = \int_0^{\pi/4} z\log(\cos(2z))\,\mathrm{d}z.$$

Therefore we have

$$-4\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin(z)\log(1-2z^2)}{\sqrt{1-z^2}}\,\mathrm{d}z = -\int_0^{\pi/2} z\log(\cos z)\,\mathrm{d}z.$$

Next, we use the Fourier series expansion of $\log(\cos z)$ to finally deduce that

$$\int_0^{\pi/2} z\log(\cos z)\,\mathrm{d}z = -\frac{7}{16}\zeta(3) - \frac{3}{4}\zeta(2)\log 2.$$

Putting all things together gives us the desired result.

2.3. Proof of identity 3

Following similar steps as above gives

$$\sum_{n=1}^{\infty} \frac{2^n H_{2n}}{n^2\binom{2n}{n}} = -4\int_0^{\pi/4} z\log\left(1-\sqrt{2}\sin z\right)\mathrm{d}z.$$

Now some routine trigonometric substitutions and simple manipulations yield

$$\begin{aligned}
&\int_0^{\pi/4} z\log\left(1-\sqrt{2}\sin z\right)\mathrm{d}z\\
&= \int_0^{\pi/4} z\log(\cos 2z)\,\mathrm{d}z - \int_0^{\pi/4} z\log\left(1+\sqrt{2}\sin z\right)\mathrm{d}z\\
&= \frac{1}{2}\int_0^{\pi/4} z\log(\cos 2z)\,\mathrm{d}z + \frac{1}{2}\int_{-\pi/4}^{\pi/4} z\log\left(1-\sqrt{2}\sin z\right)\mathrm{d}z\\
&= \int_0^{\pi/4}\left[\left(\frac{\pi}{4}-2z\right)\log\left(2\sqrt{2}\right) + \frac{z}{2}\log(\cos 2z) - \frac{1}{4}\pi\log(\cot z) - 2z\log(\tan z)\right]\mathrm{d}z\\
&= \frac{1}{2}\int_0^{\pi/4} z\log(\cos 2z)\,\mathrm{d}z - \frac{\pi}{4}\int_0^{\pi/4}\log(\cot z)\,\mathrm{d}z - 2\int_0^{\pi/4} z\log(\tan z)\,\mathrm{d}z\\
&= \frac{1}{8}\int_0^{\pi/2} z\log(\cos z)\,\mathrm{d}z - \frac{\pi}{4}\int_0^{\pi/4}\log(\cot z)\,\mathrm{d}z - 2\int_0^{\pi/4} z\log(\tan z)\,\mathrm{d}z\\
&= -\frac{7}{128}\zeta(3) - \frac{7}{8}\zeta(3) - \frac{3}{32}\zeta(2)\log 2 + \frac{\pi G}{2} - \frac{\pi G}{4} = -\frac{119}{128}\zeta(3) - \frac{3}{32}\zeta(2)\log 2 + \frac{\pi G}{4},
\end{aligned}$$

where we have simply used the fact that

$$\int_0^{\pi/4} z\log(\tan z)\,\mathrm{d}z = \frac{7}{16}\zeta(3) - \frac{\pi G}{4}, \quad \int_0^{\pi/2} z\log(\cos z)\,\mathrm{d}z = -\frac{7}{16}\zeta(3) - \frac{3}{4}\zeta(2)\log 2,$$

and the integral representation of Catalan's constant, that is

$$G = \int_0^{\pi/4}\log(\cot z)\,\mathrm{d}z.$$

Therefore, putting all things together gives us the desired result.

2.4. Proof of identity 4

Using the series exansion 5, we get

$$\sum_{n=1}^{\infty} \frac{H_{2n}}{n^2\binom{2n}{n}} = -4\int_0^{\pi/6} z\log(1-2\sin(z))\,\mathrm{d}z.$$

Now following the same steps as we did above for a similar integral yields

$$\int_0^{\pi/6} z\log(1-2\sin z)\,\mathrm{d}z = \frac{17}{36}\zeta(3) - \frac{\pi}{36\sqrt{3}}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right).$$

Combining all things together gives us the desired result.

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