Open Journal of Mathematical Sciences

PSR Press

Article

Values of zeta-one functions at positive even integers

Masato Kobayashi[1,*] and Shunji Sasaki[2]

1 Department of Engineering Kanagawa University, 3-27-1 Rokkaku-bashi, Yokohama 221-8686, Japan.
2 Kawaguchi public Kamiaoki junior high school 3-9-1 Kamiaoki-Nishi, Kawaguchi 333-0845, Japan.
* Correspondence: masato210@gmail.com

Academic Editor: Zahid Raza
Received: 28 May 2022; Accepted: 12 January 2023; Published: 31 March 2023.

Abstract: Motivated by Euler-Goldbach and Shallit-Zikan theorems, we introduce zeta-one functions with infinite sums of $n^s \pm 1$ as an analogy of the Riemann zeta function. Then we compute values of these functions at positive even integers by the residue theorem.

Keywords: Euler-Goldbach theorem; Infinite series; Residue theorem; Riemann zeta function.

MSC: 11M06; 30B10.

---

1. Introduction: Euler-Goldbach Theorem

Let us start with the celebrated *Euler-Goldbach Theorem*. Say that a natural number p is a *perfect power* if $p = n^m$ for some natural numbers $m, n \geq 2$.

Theorem 1 (Euler-Goldbach)**.**

$$\sum_{p:\text{perfect power}} \frac{1}{p-1} = 1.$$

See Bibiloni-Paradis-Viader [1] for history of this theorem. Recently, Shallit-Zikan [2] (1983) reinterpreted it in terms of *Riemann's zeta function*: the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

is convergent for all complex numbers s such that $\mathrm{Re}(s) > 1$. Indeed, Euler proved that

$$\zeta(2k) = -\frac{1}{2}\frac{(2\pi i)^{2k}}{(2k)!}B_{2k},$$

where $\{B_n\}$ are signed Bernoulli numbers; refer to Ayoub [3] for history of this function. Since $\zeta(s) = 1 + \frac{1}{2^s} + \cdots > 1$ and

$$2 > \frac{\pi^2}{6} = \zeta(2) > \zeta(3) > \zeta(4) > \zeta(5) > \cdots,$$

we have $1 < \zeta(s) < 2$ for all $s \geq 2$. That is, $\zeta(s) - 1$ is the *fractional part* of $\zeta(s)$. For example,

$$\begin{aligned}
\zeta(2) - 1 &= 0.6449\ldots, \\
\zeta(3) - 1 &= 0.2020\ldots, \\
\zeta(4) - 1 &= 0.0823\ldots, \\
\zeta(5) - 1 &= 0.0369\ldots.
\end{aligned}$$

Theorem 2 (Shallit-Zikan [2])**.**

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1.$$

Let us now see a similar result.

Theorem 3 (See J.M. Borwein-Bradley-Crandall [4, p. 262])**.**

$$\sum_{k=1}^{\infty}(\zeta(2k)-1)=\frac{3}{4}.$$

Here we give a proof since it suggests some ideas for our main results.

Proof. Consider the double sequence $a_{nk}=\left(\frac{1}{n^{2k}}\right)_{n\geq 2,k\geq 1}$ and positive series $\sum_{n\geq 2,k\geq 1} a_{nk}$. We find that

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^{2k}} &= \sum_{n=2}^{\infty}\frac{1}{n^2}\sum_{k=0}^{\infty}\left(\frac{1}{n^2}\right)^k \\ &= \sum_{n=2}^{\infty}\frac{1}{n^2}\frac{1}{1-\frac{1}{n^2}}=\sum_{n=2}^{\infty}\frac{1}{n^2-1} \\ &= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)=\frac{3}{4},\end{aligned}$$

so that we can freely switch order of this series. As a consequence,

$$\sum_{k=1}^{\infty}(\zeta(2k)-1)=\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n^{2k}}=\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^{2k}}=\frac{3}{4}.$$

□

In this proof, the infinite series $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ appeared. It is now natural to think of analogous sums $\sum\frac{1}{n^s\pm 1}$. With this simple idea, this article introduces *zeta-one functions* $\zeta_{+1}(s)$, $\zeta_{-1}(s)$ and we compute values of $\zeta_{+1}(2m)$ and $\zeta_{-1}(2m)$ as main Theorems 4 and 5.

Theorem 4.

$$\zeta_{+1}(2m)=-\frac{1}{2}+\frac{1}{2m}\sum_{k=1}^{m}\pi\alpha^{2k-1}\cot(\pi\alpha^{2k-1}).$$

Theorem 5.

$$\zeta_{-1}(2m)=\frac{1}{2}+\frac{2m-1}{4m}-\frac{\pi}{4m}\sum_{1\leq k\leq 2m-1,k\neq m}\beta^k\cot(\pi\beta^k).$$

2. Zeta-one functions

Throughout k,m,n,N and s each denote a nonnegative integer unless otherwise specified. Further, we assume that $s\geq 2$.

Definition 1. Define the *zeta-one functions* by

$$\zeta_{+1}(s)=\sum_{n=1}^{\infty}\frac{1}{n^s+1}\quad\text{and}\quad\zeta_{-1}(s)=\sum_{n=2}^{\infty}\frac{1}{n^s-1}.$$

(For $s\geq 2$, these sums are indeed convergent as mentioned below). Call each *zeta-plus-one* and *zeta-minus-one* function, respectively.

Example 1. As seen above, $\zeta_{-1}(2)=\frac{3}{4}$. Moreover, since

$$\coth(\pi z)=\frac{1}{\pi z}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2z}{z^2+n^2},\quad z\in\mathcal{C},$$

the substitution $z = 1$ implies that

$$\zeta_{+1}(2) = \sum_{n=1}^{\infty} \frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2}\coth(\pi).$$

As a consequence,

$$\zeta_{+1}(s) \leq \zeta_{+1}(2) < \infty, \ \ \zeta_{-1}(s) \leq \zeta_{-1}(2) < \infty$$

for all $s \geq 2$.

3. Proof of Theorem 4

Toward the proof of Theorem 4 on $\zeta_{+1}(2m)$, we need lemmas.

For $m \geq 1$, set

$$f(z) = \frac{\cot(\pi z)}{z^{2m}+1} \ \text{ and } \ \alpha = \exp\left(\frac{\pi i}{2m}\right).$$

Recall that

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \ \ z \in \mathcal{C}.$$

Thus, f is a meromorphic function with simple poles $z = 0, \pm 1, \pm 2, \pm 3, \cdots$ and

$$\alpha, \alpha^3, \alpha^5, \cdots, \alpha^{2m-1}, -\alpha, -\alpha^3, -\alpha^5, \cdots, -\alpha^{2m-1}$$

as all the roots of $z^{2m} + 1 = 0$. Let us compute the residue of f at each pole.

Lemma 1. *For $n = 0, \pm 1, \pm 2, \pm 3, \cdots$, we have*

$$Res(f, n) = \frac{1}{\pi(n^{2m}+1)},$$

and for $1 \leq k \leq m$,

$$Res(f, \pm\alpha^{2k-1}) = -\frac{\alpha^{2k-1}\cot(\pi\alpha^{2k-1})}{2m}.$$

Proof. First, we have

$$\text{Res}(f, n) = \lim_{z\to n}(z-n)f(z) = \lim_{z\to n} \frac{z-n}{\sin\pi(z-n)} \cdot \frac{\cos(\pi z)}{z^{2m}+1} = \frac{1}{\pi(n^{2m}+1)}.$$

Second, for $1 \leq k \leq m$,

$$\begin{aligned}
\text{Res}(f, \alpha^{2k-1}) &= \lim_{z\to\alpha^{2k-1}} (z-\alpha^{2k-1})f(z) \\
&= \lim_{z\to\alpha^{2k-1}} (z-\alpha^{2k-1})\frac{\cot(\pi z)}{z^{2m}+1} \\
&= \lim_{z\to\alpha^{2k-1}} \cot(\pi z) \lim_{z\to\alpha^{2k-1}} \frac{z-\alpha^{2k-1}}{z^{2m}+1} \\
&= \cot(\pi\alpha^{2k-1}) \lim_{z\to\alpha^{2k-1}} \frac{1}{2mz^{2m-1}} \ \ \text{(L'Hôpital's rule)} \\
&= \cot(\pi\alpha^{2k-1}) \frac{1}{2m(\alpha^{2k-1})^{2m-1}} \\
&= -\frac{\alpha^{2k-1}\cot(\pi\alpha^{2k-1})}{2m} \ \ \left((\alpha^{2k-1})^{2m} = -1\right).
\end{aligned}$$

It is quite similar to show that

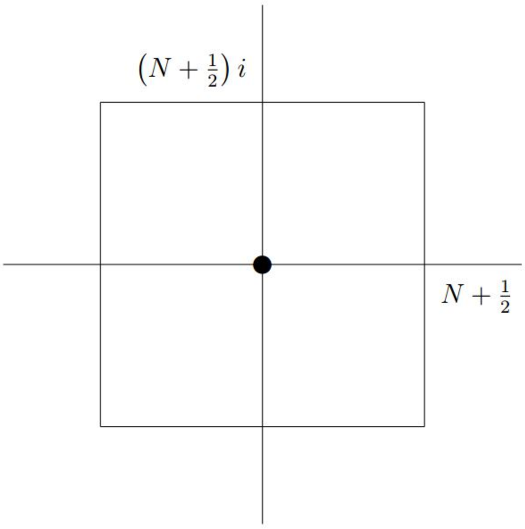
$$\text{Res}(f, -\alpha^{2k-1}) = -\frac{\alpha^{2k-1}\cot(\pi\alpha^{2k-1})}{2m}.$$

□

Lemma 2. *For a positive integer N, consider line segments on the complex plane*

$$C_1(N) = \left\{\left(N+\frac{1}{2}\right)+yi\right\} - \left(N+\frac{1}{2}\right) \le y \le N+\frac{1}{2},$$
$$C_2(N) = \left\{x+\left(N+\frac{1}{2}\right)i\right\} - \left(N+\frac{1}{2}\right) \le x \le N+\frac{1}{2},$$
$$C_3(N) = \left\{-\left(N+\frac{1}{2}\right)+yi\right\} - \left(N+\frac{1}{2}\right) \le y \le N+\frac{1}{2},$$
$$C_4(N) = \left\{x-\left(N+\frac{1}{2}\right)i\right\} - \left(N+\frac{1}{2}\right) \le x \le N+\frac{1}{2},$$

and set $C(N) = C_1(N) \cup C_2(N) \cup C_3(N) \cup C_4(N)$.



1. *If $z \in C(N)$, then $|\cot(\pi z)| \le \coth \frac{3}{2}\pi$.*
2. *If $z \in C(N)$, then*
$$\left|\frac{1}{z^{2m}+1}\right| \le \frac{1}{\left(N+\frac{1}{2}\right)^{2m}-1}.$$

Proof. 1. Suppose $z \in C(N)$. If $z \in C_1(N)$, then write

$$z = \left(N+\frac{1}{2}\right)+yi, \ -\left(N+\frac{1}{2}\right) \le y \le N+\frac{1}{2}.$$

$$\begin{aligned}|\cot \pi z| &= \frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}} = |\frac{e^{-\pi y}e^{\left(N+\frac{1}{2}\right)\pi i}+e^{\pi iy}e^{-\left(N+\frac{1}{2}\right)\pi i}}{e^{-\pi y}e^{\left(N+\frac{1}{2}\right)\pi i}-e^{\pi iy}e^{-\left(N+\frac{1}{2}\right)\pi i}}| \\ &= |\frac{e^{-\pi y}(-1)^N i+e^{\pi iy}(-1)^N(-i)}{e^{-\pi y}(-1)^N i-e^{\pi iy}(-1)^N(-i)}| \\ &= |\frac{e^{-\pi y}-e^{\pi iy}}{e^{-\pi y}+e^{\pi iy}}| = |\frac{e^{\pi iy}-e^{-\pi y}}{e^{\pi iy}+e^{-\pi y}}| \\ &= |\tanh(y)| \le 1 < \coth\frac{3}{2}\pi \ (= 1.00016\cdots).\end{aligned}$$

If $z \in C_2(N)$, then

$$z = x+\left(N+\frac{1}{2}\right)i, \ -\left(N+\frac{1}{2}\right) \le x \le N+\frac{1}{2},$$

and

$$\begin{aligned}|\cot(\pi z)| &= \left|\frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}}\right| \\ &\leq \frac{|e^{\pi i z}|+|e^{-\pi i z}|}{|e^{\pi i z}|-|e^{-\pi i z}|} \\ &= \frac{|e^{-\pi(N+1/2)}|+|e^{\pi(N+1/2)}|}{|e^{-\pi(N+1/2)}|-|e^{\pi(N+1/2)}|} \\ &= \frac{e^{\pi i(N+\frac{1}{2})}+e^{-\pi(N+\frac{1}{2})}}{e^{\pi i(N+\frac{1}{2})}-e^{-\pi(N+\frac{1}{2})}} \\ &= \coth\left(N+\frac{1}{2}\right)\pi \leq \coth\frac{3}{2}\pi,\end{aligned}$$

since $t \mapsto \coth(t)$ is decreasing for $t > 0$. For $z \in C_3(N) \cup C_4(N)$, we have $-z \in C_1(N) \cup C_2(N)$ so that

$$|\cot(\pi z)| = |-\cot(\pi(-z))| = |\cot(\pi(-z))| \leq \coth\frac{3}{2}\pi.$$

2. If $z \in C(N)$, then $|z| \geq N + \frac{1}{2}$. Consequently, $|z|^{2m} - 1 \geq \left(N+\frac{1}{2}\right)^{2m} - 1$,

$$\frac{1}{|z|^{2m}-1} \leq \frac{1}{\left(N+\frac{1}{2}\right)^{2m}-1},$$

$\frac{1}{|z^{2m}+1|} \leq \frac{1}{|z|^{2m}-1} \leq \frac{1}{\left(N+\frac{1}{2}\right)^{2m}-1}$.

□

Proof of Theorem 4. Let s be a positive even integer, say $s = 2m$, $m \geq 1$. Further, let $\alpha = \alpha_{2m} = \exp\left(\frac{\pi i}{2m}\right)$ for convenience. View $C(N) = C_1(N) + C_2(N) + C_3(N) + C_4(N)$ above as the sum of four paths with counterclockwise orientation. Notice that any pole of f does not lie on $C(N)$. We are going to compute the integral

$$I_N = \int_{C(N)} f(z)dz.$$

Let $D(N)$ be the domain enclosed by $C(N)$. Then, the residue theorem with Lemma 1 claims that

$$\begin{aligned}I_N &= \int_{C(N)} f(z)dz \\ &= 2\pi i \sum_{a:\text{pole of} f(z)|a\in D(N)} \operatorname{Res}(f,a) \\ &= 2\pi i\left(\operatorname{Res}(f,0)+\sum_{n=1}^{N}(\operatorname{Res}(f,n)+\operatorname{Res}(f,-n))+\sum_{k=1}^{m}\left(\operatorname{Res}(f,\alpha^{2k-1})+\operatorname{Res}(f,-\alpha^{2k-1})\right)\right) \\ &= 2\pi i\left(\frac{1}{\pi}+2\sum_{n=1}^{N}\frac{1}{\pi(n^{2m}+1)}+2\left(-\sum_{k=1}^{m}\frac{\alpha^{2k-1}\cot(\pi\alpha^{2k-1})}{2m}\right)\right),\end{aligned}$$

while Lemma 2 implies

$$\begin{aligned}|I_N| &= \left|\int_{C(N)} f(z)dz\right| \leq \int_{C(N)} |f(z)|dz \leq \frac{\coth\frac{3}{2}\pi}{\left(N+\frac{1}{2}\right)^{2m}-1}\int_{C(N)} dz \\ &= \frac{\coth\frac{3}{2}\pi}{\left(N+\frac{1}{2}\right)^{2m}-1}\cdot 8\left(N+\frac{1}{2}\right) \to 0 \ \ (N\to\infty).\end{aligned}$$

Therefore, taking the limit $N \to \infty$ for I_N yields

$$0 = 2\pi i \left(\frac{1}{\pi} + \frac{2}{\pi}\zeta_{+1}(2m) - 2\sum_{k=1}^{m} \frac{\alpha^{2k-1}\cot(\pi\alpha^{2k-1})}{2m} \right).$$

Conclude that

$$\zeta_{+1}(2m) = -\frac{1}{2} + \frac{1}{2m}\sum_{k=1}^{m} \pi\alpha^{2k-1}\cot(\pi\alpha^{2k-1}).$$

□

Of course, $\zeta_{+1}(2m)$ is a real number so that there should be some expression of $\zeta_{+1}(2m)$ in terms of only real trigonometric functions.

Example 2. Let $s = 4, m = 2$ and $\alpha = \alpha_4 = \exp\left(\frac{\pi i}{4}\right)$. Then

$$\begin{aligned} \zeta_{+1}(4) = \sum_{n=1}^{\infty} \frac{1}{n^4+1} &= -\frac{1}{2} + \frac{\pi}{4}\sum_{k=1}^{2} \alpha\cot(\pi\alpha) \\ &= -\frac{1}{2} + \frac{\pi}{4}\left(\alpha\cot(\pi\alpha) + \alpha^3\cot(\pi\alpha^3)\right) \\ &= -\frac{1}{2} + \frac{\pi}{4}\left(\alpha\cot(\pi\alpha) + \alpha^{-1}\cot(\pi\alpha^{-1})\right). \end{aligned}$$

Now, it follows from the facts

$$\pi\exp\left(\frac{\pi i}{4}\right) = \frac{1 \pm i}{\sqrt{2}}\pi \quad \text{and} \quad \cot(x+yi) = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x} \quad x, y \in \mathcal{R},$$

that

$$\begin{aligned} \zeta_{+1}(4) &= -\frac{1}{2} + \frac{\pi}{4}\left(\alpha\cot(\pi\alpha) + \alpha^{-1}\cot(\pi\alpha^{-1})\right) \\ &= -\frac{1}{2} + \frac{\pi}{4}\left(\frac{1+i}{\sqrt{2}} \cdot \frac{\sin\sqrt{2}\pi - i\sinh\sqrt{2}\pi}{\cosh\sqrt{2}\pi - \cos\sqrt{2}\pi} + \frac{1-i}{\sqrt{2}} \cdot \frac{\sin\sqrt{2}\pi + i\sinh\sqrt{2}\pi}{\cosh\sqrt{2}\pi - \cos\sqrt{2}\pi} \right) \\ &= -\frac{1}{2} + \frac{\sqrt{2}\pi}{4}\left(\frac{\sin\sqrt{2}\pi + \sinh\sqrt{2}\pi}{\cosh\sqrt{2}\pi - \cos\sqrt{2}\pi} \right). \end{aligned}$$

Let $s = 6$, $m = 3$ and $\alpha = \alpha_6 = \exp\left(\frac{\pi i}{6}\right) = \frac{\sqrt{3}+i}{2}$. With $\alpha^3 = i$, $\alpha^5 = \alpha^{-1} = \frac{\sqrt{3}-i}{2}$ and $i\cot(\pi i) = \coth(\pi)$, we observe that

$$\begin{aligned} \zeta_{+1}(6) &= -\frac{1}{2} + \frac{\pi}{6}\left(\alpha\cot(\pi\alpha) + \alpha^3\cot(\pi\alpha^3) + \alpha^5\cot(\pi\alpha^5)\right) \\ &= -\frac{1}{2} + \frac{\pi}{6}\left(\alpha\cot(\pi\alpha) + \alpha^{-1}\cot(\pi\alpha^{-1}) + i\cot(\pi i)\right) \\ &= -\frac{1}{2} + \frac{\pi}{6}\left(\frac{\sqrt{3}+i}{2} \cdot \frac{\sin\sqrt{3}\pi - i\sinh\pi}{\cosh\pi - \cos\sqrt{3}\pi} + \frac{\sqrt{3}-i}{2} \cdot \frac{\sin\sqrt{3}\pi + i\sinh\pi}{\cosh\pi - \cos\sqrt{3}\pi} + \coth(\pi) \right) \\ &= -\frac{1}{2} + \frac{\pi}{6}\left(\frac{\sqrt{3}\sin\sqrt{3}\pi + \sinh\pi}{\cosh\pi - \cos\sqrt{3}\pi} + \coth(\pi) \right). \end{aligned}$$

4. Proof of Theorem 5

Next, we prove Theorem 5 on $\zeta_{-1}(2m)$. Ideas are quite same.

For $m \geq 1$, let

$$g(z) = \frac{\cot(\pi z)}{z^{2m} - 1} \quad \text{and} \quad \beta = \exp\left(\frac{\pi i}{m}\right).$$

It has poles at $z = 0, \pm 1, \pm 2, \pm 3, \ldots$, and $z = \beta^k$, $1 \leq k \leq 2m-1, k \neq m$. The order of the poles $z = \pm 1$ is 2 and the all others are simple.

Lemma 3. *For $n = 0, \pm 2, \pm 3, \ldots$, we have*

$$Res(g,n) = \frac{1}{\pi(n^{2m}-1)},$$

for $1 \le k \le 2m-1, k \neq m$,

$$Res(\pi, \beta^k) = \frac{\beta^k}{2m}\cot(\pi\beta^k),$$

and moreover

$$Res(g, \pm 1) = -\frac{2m-1}{4m\pi}.$$

Proof. The proofs of the first two equalities are almost similar to ones for Lemma 1. Thus we only need to show $\mathrm{Res}(g, \pm 1) = -\frac{2m-1}{4m\pi}$. Let $\phi(z) = \sum_{k=0}^{2m-1} z^k$. Notice that $\phi(z) = (z^{2m+1}-1)/(z-1)$. Then

$$\begin{aligned}
\mathrm{Res}(g,1) &= \lim_{z\to 1}\frac{d}{dz}(z-1)^2 g(z) \\
&= \lim_{z\to 1}\frac{d}{dz}(z-1)\cot(\pi z)\cdot\frac{1}{\phi(z)} \\
&= \lim_{z\to 1}\left(\frac{\cot(\pi z) - \pi(z-1)(\cot^2 \pi z + 1)}{\phi(z)} - (z-1)\cot(\pi z)\frac{\phi'(z)}{\phi(z)^2}\right).
\end{aligned}$$

Let us see the first term. Immediately, $\lim_{z\to 1}\phi(1) = 2m$ and

$$\begin{aligned}
\lim_{z\to 1}\left(\cot(\pi z) - \pi(z-1)(\cot^2 \pi z + 1)\right) &= \lim_{w\to 0}\left(\cot(\pi w) - \pi w(\cot^2 \pi w + 1)\right) \\
&= \lim_{w\to 0}\left(\cot(\pi w)(1 - \pi w\cot(\pi w))\right) - \lim_{w\to 0}\pi w \\
&= \lim_{w\to 0}\frac{\tan(\pi w) - \pi w}{\tan^2 \pi w} - 0 \\
&= \lim_{w\to 0}\frac{\pi(1+\tan^2 \pi w) - \pi}{2\pi \tan \pi w(1+\tan^2 \pi w)} = 0. \quad \text{(L'Hôpital's rule)}
\end{aligned}$$

In addition, since

$$\phi'(1) = \sum_{k=0}^{2m-1} k = m(2m-1),$$

the limit $z \to 1$ for the second term is

$$\begin{aligned}
\lim_{z\to 1}\left(-(z-1)\cot(\pi z)\frac{\phi'(z)}{\phi(z)^2}\right) &= -\lim_{w\to 0} w\cot(\pi w)\frac{\phi'(w-1)}{\phi(w-1)^2} \\
&= -\frac{1}{\pi}\frac{m(2m-1)}{(2m)^2} = -\frac{2m-1}{4m\pi}.
\end{aligned}$$

□

Proof of Theorem 5. Let $\beta = \exp(\frac{\pi i}{m})$ be as above. Further, let $N, C(N), D(N)$ be as in the previous section. Again, the residue theorem claims that

$$\int_{C(N)} g(z)\,dz = 2\pi i \sum_{\substack{a:\text{pole of } g \\ a\in D(N)}} \mathrm{Res}(g,a).$$

Taking the limit $N \to \infty$, the integral converges to 0 likewise. It follows from Lemma 3 that

$$0 = 2\pi i \sum_{a:\text{pole of } g} \mathrm{Res}(g,a),$$

$$
\begin{aligned}
0 &= \mathrm{Res}(g,0) + \mathrm{Res}(g,1) + \mathrm{Res}(g,-1) + \sum_{n=2}^{\infty} (\mathrm{Res}(g,n) + \mathrm{Res}(g,-n)) + \sum_{1\le k\le 2m-1, k\ne m} \frac{\pi}{2m}\beta^k \cot(\pi\beta^k) \\
&= -\frac{1}{\pi} - \frac{2m-1}{2m}\pi + \frac{2}{\pi}\zeta_{-1}(2m) + \sum_{1\le k\le 2m-1, k\ne m} \frac{\pi}{2m}\beta^k \cot(\pi\beta^k).
\end{aligned}
$$

Conclude that

$$\zeta_{-1}(2m) = \frac{1}{2} + \frac{2m-1}{4m} - \frac{\pi}{4m}\sum_{1\le k\le 2m-1, k\ne m} \beta^k \cot(\pi\beta^k).$$

□

Example 3. For $s = 4, m = 2, \beta = \exp\left(\frac{\pi i}{2}\right) = i$, an expression of real trigonometric function for $\zeta_{-1}(4)$ is

$$\zeta_{-1}(4) = \frac{1}{2} + \frac{3}{8} - \frac{\pi}{8}\left(i\cot(\pi i) + i^3\cot(\pi i^3)\right) = \frac{7}{8} - \frac{\pi}{4}\coth(\pi).$$

Notice that this also shows that

$$\sum_{k=1}^{\infty}(\zeta(4k) - 1) = \frac{7}{8} - \frac{\pi}{4}\coth(\pi)$$

as in [4, p.263]. For $s = 6, m = 3, \beta = \exp\left(\frac{\pi i}{3}\right)$, we see that

$$
\begin{aligned}
\zeta_{-1}(6) &= \frac{1}{2} + \frac{5}{12} - \frac{\pi}{12}\left(e^{\pi i/3}\cot(\pi e^{\pi i/3}) + e^{-\pi i/3}\cot(\pi e^{-\pi i/3}) + e^{2\pi i/3}\cot(\pi e^{2\pi i/3}) + e^{-2\pi i/3}\cot(\pi e^{-2\pi i/3})\right) \\
&= \frac{11}{12} - \frac{\pi}{12}\left(\frac{\sqrt{3}\sinh\sqrt{3}\pi}{\cosh\sqrt{3}\pi + 1} + \frac{\sqrt{3}\sinh\sqrt{3}\pi}{\cosh\sqrt{3}\pi + 1}\right) \\
&= \frac{11}{12} - \frac{\pi}{12}\left(2\sqrt{3}\,\frac{2\sinh\frac{\sqrt{3}}{2}\pi\cosh\frac{\sqrt{3}}{2}\pi}{2\cosh^2\frac{\sqrt{3}}{2}\pi}\right) \\
&= \frac{11}{12} - \frac{\sqrt{3}}{6}\pi\tanh\frac{\sqrt{3}}{2}\pi.
\end{aligned}
$$

Acknowledgments: The first author thanks Satomi Abe, Shoko Asami and Michihito Tobe for sincerely supporting him. Also, the authors would like to thank Shigeru Iitaka. This research project arose from his online seminar in 2020-2022.

Author Contributions: All authors contributed equally in this paper. All authors read and approved final version of this paper.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Bibiloni, L., Viader, P., & Paradís, J. (2006). On a series of Goldbach and Euler. *The American Mathematical Monthly, 113*(3), 206-220.

[2] Shallit, J. D. Zikan, K. (1986). A theorem of Goldbach. *The American Mathematical Monthly, 93*, 402-403.

[3] Ayoub, R. (1974). Euler and the zeta function. *The American Mathematical Monthly, 81*(10), 1067-1086.

[4] Borwein, J. M., Bradley, D. M., & Crandall, R. E. (2000). Computational strategies for the Riemann zeta function. *Journal of Computational and Applied Mathematics, 1-2*, 247-296.