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A version of the Hermite-Hadamard inequality for Quasi $F-(h,g,m)$-convex functions

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Abstract: This paper aims to present Hermite-Hadamard type inequalities for a new class of functions, which will be denoted by $Q_m^{h,g}(F;I)$ an and called class of quasi $F-(h,g;m)$-convex functions defined on interval I. Many well known classes of functions can be recaptured from this new quasi convexity in particular cases. Also, several publish results are obtained along with new kinds of inequalities.

Keywords: Convex function; $(h,g;m)$-convex function; Hermite-Hadamard inequality

MSC: 26A51, 26D15.

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1. Introduction and Preliminaries

A function f defined on $[a,b]$ and satisfying the inequality (1), is called convex function and it leads to several new definitions and notions due to this analytical presentation. A convex function is also defined in many other ways, but inequality (1) and the Hermite-Hadamard inequality (2) are the most acknowledged and celebrated definitions. Since the definition of convex function is introduced, it got special attention in mathematical analysis, because of many interesting properties and characterizations, see [1–3].

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y);\ t\in[0,1],\ x,y\in[a,b]. \tag{1}$$

The following inequality (2) is due to Hermite [4] and Hadamard [5], holds for convex functions.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

The above inequality gives the upper and lower estimations of integral mean of a convex function. Inequality (1) preserves in different settings of new and existing parameter t, for example by involving m, $m\in[0,1]$ under argument of f in left hand side m-convex function is defined by the following inequality, see [6]:

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y);\ t\in[0,1],\ x,y\in[0,b]. \tag{3}$$

By replacing t and $1-t$ with one on the right hand side of (1) P-convex function is defined as follows:

$$f(tx+(1-t)y) \leq f(x)+f(y);\ t\in[0,1],\ x,y\in[a,b]. \tag{4}$$

By replacing t and $1-t$ with t^s and $(1-t)^s$, $s\in(0,1]$ on the right hand side of (1) s-convex function is defined as follows:

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y);\ t\in[0,1],\ x,y\in[0,\infty). \tag{5}$$

By replacing t and $1-t$ with $h(t)$ and $h(1-t)$ respectively on the right hand side of (1) h-convex function is defined as follows:

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y);\ t\in[0,1],\ x,y\in[a,b]. \tag{6}$$

Likewise, (s,m)-, (α,m)-, $(h-m)$-, (p,h)-, $(\alpha,h-m)$-convex functions and many other such names have been introduced in the literature by modifying (1) in different ways. All such types of functions are defined for generalizing the Hermit-Hadamard inequality (2). In [7], $(h,g;m)$-convex function was defined and general Hermite-Hadamard type inequality was presented. By replacing y with my, $m \in [0,1]$ in argument of f on the left hand side and replacing t and $1-t$ with $h(t)g(x)$ and $h(1-t)g(y)$ respectively on the right hand side of (1), the following inequality is obtained

$$f(tx+m(1-t)y) \leq h(t)f(x)g(x)+mh(1-t)f(y)f(y). \tag{7}$$

A function f satisfying (7) is called $(h,g;m)$-convex function, see [7]. Next, we state version of the Hermite-Hadamard inequality for $(h,g;m)$-convex functions.

Theorem 1. *Let $f:[a,b] \subset I \to \mathbb{R}$ be (h,g,m)-convex function. Then the following inequality holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f.g(z)+mf.g\left(\frac{z}{m}\right)\right)dz \\ &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\left\{f.g(a)\int_a^b h\left(\frac{b-z}{b-a}\right)g(z)dz+mf.g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{z-a}{b-a}\right)g(z)dz\right. \\ &\left.+mf.g\left(\frac{a}{m}\right)\int_a^b h\left(\frac{b-z}{b-a}\right)g\left(\frac{z}{m}\right)dz+m^2f.g\left(\frac{b}{m^2}\right)\int_a^b h\left(\frac{z-a}{b-a}\right)g\left(\frac{z}{m}\right)dz\right\}. \end{aligned} \tag{8}$$

The above inequality actually generates almost all versions of the Hermite-Hadamard inequality for convex and non-convex functions linked with (1).

The goal of this paper is to establish a version of the Hermite-Hadamard inequality involving quasi arithmetic mean in the place of geometric mean. We define a new class of functions which will be called quasi $F-(h,g;m)$-convex functions. Riemann integrals of such kinds of functions are estimated.

2. Auxiliary Definitions

We give the definition of quasi $F-(h,g;m)$-convex function and its consequences. First, we define quasi arithmetic mean. For a continuous and strictly monotone function $F: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, the quasi arithmetic mean denoted with $Q(\mathbf{p},\mathbf{x})$ is defined by;

$$Q(\mathbf{p},\mathbf{x}) := F^{-1}\left(\sum_1^n p_iF(x_i)\right),$$

where $\mathbf{x}=(x_1,...,x_n)$, $\mathbf{p}=(p_1,...,p_n)$ and $x_i, p_i \geq 0; \sum_i^n p_i = 1$.

Definition 1. Let h be a non-negative function on $J \subset \mathbb{R}$, $(0,1) \subset J$, $h \neq 0$ and let g be a positive function on $I \subset \mathbb{R}$. Furthermore, let $m \in (0,1]$. A function $f: I \to R$ is said to be quasi $F-(h,g;m)$-convex function if it is non-negative and if

$$f(F^{-1}(\lambda F(x)+(1-\lambda)F(my)) \leq h(\lambda)f(x)g(x)+mh(1-\lambda)f(y)g(y) \tag{9}$$

provided $F: I \to \mathbb{R}$ is strictly monotone, where $\lambda \in [0,1]$, $x,y \in I$.

Remark 1. It is noted that for $F(x)=x$, the inequality (9) reduces to the inequality (7). All classes of functions associated with convex functions such as exponentially convex, exponentially s-convex, exponentially (s,m)-convex, exponentially $(h-m)$-convex, s-convex, (s,m)-convex and $(h-m)$-convex can be recovered from the above definition.

By setting $F(x)=x^p$ in (9), we can give the following definition of $(h,g;m)-p$-convex function.

Definition 2. Let h be a non-negative function on $J \subset \mathbb{R}$, $(0,1) \subset J$, $h \neq 0$ and let g be a positive function on $I \subset \mathbb{R}$. Furthermore, let $m \in (0,1]$. A function $f : I \to \mathbb{R}$ is said to be $(h,g;m) - p$-convex if it is non-negative and satisfy the following inequality

$$f\left((\lambda x^p + (1-\lambda)(my)^p)^{\frac{1}{p}}\right) \leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y), \tag{10}$$

where $p \in \mathbb{R} - \{0\}, \lambda \in [0,1]$, $x, y \in I$.

It is noted that almost all kinds of p-convexities and exponentially p-convexities can be recovered by setting suitable values of function $g(x)$ and parameter m. For instance the definition of exponentially $h-p$-convex is obtained by setting $g(x) = \exp(-\alpha x)$, $m = 1$ as follows:

$$f\left((\lambda x^p + (1-\lambda)y^p)^{\frac{1}{p}}\right) \leq \frac{h(\lambda)f(x)}{e^{\alpha x}} + \frac{h(1-\lambda)f(y)}{e^{\alpha y}}. \tag{11}$$

Further, definitions of exponentially (p,P)-convex, Godunova–Levin type exponentially harmonic convex, exponentially harmonic s-convex and exponentially HA-convex functions given in [8] can be obtained.
By setting $F(x) = \log x$ in (9), we can give the following definition of geometric $(h,g;m)$-convex function.

Definition 3. Let h be a non-negative function on $J \subset \mathbb{R}$, $(0,1) \subset J$, $h \neq 0$ and let g be a positive function on $I \subset \mathbb{R}$. Furthermore, let $m \in (0,1]$. A function $f : I \to \mathbb{R}$ is said to be geometric $(h,g;m)$-convex if it is non-negative and holds the following inequality

$$f\left(x^\lambda y^{1-\lambda}\right) \leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y), \tag{12}$$

where $\lambda \in [0,1]$, $x, y \in I$.

Next, we give new definitions deducible from Definition 1.

- By setting $m = 1$, we will say f is quasi $F - (g,h)$-convex function.
- By setting $h(x) = x$, we will say f is quasi $F - (g;m)$-convex function.
- By setting $g(x) = 1 = m$, we will say f is quasi $F - h$-convex function.
- By setting $g(x) = 1$, we will say f is quasi $F - (h-m)$-convex function.
- By setting $g(x) = 1$, $h(x) = x$, we will say f is quasi $F - m$-convex function.
- By setting $g(x) = 1$, $h(x) = x^s$, we will say f is quasi $F - (s,m)$-convex function.
- By setting $g(x) = 1 = m$, $h(x) = x^s$, we will say f is quasi $F - s$-convex function.
- By setting $g(x) = 1 = m$, $h(x) = 1$, we will say f is quasi $F - P$-convex function.
- By setting $g(x) = 1$, $h(x) = \frac{1}{x^s}$, we will say f is quasi Godunova Levin $F - (s,m)$-convex function.
- By setting $g(x) = 1 = m$, $h(x) = \frac{1}{x}$, we will say f is quasi Godunova Levin F-convex function.
- By setting $g(x) = \exp(-\alpha x)$, we will say f is quasi exponentially $F - (h-m)$-convex function.
- By setting $g(x) = \exp(-\alpha x)$, $h(x) = x^s$, we will say f is quasi exponentially $F - (s-m)$-convex function.
- By setting $g(x) = \exp(-\alpha x)$, $m = 1$, we will say f is quasi exponentially $F - h$-convex function.
- By setting $g(x) = \exp(-\alpha x)$, $m = 1$, $h(x) = \frac{1}{x}$, we will say f is quasi Godunova Levin type exponentially F-convex function.
- By setting $g(x) = \exp(-\alpha x)$, $m = 1$, $h(x) = x$, we will say f is quasi exponentially F-convex function.
- By setting $g(x) = 1 = m$, $h(x) = x$, we will say f is quasi F-convex function.

The forthcoming section contains the Hermite-Hadamard inequality and associated results, the estimations of integral mean of $F - (h,g;m)$-convex function.

3. Main Results

First, we state and prove the following Hermite-Hadamard type inequality for $F - (h,g;m)$-convex functions.

Theorem 2. *Let $f : [a,b] \subset I \to \mathbb{R}$ be a quasi $F-(h,g,m)$-convex function. Then the following inequality holds:*

$$
\begin{aligned}
& f\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right) \leq \frac{h\left(\frac{1}{2}\right)}{F(b)-F(a)} \int_a^b \left(f.g(z) + mf.g\left(\frac{z}{m}\right)\right) d(F(z)) \\
& \leq \frac{h\left(\frac{1}{2}\right)}{F(b)-F(a)} \left\{ f.g(a) \int_a^b h\left(\frac{F(b)-F(z)}{F(b)-F(a)}\right) g(z) d(F(z)) \right. \\
& \left. + mf.g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{F(z)-F(a)}{F(b)-F(a)}\right) g(z) d(F(z)) + m \int_a^b f.g\left(\frac{z}{m}\right) d(F(z)) \right\}.
\end{aligned} \tag{13}
$$

Proof. By setting $\lambda = \frac{1}{2}$ and $y \to \frac{y}{m}$ in (9), we get

$$
f\left(F^{-1}\left(\frac{F(x)+F(y)}{2}\right)\right) \leq h\left(\frac{1}{2}\right) \left(f(x)g(x) + mf\left(\frac{y}{m}\right) g\left(\frac{y}{m}\right)\right). \tag{14}
$$

Further, by setting $F(x) = \lambda F(a) + (1-\lambda)F(b)$, $F(y) = \lambda F(b) + (1-\lambda)F(a)$ $\lambda \in [0,1]$, we find the following inequality:

$$
\begin{aligned}
& f\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right) \leq h\left(\frac{1}{2}\right) \left[f\left(F^{-1}\left(\lambda F(a) + (1-\lambda)F(b)\right)\right) g\left(F^{-1}\left(\lambda F(a) + (1-\lambda)F(b)\right)\right) \right. \\
& \left. + mf\left(\frac{F^{-1}\left(\lambda F(b) + (1-\lambda)F(a)\right)}{m}\right) g\left(\frac{F^{-1}\left(\lambda F(b) + (1-\lambda)F(a)\right)}{m}\right) \right].
\end{aligned} \tag{15}
$$

After integrating over $[0,1]$, we get

$$
\int_0^1 f\left(F^{-1}\left(\lambda F(a) + (1-\lambda)F(b)\right)\right) g\left(F^{-1}\left(\lambda F(a) + (1-\lambda)F(b)\right)\right) d\lambda = \frac{1}{F(b)-F(a)} \int_a^b (f.g)(z) F'(z) dz, \tag{16}
$$

where we have used the substitution $F(z) = \lambda F(a) + (1-\lambda)F(b)$. Also, after integration we have

$$
\begin{aligned}
& \int_0^1 f\left(\frac{F^{-1}\left(\lambda F(b) + (1-\lambda)F(a)\right)}{m}\right) g\left(\frac{F^{-1}\left(\lambda F(b) + (1-\lambda)F(a)\right)}{m}\right) d\lambda \\
& = \frac{1}{F(b)-F(a)} \int_a^b (f.g)\left(\frac{z}{m}\right) F'(z) dz,
\end{aligned} \tag{17}
$$

where we have used the substitution $F(z) = \lambda F(b) + (1-\lambda)F(a)$. By integrating the inequality (15) over $[0,1]$ and putting values of integrals from (16) and (17), the first inequality in (8) is achieved. For getting the second inequality we proceed as follows: By definition we have

$$
f(F^{-1}(\lambda F(a) + (1-\lambda)F(b)) \leq h(\lambda) f(a) g(a) + mh(1-\lambda) f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right). \tag{18}
$$

Multiplying the above inequality (18) by $g(F^{-1}(\lambda F(a) + (1-\lambda)F(b)))$ and integrating over $[0,1]$, we get

$$
\begin{aligned}
\frac{1}{F(b)-F(a)} \int_a^b (f.g)(z) F'(z) dz \leq & \frac{f.g(a)}{F(b)-F(a)} \int_a^b h\left(\frac{F(b)-F(z)}{F(b)-F(a)}\right) g(z) d(F(z)) \\
& + \frac{mf.g\left(\frac{b}{m}\right)}{F(b)-F(a)} \int_a^b h\left(\frac{F(z)-F(a)}{F(b)-F(a)}\right) g(z) d(F(z)),
\end{aligned} \tag{19}
$$

here we have used the substitution $F(z) = \lambda F(b) + (1-\lambda)F(a)$. By using (19), one can get the second inequality in (8). □

Some of the results associated with the inequality (8) are given in the following corollaries and remarks.

Remark 2. Setting of $g(x) = 1$ gives [9, Corollary 3.1], and with $\int_0^1 h(\lambda) d\lambda \leq 1$, [10, Theorem 9] is obtained. By setting $F(x) = x, g(x) = 1 = m$, the Hermite-Hadamard inequality for h-convex functions given in [11] is obtained. By setting $F(x) = x, g(x) = 1, h(x) = x$, the Hermite-Hadamard inequality for m-convex functions given in [12] is obtained. For $F(x) = x, h(x) = x$, Theorem 2 reduces to [9, Corollary 3.4].

Corollary 3. *Under the assumptions of Theorem 2, if we select $F(x) = x$, then the inequality (8) can be obtained.*

Proof. By setting $F(x) = x$ in (13), the following inequality is yielded:

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a} \int_a^b \left(f.g(z) + mf.g\left(\frac{z}{m}\right)\right) dz \leq \frac{h\left(\frac{1}{2}\right)}{b-a} \left\{ f.g(a) \int_a^b h\left(\frac{b-z}{b-a}\right) g(z) dz \right.$$
$$\left. + mf.g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{z-a}{b-a}\right) g(z) dz + m \int_a^b f.g\left(\frac{z}{m}\right) dz \right\}.$$

The integral in last term of the right hand side of last inequality in (**??**) can be estimated as follows: By using $F(x) = x$ in (9), we get that f is $(h, g; m)$-convex function which yielded th forthcoming inequality

$$f\left(\lambda \frac{a}{m} + (1-\lambda)\frac{b}{m}\right) \leq h(\lambda) f.g\left(\frac{a}{m}\right) + mh(1-\lambda) f.g\left(\frac{b}{m^2}\right). \tag{20}$$

By multiplying with $g\left(\lambda \frac{a}{m} + (1-\lambda)\frac{b}{m}\right)$ on both sides of the above inequality and integrating over $[0, 1]$, we get

$$\int_a^b f.g\left(\frac{z}{m}\right) dz \leq f.g\left(\frac{a}{m}\right) \int_a^b h\left(\frac{b-z}{b-a}\right) g\left(\frac{z}{m}\right) dz + mf.g\left(\frac{b}{m^2}\right) \int_a^b h\left(\frac{z-a}{b-a}\right) g\left(\frac{z}{m}\right) dz.$$

By using (**??**) in (**??**), the required inequality (8) is achieved. □

Remark 3. By setting $F(x) = x$, $g(x) = 1$ in (13), we get [9, Corollary 3.1].

The Hermite-Hadamard inequality proved in Theorem 2 can be obtained for all new and classical definitions given in Section 2. Here we give some consequences for new definitions.

Theorem 4. *The following inequality holds for quasi $F - (h - m)$-convex functions*

$$\begin{aligned} f\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right) &\leq \frac{h\left(\frac{1}{2}\right)}{F(b)-F(a)} \int_a^b \left(f(z) + mf\left(\frac{z}{m}\right)\right) d(F(z)) \\ &\leq \frac{h\left(\frac{1}{2}\right)}{F(b)-F(a)} \left\{ f(a) \int_a^b h\left(\frac{F(b)-F(z)}{F(b)-F(a)}\right) d(F(z)) \right. \\ &\left. + mf\left(\frac{b}{m}\right) \int_a^b h\left(\frac{F(z)-F(a)}{F(b)-F(a)}\right) d(F(z)) + m \int_a^b f\left(\frac{z}{m}\right) d(F(z)) \right\}. \end{aligned} \tag{21}$$

Proof. The required inequality can be obtained by setting $g(x) = 1$ in (13). □

Theorem 5. *The following inequality holds for $(h, g; m) - p$-convex functions*

$$\begin{aligned} f\left(\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}} &\leq \frac{ph\left(\frac{1}{2}\right)}{b^p-a^p} \int_a^b \left(f.g(z) + mf.g\left(\frac{z}{m}\right)\right) z^{p-1} dz \\ &\leq \frac{ph\left(\frac{1}{2}\right)}{b^p-a^p} \left\{ f.g(a) \int_a^b h\left(\frac{b^p-z^p}{b^p-a^p}\right) g(z) z^{p-1} dz \right. \\ &+ mf.g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{z^p-a^p}{b^p-a^p}\right) g(z) z^{p-1} dz + mf.g\left(\frac{a}{m}\right) \int_a^b h\left(\frac{b^p-z^p}{b^p-a^p}\right) g\left(\frac{z}{m}\right) z^{p-1} dz \\ &\left. + m^2 f.g\left(\frac{b}{m^2}\right) \int_a^b h\left(\frac{z^p-a^p}{b^p-a^p}\right) g\left(\frac{z}{m}\right) z^{p-1} dz \right\}. \end{aligned} \tag{22}$$

Proof. Let $F(x) = x^p$, $p \neq 0$. Then $F'(x) = px^{p-1}$, and by considering these settings in (13), one can have

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{ph\left(\frac{1}{2}\right)}{b^p-a^p}\int_a^b \left(f(g(z)) + mf\left(g\left(\frac{z}{m}\right)\right)\right) z^{p-1}\,dz$$
$$\leq \frac{ph\left(\frac{1}{2}\right)}{b^p-a^p}\Bigg\{f(g(a))\int_a^b h\left(\frac{b^p-z^p}{b^p-a^p}\right) g(z)z^{p-1}\,dz + mf\left(g\left(\frac{b}{m}\right)\right)\int_a^b h\left(\frac{z^p-a^p}{b^p-a^p}\right) g(z)z^{p-1}\,dz$$
$$+ m\int_a^b f\left(g\left(\frac{z}{m}\right)\right) z^{p-1}\,dz\Bigg\}. \tag{23}$$

By definition we have

$$f\left(\frac{(\lambda a^p+(1-\lambda)b^p)^{\frac{1}{p}}}{m}\right) = f\left(\left(\lambda\left(\frac{a}{m}\right)^p + (1-\lambda)\left(\frac{b}{m}\right)^p\right)^{\frac{1}{p}}\right) \leq h(\lambda)f\left(g\left(\frac{a}{m}\right)\right) + h(1-\lambda)f\left(g\left(\frac{b}{m}\right)\right).$$

Multiplying both sides of (**??**) by $g\left(\frac{(\lambda a^p+(1-\lambda)b^p)^{\frac{1}{p}}}{m}\right)$ and integrating over $[0,1]$, we have

$$\int_0^1 f.g\left(\frac{(\lambda a^p+(1-\lambda)b^p)^{\frac{1}{p}}}{m}\right) d\lambda$$
$$\leq \int_0^1 \left(h(\lambda)f.g\left(\frac{a}{m}\right) + mh(1-\lambda)f.g\left(\frac{b}{m^2}\right)\right) g\left(\frac{(\lambda a^p+(1-\lambda)b^p)^{\frac{1}{p}}}{m}\right) d\lambda.$$

The substitution $(\lambda a^p + (1-\lambda)b^p)^{\frac{1}{p}} = z$ gives the following integral inequality:

$$\int_a^b f.g\left(\frac{z}{m}\right) z^{p-1}dz \leq f.g\left(\frac{a}{m}\right)\int_a^b h\left(\frac{b^p-z^p}{b^p-a^p}\right) g\left(\frac{z}{m}\right) z^{p-1}dz + mf.g\left(\frac{b}{m^2}\right)\int_a^b h\left(\frac{z^p-a^p}{b^p-a^p}\right) g\left(\frac{z}{m}\right) z^{p-1}dz.$$

By using the estimation of above integral in (23), the required inequality is obtained. □

Remark 4. From the above inequality (22), one can obtain results for p-convex, (p,h)-convex, $(p,h-m)$-convex functions by setting $g(x) = 1$. On the other hand by setting $g(x) = \exp(-\alpha x)$ results for several kinds of exponentially convexities can be obtained. We leave them for readers.

It is interesting to note that the class of for $m = 1$, the class of quasi $F-(g,h)$-convex functions is obtained which is closely connected to the class of (g,h)-convex functions as follows: Let f is quasi $F-(g,h)$-convex function. Then we have the following inequality

$$f(F^{-1}(\lambda F(x) + (1-\lambda)F(y)) \leq h(\lambda)f(x)g(x) + h(1-\lambda)f(y)g(y). \tag{24}$$

If we set $X = F(x)$ and $Y = F(y)$ in the inequality (24), then we get

$$f(F^{-1}(\lambda X + (1-\lambda)Y)) \leq h(\lambda)f(F^{-1}(X))g(F^{-1}(X)) + h(1-\lambda)f(F^{-1}(Y))g(F^{-1}(Y)). \tag{25}$$

Further, by putting $U = f(F^{-1})$ and $V = g(F^{-1})$ in (25), we get

$$U(\lambda X + (1-\lambda)Y) \leq h(\lambda)U(X)V(X) + h(1-\lambda)U(Y)V(Y).$$

The above inequality shows that U is (V,h)-convex function. Next, we intend to give the cases when $m = 1$, in inequalities of aforementioned theorems.

Theorem 6. *The following inequality holds for quasi $F-(g,h)$-convex functions:*

$$f\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right) \leq \frac{2h\left(\frac{1}{2}\right)}{F(b)-F(a)} \int_a^b f.g\,(z)\,d(F(z))$$
$$\leq \frac{2h\left(\frac{1}{2}\right)}{F(b)-F(a)} \left\{ f.g(a) \int_a^b h\left(\frac{F(b)-F(z)}{F(b)-F(a)}\right) g(z)d(F(z)) \right.$$
$$\left. + f.g\,(b) \int_a^b h\left(\frac{F(z)-F(a)}{F(b)-F(a)}\right) g(z)d(F(z)) \right\}.$$

Proof. By setting $m=1$ in (13), the following result holds:

$$f\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right) \leq \frac{2h\left(\frac{1}{2}\right)}{F(b)-F(a)} \int_a^b f.g\,(z)\,d(F(z))$$
$$\leq \frac{h\left(\frac{1}{2}\right)}{F(b)-F(a)} \left\{ f.g(a) \int_a^b h\left(\frac{F(b)-F(z)}{F(b)-F(a)}\right) g(z)d(F(z)) \right. \tag{26}$$
$$\left. + f.g\,(b) \int_a^b h\left(\frac{F(z)-F(a)}{F(b)-F(a)}\right) g(z)d(F(z)) + \int_a^b f.g\,(z)\,d(F(z)) \right\}.$$

By using the estimation of integral $\int_a^b f.g\,(z)\,d(F(z))$ from (19) for $m=1$, we get the required inequality. □

Remark 5. If $F(t)=t$, $g(x)=1$ in (26), we get the Hermite-Hadamard inequality proved in [11]. The setting $F(t)=t$, $h(x)=x$ and $g(x)=e^{-\alpha x}$ in (26), gives [9, Corollary 3.6].

In the following we give the Hermite-Hadamard inequality for geometric (h,g,m)-convex function.

Theorem 7. *Let $f:[a,b]\subset I \to \mathbb{R}$ be a quasi $F-(h,g,m)$-convex function. Then the following inequality holds:*

$$f\left(\sqrt{ab}\right) \leq \frac{h\left(\frac{1}{2}\right)}{\log\left(\frac{b}{a}\right)} \int_a^b \frac{f.g(z)+mf.g\left(\frac{z}{m}\right)}{z} dz \leq \frac{h\left(\frac{1}{2}\right)}{\log\left(\frac{b}{a}\right)} \left\{ f.g(a) \int_a^b h\left(\frac{\log\left(\frac{b}{z}\right)}{\log\left(\frac{b}{a}\right)}\right) \frac{g(z)}{z} dz \right.$$
$$\left. + mf.g\left(\frac{b}{m}\right) \int_a^b h\left(\frac{\log\left(\frac{z}{a}\right)}{\log\left(\frac{b}{a}\right)}\right) \frac{g(z)}{z} dz + m\int_a^b \frac{1}{z} f.g\left(\frac{z}{m}\right) dz \right\}.$$

Proof. By setting $F(x)=\log x$ in (13), the inequality (**??**) can be obtained. □

Theorem 8. *Let $f:[a,b]\subset I \to \mathbb{R}$ be a quasi $F-(h,g,m)$-convex function. Then the following inequality holds:*

$$\frac{1}{F(b)-F(a)} \int_a^b f(z)d(F(z)) \leq \min\left\{ f.g(a)+mf.g\left(\frac{b}{m}\right), f.g(b)+mf.g\left(\frac{a}{m}\right) \right\}. \int_0^1 h(z)dz. \tag{27}$$

Proof. By definition we have the inequalities

$$f(F^{-1}(\lambda F(a)+(1-\lambda)F(b)) \leq h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), \tag{28}$$

and

$$f(F^{-1}(\lambda F(b)+(1-\lambda)F(a)) \leq h(\lambda)f(b)g(b)+mh(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right). \tag{29}$$

On integrating the above two inequalities over $[0,1]$ with suitable substitutions on the left hand side, we have the following integral inequalities:

$$\frac{1}{F(b)-F(a)} \int_a^b f(z)d(F(z)) \leq f.g(a) \int_0^1 h(\lambda)d\lambda + mf.g\left(\frac{b}{m}\right) \int_0^1 h(1-\lambda)d\lambda, \tag{30}$$

and

$$\frac{1}{F(b)-F(a)}\int_a^b f(z)d(F(z)) \leq f.g(b)\int_0^1 h(\lambda)d\lambda + mf.g\left(\frac{a}{m}\right)\int_0^1 h(1-\lambda)d\lambda. \tag{31}$$

The required inequality (13) is trivial after keeping in view the equation $\int_0^1 h(\lambda)d\lambda = \int_0^1 h(1-\lambda)d\lambda$. □

Remark 6. By setting $F(x) = x$ in (27), we get [9, Theorem 3.3].

Theorem 9. *Let $f : [a,b] \subset I \to \mathbb{R}$ be a quasi $F-(h,g,m)$-convex function. Then the following inequality holds:*

$$\begin{aligned}&\frac{1}{F(mb)-F(a)}\int_a^{mb} f(z)d(F(z)) + \frac{1}{F(b)-F(ma)}\int_{ma}^b f(z)d(F(z))\\&\leq \min\left\{f.g(a)+mf.g\left(\frac{b}{m}\right), f.g(b)+mf.g\left(\frac{a}{m}\right)\right\}.\int_0^1 h(z)dz.\end{aligned} \tag{32}$$

Proof. By definition we have the inequalities

$$f(F^{-1}(\lambda F(a)+(1-\lambda)F(mb)) \leq h(\lambda)f.g(a) + mh(1-\lambda)f.g(b),$$

$$f(F^{-1}(\lambda F(mb)+(1-\lambda)F(a)) \leq h(1-\lambda)f.g(a) + mh(\lambda)f.g(b),$$

$$f(F^{-1}(\lambda F(b)+(1-\lambda)F(ma)) \leq h(\lambda)f.g(b) + mh(1-\lambda)f.g(a),$$

$$f(F^{-1}(\lambda F(ma)+(1-\lambda)F(b)) \leq h(1-\lambda)f.g(b) + mh(\lambda)f.g(a),$$

On adding the above inequalities we get

$$\begin{aligned}&f(F^{-1}(\lambda F(a)+(1-\lambda)F(mb)) + f(F^{-1}(\lambda F(mb)+(1-\lambda)F(a))\\&+f(F^{-1}(\lambda F(b)+(1-\lambda)F(ma)) + f(F^{-1}(\lambda F(ma)+(1-\lambda)F(b))\\&\leq (m+1)(f.g(a)+f.g(b))\int_0^1 (h(\lambda)+h(1-\lambda))d\lambda,\end{aligned} \tag{33}$$

Integrating over $[0,1]$ on the both side, we have the following integral inequality:

$$\begin{aligned}&\int_0^1 (f(F^{-1}(\lambda F(a)+(1-\lambda)F(mb))) + f(F^{-1}(\lambda F(mb)+(1-\lambda)F(a))))d\lambda\\&+\int_0^1 (f(F^{-1}(\lambda F(b)+(1-\lambda)F(ma))) + f(F^{-1}(\lambda F(ma)+(1-\lambda)F(b))))d\lambda\\&\leq (m+1)(f.g(a)+f.g(b))\int_0^1 (h(\lambda)+h(1-\lambda))d\lambda.\end{aligned} \tag{34}$$

It can be verified that

$$\begin{aligned}&\int_0^1 f(F^{-1}(\lambda F(a)+(1-\lambda)F(mb)))d\lambda = \int_0^1 f(F^{-1}(\lambda F(mb)+(1-\lambda)F(a)))d\lambda\\&=\frac{1}{F(mb)-F(a)}\int_a^{mb} f(z)d(F(z))\end{aligned} \tag{35}$$

$$\begin{aligned}&\int_0^1 f(F^{-1}(\lambda F(b)+(1-\lambda)F(ma))d\lambda = \int_0^1 f(F^{-1}(\lambda F(ma)+(1-\lambda)F(b)))d\lambda\\&=\frac{1}{F(b)-F(ma)}\int_{ma}^b f(z)d(F(z)).\end{aligned} \tag{36}$$

The required inequality (27) can be obtained by using equations $\int_0^1 h(\lambda)d\lambda = \int_0^1 h(1-\lambda)d\lambda$, (35) and (36) in (34). □

Remark 7. By setting $F(x) = x$ in (32), we get [9, Theorem 3.4].

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