Open Journal of Mathematical Sciences

PSR Press

Article

Distribution of prime numbers and Fibonacci polynomials

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Communicated By: Waqas Nazeer
Received: 14 June 2023; Accepted: 6 April 2024; Published: 16 May 2024.

Abstract: Squares of odd index Fibonacci polynomials are used to define a new function $\Phi\left(10^n\right)$ to approximate the number $\pi\left(10^n\right)$ of primes less than 10^n. Multiple of 4 index Fibonacci polynomials are further used to define another new function $\Psi\left(10^n\right)$ to approximate the number $\Delta\left(\pi\left(10^n\right)\right)$ of primes having n digits and compared to a third function $\Psi'\left(10^n\right)$ defined as the difference of the first function $\Phi\left(10^n\right)$ based on odd index Fibonacci polynomials. These three functions provide better approximations of $\pi\left(10^n\right)$ than those based on the classical $\left(\frac{x}{log(x)}\right)$, Gauss' approximation $Li\left(x\right)$, and the Riemann $R\left(x\right)$ functions.

Keywords: Distribution of prime numbers; Fibonacci numbers; Fibonacci polynomials

MSC: Primary: 11, Secondary: 11N05, 11B39, 11Y55

1. Introduction and Preliminaries

The Prime Number Theorem states (see e.g. [1]) that if $\pi\left(x\right)$ is the number of primes $\leq x$ ($x \in \mathbb{Z}^+$), then

$$\lim_{x\to\infty}\left(\frac{\pi(x)}{\frac{x}{log(x)}}\right) = 1 \tag{1}$$

which means in all generality that the relative error of the approximation of $\pi(x)$ by $\left(\frac{x}{log(x)}\right)$ approaches 0 as x approaches infinity. Several better approximations to $\pi(x)$ are given for example by Gauss' approximation, i. e. the offset logarithmic integral or Eulerian logarithmic integral (see e.g. [2])

$$Li(x) = li\left(x\right) - li\left(2\right) = \int_2^x \frac{dt}{log\left(t\right)} \tag{2}$$

in function of the logarithmic integral $li(x) = \int_0^x \frac{dt}{log(t)}$, or by the Riemann function (see e.g. [3])

$$R\left(x\right) = \sum_{i=1}^{\infty} \frac{\mu\left(i\right)}{i} li\left(x^{\frac{1}{i}}\right), \tag{3}$$

where $\mu\left(i\right)$ is the Moebius function and $i \in \mathbb{Z}^+$, or by an even better function

$$R\left(x\right) - \frac{1}{log\left(x\right)} + \frac{1}{\pi} arctan\left(\frac{\pi}{log\left(x\right)}\right). \tag{4}$$

Interestingly when looking at the first values of $\pi\left(10^n\right)$ for $1 \leq n \leq 4$ ($n \in \mathbb{Z}^+$), it appears that they are equal or close to the square of odd index Fibonacci numbers F_{2n+1}, while the number of prime numbers having n digits, i.e. the difference $\pi\left(10^n\right) - \pi\left(10^{n-1}\right)$, is equal or close to Fibonacci numbers F_{4n} of indices multiple of 4.

We propose in this paper new functions giving a better representation of the sequence of $\pi\left(10^n\right)$ based on the square of odd index Fibonacci polynomials of a function ξ of n and two other representations based on Fibonacci polynomials for the difference $\pi\left(10^n\right) - \pi\left(10^{n-1}\right)$.

2. Number of primes and Fibonacci numbers

The values of $\pi(10^n)$ are known for $1 \leq n \leq 29$ (see [4]). The second column of Table 1 gives the first

Table 1. First five values of $\pi(10^n)$, F_{2n+1}^2, $\Delta(\pi(10^n))$ and F_{4n}

n	$\pi(10^n)$	F_{2n+1}^2	$\Delta(\pi(10^n))$	F_{4n}
1	4	4	4	3
2	25	25	21	21
3	168	169	143	144
4	1229	1156	1061	987
5	9592	7921	8363	6765

five values of $\pi(10^n)$. Interestingly,the first two terms of the sequence of $\pi(10^n)$ are integer squares while following terms are close to squares. With the notation $|[x]| = \text{round}(x)$, taking the nearest integer ρ_n to the square root of $\pi(10^n)$ for $1 \leq n \leq 29$

$$\rho_n = \left| \left[\sqrt{\pi(10^n)} \right] \right| \tag{5}$$

yields then the values 2, 5, 13, 35, 98, 280, ... [5].The first values are equal or close to those of odd index Fibonacci numbers F_{2n+1}, i.e. 2, 5, 13, 34, 89, 233, ... [6](see third column of Table 1). However, they diverge quite rapidly thereafter and $F_{2n+1} < \pi(10^n)$ for larger values of n. Figure 1 shows, for $1 \leq n \leq 25$, the of the relative difference

$$\delta_n = \frac{\rho_n - F_{2n+1}}{\rho_n}. \tag{6}$$

On the other hand, if the first values of $\pi(10^n)$ can be represented by the square of odd index Fibonacci

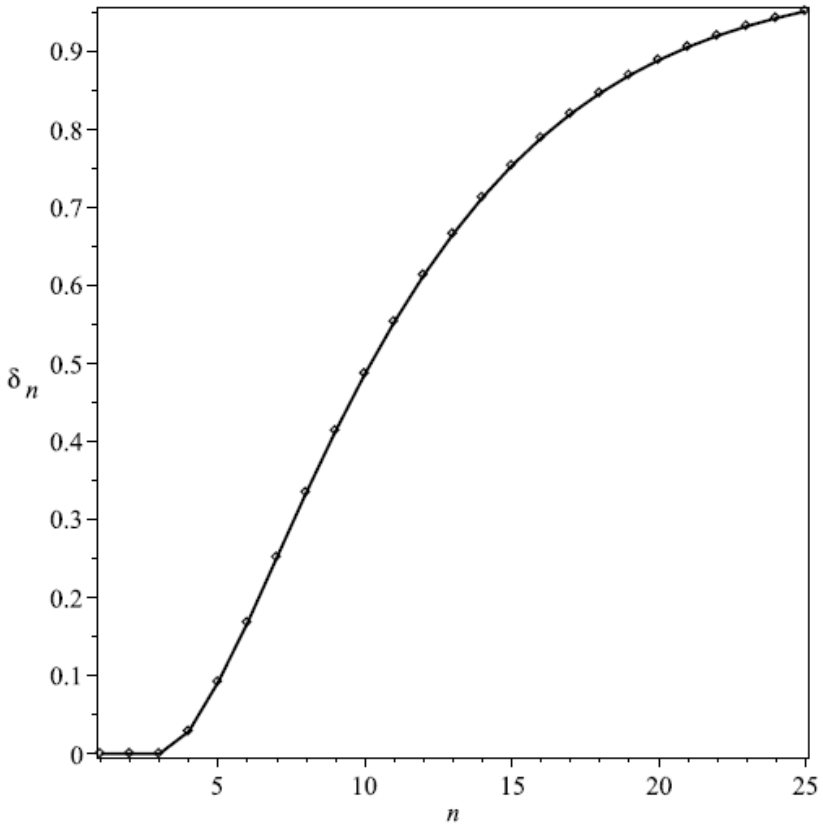

Figure 1. δ_n in function of n

numbers F_{2n+1}, and as

$$F_{4n} = F_{2n+1}^2 - F_{2n-1}^2, \tag{7}$$

(see e.g. [7]), one can expect that the difference of two successive values of $\pi\left(10^n\right)$, i.e.,

$$\Delta\left(\pi\left(10^n\right)\right)=\pi\left(10^n\right)-\pi\left(10^{n-1}\right) \tag{8}$$

can be represented by multiple of 4 index Fibonacci numbers F_{4n},i.e., 3, 21, 144, 987, 6765, ... In fact, $\Delta\left(\pi\left(10^n\right)\right)$ gives the number of primes between 10^n and 10^{n-1} or the number of primes with n digits ([8], see fourth column of Table 1 (with $\pi(1)=0$), i.e. there are 4 primes between 1 and 10, 21 primes between 11 and 100, 143 primes between 101 and 1000, etc. And these values are indeed equal or close to the first Fibonacci numbers F_{4n} of indices multiple of 4 (fifth column of Table 1). Again, they diverge quite rapidly thereafter and $F_{4n}<\Delta\left(\pi\left(10^n\right)\right)$ for larger values of n.

3. Number of primes and functions of Fibonacci polynomials

Next, we ask whether the square of odd index Fibonacci polynomials $F_{2n+1}(x)$ (see e.g. [9]) (with now $x\in\mathbb{R}$ onward) could approximate better the sequence of values of $\pi\left(10^n\right)$. The first odd index Fibonacci polynomials $F_{2n+1}(x)$ are given in Table 2. The roots of

Table 2. First five odd index Fibonacci polynomials $F_{2n+1}(x)$

n	$F_{2n+1}(x)$
1	x^2+1
2	x^4+3x^2+1
3	$x^6+5x^4+6x^2+1$
4	$x^8+7x^6+15x^4+10x^2+1$
5	$x^{10}+9x^8+28x^6+35x^4+15x^2+1$

$$F_{2n+1}(x)-\sqrt{\pi\left(10^n\right)}=0 \tag{9}$$

are then computed[1] for $1\leq n\leq 25$. All polynomials (9) have two real roots, one positive and one negative, and $2(n-1)$ complex roots of no interest here. Only the positive real root x_n^+ is considered in each case and, remarkably, all are close to unity, slowly increasing with increasing n, $1\lesssim x_n^+\lesssim 1.13...$ (see Figure 2).

The next step is to find a function that can approximate the sequence of real roots x_n^+ in function of n. Using a nonlinear regression curve fitting algorithm, such a function is given by

$$\xi(n)=\sum_{i=0}^{2}C_i\left(log\left(log\left(A\left(B+n^2\right)\right)\right)\right)^{2i}, \tag{10}$$

where $i\in\mathbb{Z}^+$, A, B, $C_i\in\mathbb{R}$ with $A=0.1641239$, $B=10.0861$, $C_0=0.9976796712309498$, $C_1=7.445960495\times 10^{-2}$, $C_2=-6.73751166802\times 10^{-3}$. Then the distribution of $\pi\left(10^n\right)$ can be approximated by

$$\Phi\left(10^n\right)=\left|\left[\left(F_{2n+1}\left(\xi(n)\right)\right)^2\right]\right|. \tag{11}$$

This function $\Phi\left(10^n\right)$ gives exactly the first four values of $\pi\left(10^n\right)$ for $1\leq n\leq 4$ and quite closely the following values (see [10]).

For the difference $\Delta\left(\pi\left(10^n\right)\right)$, we verify whether similarly Fibonacci polynomials $F_{4n}(x)$ of indices multiple of 4 could similarly approximate the sequence of values of $\Delta\left(\pi\left(10^n\right)\right)$. The first Fibonacci polynomials $F_{4n}(x)$ are shown in Table 3. However, relation (7) does not hold for Fibonacci polynomials. The correct relation is

$$xF_{4n}(x)=F_{2n+1}^2(x)-F_{2n-1}^2(x). \tag{12}$$

Therefore we use the product $xF_{4n}(x)$ instead of $F_{4n}(x)$, and the roots of

$$xF_{4n}(x)-\Delta\left(\pi\left(10^n\right)\right)=0 \tag{13}$$

[1] using Solve routine of Maple 16.00 with 30 digits precision

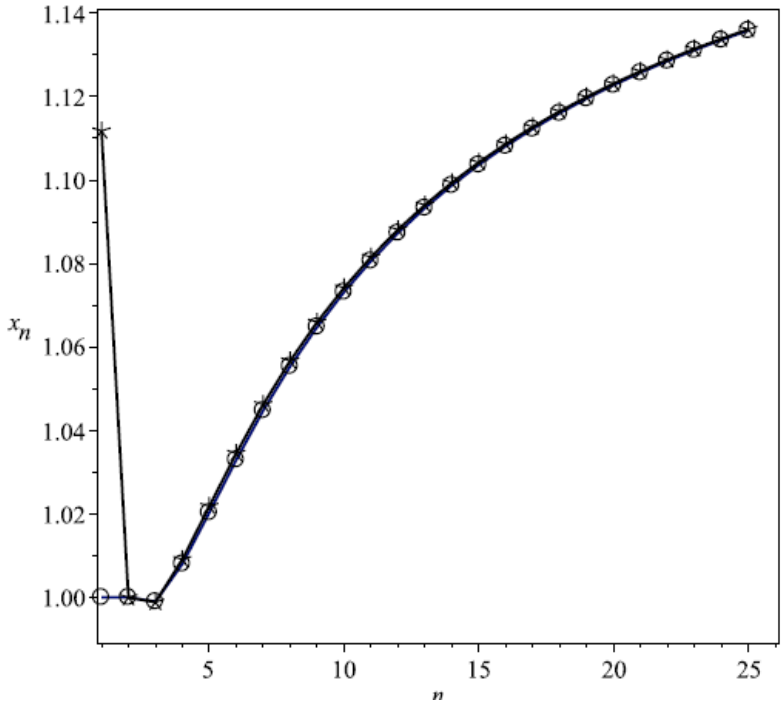

Figure 2. Positive real roots x_n^+ and $x_n^{'+}$ in function of n

Table 3. First five Fibonacci polynomials $F_{4n}(x)$

n	$F_{4n}(x)$
1	x^3+2x
2	$x^7+6x^5+10x^3+4x$
3	$x^{11}+10x^9+36x^7+56x^5+35x^3+6x$
4	$x^{15}+14x^{13}+78x^{11}+220x^9+330x^7+252x^5+84x^3+8x$
5	$x^{19}+18x^{17}+136x^{15}+560x^{13}+1365x^{11}+2002x^9+1716x^7+792x^5+165x^3+10x$

are computed[2] for $1 \leq n \leq 25$. All polynomials (13) have two real roots, one positive and one negative, two imaginary roots and $4(n-1)$ complex roots of no interest here. Again, all 25 real positive roots $x_n^{'+}$ are close to unity; except for the first value (for $n=1$), all values of $x_n^{'+}$ are slowly increasing with increasing n, $1 \lesssim x_n^{'+} \lesssim 1.13...$ (see Figure 2). A function approximating the sequence of roots $x_n^{'+}$ in function of n is found to be

$$\xi'(n) = \sum_{i=0}^{3} C_i \left(log\left(log\left(A\left(B+n^2\right)\right)\right)\right)^i, \tag{14}$$

where $i \in \mathbb{Z}^+$, A, B, $C_j \in \mathbb{R}$ with $A = 5.55206477803854105 \times 10^{-3}$, $B = 179.5601349965941682$, $C_0 = 1.1152639923333283653$, $C_1 = 5.11592774905246001 \times 10^{-2}$, $C_2 = 0$, $C_3 = -1.4047014763335134 \times 10^{-3}$. The distribution of $\Delta(\pi(10^n))$ can then be approximated by

$$\Psi(10^n) = \left|\left[\xi'(n) F_{4n}\left(\xi'(n)\right)\right]\right|. \tag{15}$$

This function $\Psi(10^n)$ gives exactly the first three values of $\Delta(\pi(10^n))$ for $1 \leq n \leq 3$ and quite closely the following values.

Another function $\Psi'(10^n)$ approximating $\Delta(\pi(10^n))$ can be obtained by successive values of $\Phi(10^n)$, i.e.,

$$\Psi'(10^n) = \Phi(10^n) - \Phi\left(10^{n-1}\right) = \left|\left[\left(F_{2n+1}(\xi(n))\right)^2\right]\right| - \left|\left[\left(F_{2n-1}(\xi(n-1))\right)^2\right]\right|. \tag{16}$$

[2] using Solve routine of Maple 16.00 with 30 digits precision

The first value for $n = 1$ is not well represented as $\Psi'(10) = 3$ (with $\left|\left[\left(F_1(\xi(0))\right)^2\right]\right| = 1$ and $\xi(0) = 1.0311467...$) while $\Delta(\pi(10)) = 4$. However $\Psi'(10^n)$ gives exactly the following three values of $\Delta(\pi(10^n))$ for $2 \leq n \leq 4$ and approximates well the following values up to $n = 25$, much closer than $\Psi(10^n)$ (see further discussion).

4. Discussion

For the distribution of $\pi(10^n)$, it is interesting to compare the values obtained with $\Phi(10^n)$ to the nearest integers to those values obtained with the approximating functions of Section 1.

Table 4. First ten values of $\pi(10^n)$, the three distributions $f(n)$ and $\Phi(10^n)$

n	$\pi(10^n)$	$\left\|\left[\frac{10^n}{log(10^n)}\right]\right\|$	$\|[Li(10^n)]\|$	$\|[R(10^n)]\|$	$\Phi(10^n)$
1	4	4	5	5	4
2	25	22	29	26	25
3	168	145	177	168	168
4	1229	1086	1245	1227	1229
5	9592	8686	9629	9587	9595
6	78498	72382	78627	78527	78527
7	664579	620421	664917	664667	664408
8	5761455	5428681	5762208	5761552	5759130
9	50847534	48254942	50849234	50847455	50833725
10	455052511	434294482	455055614	455050683	455019102

Table 4 shows the first ten values of $\pi(10^n)$, $\left|\left[\frac{10^n}{log(10^n)}\right]\right|$ (see [11]), $|[Li(10^n)]|$ (from (2), see [12]), $|[R(10^n)]|$ (from (3), see [13]), or

$$f(n) = \text{either } \left|\left[\frac{10^n}{log(10^n)}\right]\right| \text{ or } |[Li(10^n)]| \text{ or } |[R(10^n)]|, \tag{17}$$

and $\Phi(10^n)$ (11). Note that in calculating $R(10^n)$, the summation in (3) is made for $1 \leq i \leq 1000$, as the value of $R(10^n)$ does not change for higher values of i, due mainly to the operation of rounding to the nearest integer. The function (4) is not considered as the values obtained by rounding to the nearest integer the function (4) does not differ (except for $n = 1$) from those obtained by rounding $R(10^n)$ for the same reason.

Computing the absolute value of the relative differences

$$\delta_n' = \left|\frac{\pi(10^n) - d'(n)}{\pi(10^n)}\right| \tag{18}$$

for $1 \leq n \leq 25$, where $d'(n)$ is either $f(n)$ (17) or $\Phi(10^n)$ (11), Figure 3 shows the , in function of n, of δ_n' for these four distributions on a log scale (recall that the first four values of δ_n' are for $\Phi(10^n)$) and Table 5 shows the averages $\mu\left(\delta_n'\right)$ and standard deviations $\sigma\left(\delta_n'\right)$ of δ_n' for $1 \leq n \leq 25$ for the four distributions.

Table 5. Averages and standard deviations of δ_n' for the four distributions $d'(n)$

$d'(n)$	$\left\|\left[\frac{10^n}{log(10^n)}\right]\right\|$	$\|[Li(10^n)]\|$	$\|[R(10^n)]\|$	$\Phi(10^n)$
$\mu\left(\delta_n'\right)$	4.65309×10^{-2}	1.93110×10^{-2}	1.17069×10^{-2}	1.58269×10^{-4}
$\sigma\left(\delta_n'\right)$	3.60695×10^{-2}	5.83932×10^{-2}	5.02812×10^{-2}	1.28997×10^{-4}

It is seen that the distribution $\left|\left[\frac{10^n}{log(10^n)}\right]\right|$ provides a relatively poor approximation for small and large value (up to $n = 25$) of $\pi(10^n)$. The distributions $|[Li(10^n)]|$ and $|[R(10^n)]|$ provide both a relatively poor approximation of $\pi(10^n)$ for small values of n, but increasingly better with increasing values of n. The distribution $\Phi(10^n)$ gives a better approximation of $\pi(10^n)$ for small values of $n \leq 6$ but not as good as $|[Li(10^n)]|$ and $|[R(10^n)]|$ for larger values of n. However the distribution $\Phi(10^n)$ gives on average a better approximation of the 25 values of $\pi(10^n)$ than the $\left(\frac{x}{log(x)}\right)$ function, the Gauss' approximation $Li(x)$ and the Riemann function $R(x)$ as shown by the average and standard deviation values given in Table 4 ($\mu\left(\delta_n'\right)$ and

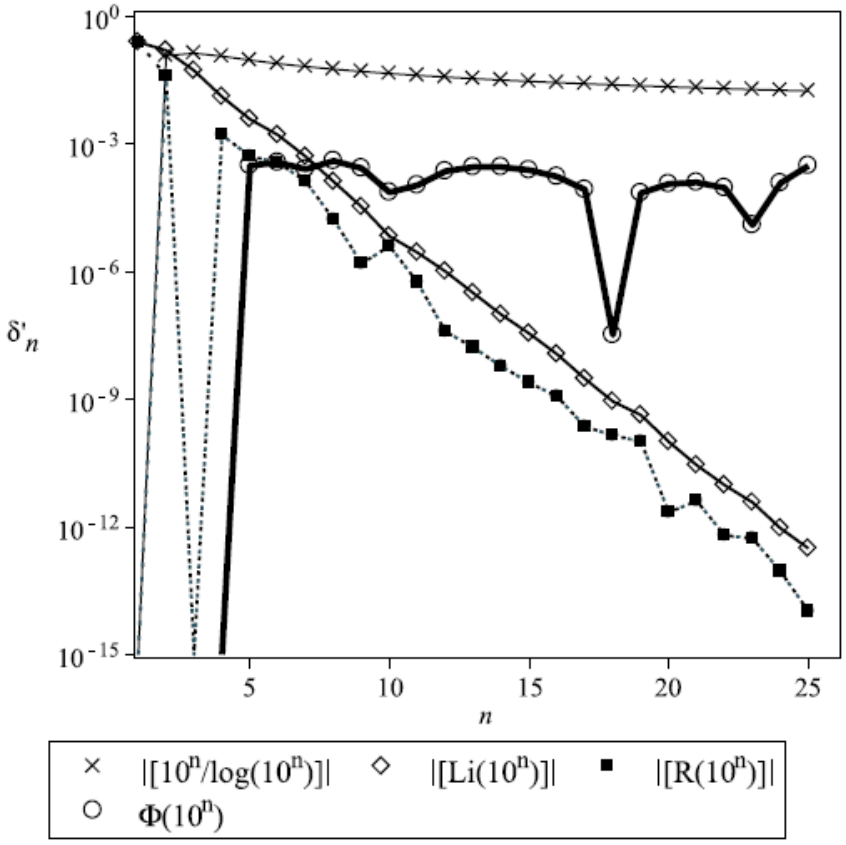

Figure 3. $\delta_n^{'}$ in function of n

$\sigma\left(\delta_n^{'}\right)$ for $\Phi\left(10^n\right)$ are approximately two orders of magnitude less that the other values), due to the fact that $\Phi\left(10^n\right)$ gives exactly the first four values of $\pi\left(10^n\right)$.

Comparing now the values for the difference $\Delta\left(\pi\left(10^n\right)\right)$ obtained by subtracting successive values of the functions of section 1, i.e.,

$$g(n) = f(n) - f(n-1) = \Delta(f(n)) \tag{19}$$

with $f(n)$ (17), to those of $\Psi\left(10^n\right)$ and $\Psi'\left(10^n\right)$, Table 6 gives the first ten values of $\Delta\left(\pi\left(10^n\right)\right)$, $g(n)$ (19), $\Psi\left(10^n\right)$ and $\Psi'\left(10^n\right)$. Note that the values of the three distributions $g(n)$ (19) are not defined for $n = 1$, as $f(n-1) = f(0)$ is not defined and, for the sake of the argument, the values of $\left|\left[\frac{1}{log(1)}\right]\right|$, $|[Li(1)]|$, $|[R(1)]|$ have been arbitrarily put equal to 0. Figure 4 shows the of the absolute value of the relative differences

Table 6. First ten values of $\Delta\left(\pi\left(10^n\right)\right)$, $g(n)$, $\Psi\left(10^n\right)$ and $\Psi'\left(10^n\right)$

n	$\Delta\left(\pi\left(10^n\right)\right)$	$\Delta\left(\left\|\left[\frac{10^n}{log(10^n)}\right]\right\|\right)$	$\Delta\left(\|[Li\left(10^n\right)]\|\right)$	$\Delta\left(\|[R\left(10^n\right)]\|\right)$	$\Psi\left(10^n\right)$	$\Psi'\left(10^n\right)$
1	4	4	5	5	4	3
2	21	18	24	21	21	21
3	143	123	148	142	143	143
4	1061	941	1068	1059	1063	1061
5	8363	7600	8384	8360	8385	8366
6	68906	63696	68998	68940	68929	68932
7	586081	548039	586290	586140	584467	585881
8	5096876	4808260	5097291	5096885	5074924	5094722
9	45086079	42826261	45087026	45085903	44885325	45074595
10	404204977	386039540	404206380	404203228	402777151	404185377

$$\delta_n^{''} = \left|\frac{\Delta\left(\pi\left(10^n\right)\right) - d''(n)}{\Delta\left(\pi\left(10^n\right)\right)}\right| \tag{20}$$

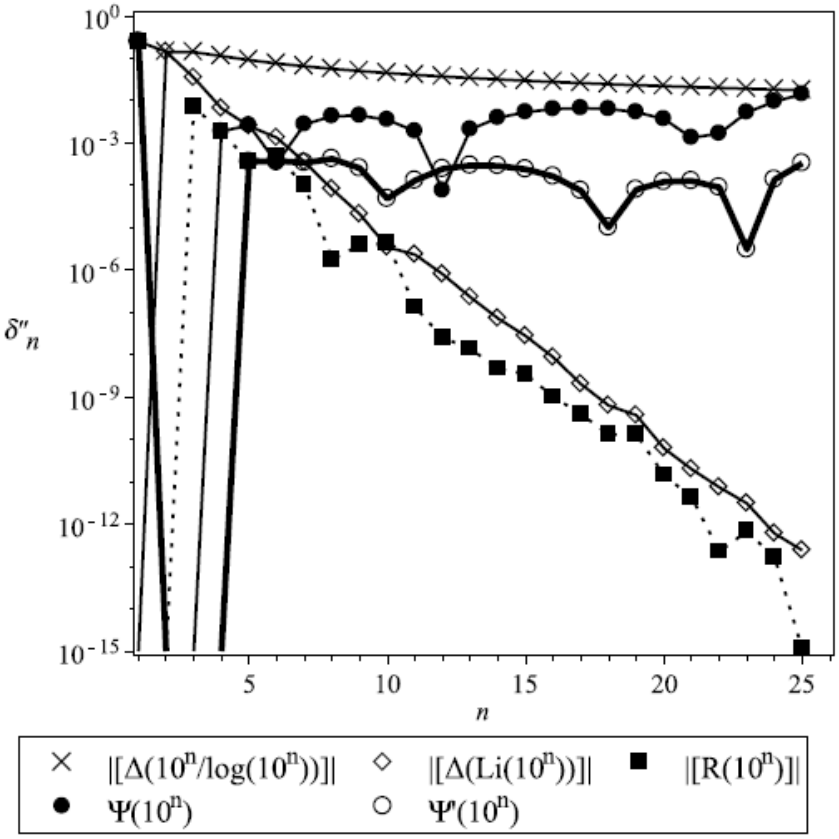

Figure 4. $\delta_n^{''}$ in function of n

in function of n for $1 \leq n \leq 25$, where $d''(n)$ is either $g(n) = \Delta(f(n))$, or $\Psi(10^n)$ or $\Psi'(10^n)$, and Table 7 shows the averages $\mu(\delta_n^{''})$ and standard deviations $\sigma(\delta_n^{''})$ of $\delta_n^{''}$ for the five distributions with and without the first value for $n = 1$ (i.e. respectively for $1 \leq n \leq 25$ and for $2 \leq n \leq 25$).

Table 7. Averages and standard deviations of $\delta_n^{''}$ for $g(n)$, $\Psi(10^n)$ and $\Psi'(10^n)$

	$1 \leq n \leq 25$		$2 \leq n \leq 25$	
$d''(n)$	$\mu(\delta_n^{''})$	$\sigma(\delta_n^{''})$	$\mu(\delta_n^{''})$	$\sigma(\delta_n^{''})$
$\Delta\left(\left\|\left[\frac{10^n}{log(10^n)}\right]\right\|\right)$	4.69094×10^{-2}	3.80699×10^{-2}	4.88640×10^{-2}	3.75855×10^{-2}
$\Delta(\|[Li(10^n)]\|)$	1.75493×10^{-2}	5.64517×10^{-2}	7.86383×10^{-3}	2.96342×10^{-2}
$\Delta(\|[R(10^n)]\|)$	1.03936×10^{-2}	4.99384×10^{-2}	4.10042×10^{-4}	1.45666×10^{-3}
$\Psi(10^n)$	3.75342×10^{-3}	3.32007×10^{-3}	3.90981×10^{-3}	3.29607×10^{-3}
$\Psi'(10^n)$	1.01657×10^{-2}	4.99657×10^{-2}	1.72564×10^{-4}	1.36990×10^{-4}

Again, both functions based on Fibonacci polynomials of a function of n approximate on average better the distribution of the number of primes of n digits, mainly because the first values of $\Delta(\pi(10^n))$ are exactly or closely represented. If the first value for $n = 1$ is included, the representation of $\Delta(\pi(10^n))$ by the function $\Psi(10^n)$ of Fibonacci polynomials $(\xi'(n) F_{4n}(\xi'(n)))$ of a function $\xi'(n)$ of n (with $\xi'(n) \approx 1$) is better than by the $\left(\frac{x}{log(x)}\right)$ function, the Gauss' approximation $Li(x)$ and the Riemann function $R(x)$, with an average $\mu(\delta_n^{''})$ and a standard deviation $\sigma(\delta_n^{''})$ for $\Psi(10^n)$ approximately one order of magnitude less than those for the other functions. If the first value for $n = 1$ is discarded, the representation of $\Delta(\pi(10^n))$ by the function $\Psi'(10^n)$ of the difference of the square of odd index Fibonacci polynomials $F_{2n+1}(\xi(n))$ of a function $\xi(n)$ of n (with $\xi(n) \approx 1$) is better than by the $\left(\frac{x}{log(x)}\right)$ function, the Gauss' approximation $Li(x)$ and the Riemann function $R(x)$ (with $\mu(\delta_n^{''})$ and $\sigma(\delta_n^{''})$ for $\Psi'(10^n)$ approximately two orders of magnitude less than for $\left(\frac{x}{log(x)}\right)$ and $Li(x)$ and $\mu(\delta_n^{''})$ and $\sigma(\delta_n^{''})$ for $\Psi'(10^n)$ respectively less than and one order of magnitude less than for $R(x)$).

5. Conclusions

A function $\Phi(10^n)$ of odd index Fibonacci polynomials $F_{2n+1}(\xi(n))$ of a function $\xi(n)$ of n was found to approximate the distribution of the number $\pi(10^n)$ of primes less than 10^n. Two functions $\Psi(10^n)$ and $\Psi'(10^n)$ were found to approximate the distribution of the number $\Delta(\pi(10^n))$ of primes having n digits, where $\Psi(10^n)$ is a function of multiple of 4 index Fibonacci polynomials $(\xi'(n) F_{4n}(\xi'(n)))$ of a function $\xi'(n)$ of n and $\Psi'(10^n) = \Phi(10^n) - \Phi(10^{n-1})$ is a function of the difference of odd index Fibonacci polynomials $F_{2n+1}(\xi(n))$ for successive values of n.

On average these three functions provide better approximations of one or two orders of magnitude in the averages of relative absolute differences of the exact and calculated values than classical functions, i.e. the $\left(\frac{x}{log(x)}\right)$ function, the Gauss' approximation $Li(x)$ and the Riemann function $R(x)$.

Note that these results do not disprove the Prime Number Theorem, but provide better representations of $\pi(10^n)$ and $\Delta(\pi(10^n))$.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: "All data required for this research is included within this paper".

Funding Information: "No funding is available for this research".

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