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Euler's and the taxi cab relations and other numbers that can be written twice as sums of two cubed integers

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Abstract: We show that Euler's relation and the Taxi-Cab relation are both solutions of the same equation. General solutions of sums of two consecutive cubes equaling the sum of two other cubes are calculated. There is an infinite number of relations to be found among the sums of two consecutive cubes and the sum of two other cubes, in the form of two families. Their recursive and parametric equations are calculated.

Keywords: Sums of two consecutive cubes ; Equal sums of two cubes ; Taxi-Cab number ; Euler's relation

MSC: Primary 11D25; Secondary 11B37.

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1. Introduction and Preliminaries

The remarkable relation

$$3^3 + 4^3 + 5^3 = 6^3 \tag{1}$$

among the cubes of four successive integers is often attributed to Euler, while in fact it was already known to P. Bungus in the XVIth century [1,2]. No other similar relation can be found between cubes of four successive integers. Another well-known relation involving two different sums of two cubes is

$$1729 = 9^3 + 10^3 = 12^3 + 1^3 \tag{2}$$

often call the taxi-cab number or taxi-cab relation and attributed to Indian mathematician Ramanujan after he mentioned in 1919 to fellow British mathematician Hardy that this number is remarkable in the fact that it is the smallest integer that can be written as the sum of two positive cubes in two ways (see historical account in e.g.,[3]). However, this relation was already mentioned by French mathematician Frenicle in the XVIIth century [2,4]. Nevertheless, we will refer in this paper to (1) and (2) as Euler's relation and Ramanujan's taxi-cab relation. In fact, both relations can be deduced from a same equation, as we show in this paper. It is simple to see that one can find other taxi-cab numbers smaller than Ramanujan's by transferring one term from left to right of (1), introducing negative integers and yielding successively

$$91 = 3^3 + 4^3 = 6^3 + (-5)^3 \tag{3}$$

$$152 = 3^3 + 5^3 = 6^3 + (-4)^3 \tag{4}$$

$$189 = 4^3 + 5^3 = 6^3 + (-3)^3 \tag{5}$$

and so on. Other taxi-cab numbers can be found by multiplying each relations (3) to (5) by k^3, i.e., the cube of any integer k. Sequences A001235 and A051347 in the OEIS [5] list all taxi-cab numbers for respectively only positive integers and for positive and negative integers. Numerous mathematicians and authors have worked on sums of cubes and equal sums of cubes. Excellent summaries and numerous results can be found e.g., in [2,6]. In this paper, our interest is in numbers that can be written as sums of two cubes in at least two ways, one of them involving two consecutive cubes. In Section, 2 we show first that Euler's and the Taxi-Cab relations are solutions of the same equation. We calculate then the general case of the sum of two consecutive cubes equal to the sum of two other cubes. In Section 3, we characterize two infinite families of solutions of the sum of two consecutive cubes equaling the sum of two other cubes.

2. General equation

We show first that (1) and (2) are both solutions of the same equation. If one observes that the first term on the right-hand side in (2) and (3) is three units larger than the first term on the left-hand side, we can write

$$N = n^3 + (n+1)^3 = (n+3)^3 + (n+\alpha)^3 \tag{6}$$

with α integer and N the positive integer that can be represented in (at least) two ways by sums of two consecutive cubes and of two other cubes, one of which possibly negative. Equation (6) yields the two general solutions

$$n = \frac{-3\left(\alpha^2+8\right) \pm \sqrt{-3\left(\alpha^4+8\left(\alpha^3-6\alpha^2+13\alpha+2\right)\right)}}{6\left(\alpha+2\right)} \tag{7}$$

which produces integer solutions for $\alpha = -8$, giving $n_+ = 3$ and $n_- = 9$. Equation (6) yields then respectively (3) and (2), showing that (6) yields both Euler's relation and Ramanujan's taxi-cab number relation. Let us consider now the general equation

$$N = n^3 + (n+1)^3 = (n+a)^3 + (n+b)^3 \tag{8}$$

with a and b integers, $a > 0$ and $b < 0$. Solving for n, the third degree equation defined by the second equality in (8) reduces to a second degree equation

$$3n^2\left(a+b-1\right) + 3n\left(a^2+b^2-1\right) + \left(a^3+b^3-1\right) = 0 \tag{9}$$

whose discriminant reads

$$D = 3\left((a-b)^4 - \left(a^4+b^4+(a-1)^4+(b-1)^4\right)+1\right) \tag{10}$$

providing two real solutions for $D > 0$, namely

$$n = \frac{-3\left(a^2+b^2-1\right) \pm \sqrt{3\left((a-b)^4 - \left(a^4+b^4+(a-1)^4+(b-1)^4\right)+1\right)}}{6\left(a+b-1\right)} \tag{11}$$

As N must be positive, we limit our search to $b < 0 < a < |b|$ and discrete solutions of (8) or (9) are found, as shown in Table 1 for $n < 1000$, and arranged in increasing order of N. Note also that similar relations but with coefficients having opposite signs are obtained for negative values of a and for $a' = -b+1$, $b' = -a+1$, and $n' = -n-1$.

It is seen also that three sums of two cubes are found for $n = 121, 163, 235, 562$. Other relations are given in OEIS [5] Sequences A352133 to A352136 and cases with three sums of two cubes are given in Sequences A352220 to A352225.

3. Two Infinite Families

Figure 1 shows a plot of the couples $(n, n+a)$ for $0 < n \leq 275$ (data are from OEIS [5] Sequences A352135, A352136, A352222, A352223, A352224, A352225). Two families are clearly visible along two curves.

The first top curve (or first family) includes all couples $(n, n+a)$ such that $\eta = (n+a)+(n+b) = 2n+a+b$ are regularly increasing odd integers as shown in Table 2 for the first twenty cases, while for the second below curve (or second family), $\eta = 2n+a+b$ are regularly increasing odd multiples of 3.

3.1. Recursive relations

The values of n, $n+a$ and $n+b$ of both the first and second families can be found by the recurrence relations

$$n_i = 3n_{i-1} - 3n_{i-2} + n_{i-3} + \kappa \tag{12}$$

$$(n+a)_i = 3\,(n+a)_{i-1} - 3\,(n+a)_{i-2} + (n+a)_{i-3} + \lambda \tag{13}$$

$$(n+b)_i = 3\,(n+b)_{i-1} - 3\,(n+b)_{i-2} + (n+b)_{i-3} - \lambda \tag{14}$$

Table 1. Values of a, b, n_+, n_-, solutions of (8) for $n < 1000$ and $b < 0 < a < |b|$

a	b	n_+	n_-	$N = n^3 + (n+1)^3 = (n+a)^3 + (n+b)^3$
3	-8	3	–	$91 = 3^3 + 4^3 = 6^3 + (-5)^3$
2	-7	4	–	$189 = 4^3 + 5^3 = 6^3 + (-3)^3$
3	-8	–	9	$1729 = 9^3 + 10^3 = 12^3 + 1^3$
10	-39	18	–	$12691 = 18^3 + 19^3 = 28^3 + (-21)^3$
9	-38	–	32	$68705 = 32^3 + 33^3 = 41^3 + (-6)^3$
10	-39	–	36	$97309 = 36^3 + 37^3 = 46^3 + (-3)^3$
105	-194	46	–	$201159 = 46^3 + 47^3 = 151^3 + (-148)^3$
32	-127	58	–	$400491 = 58^3 + 59^3 = 90^3 + (-69)^3$
64	-243	107	–	$2484755 = 107^3 + 108^3 = 171^3 + (-136)^3$
73	-258	108	–	$2554741 = 108^3 + 109^3 = 181^3 + (-150)^3$
32	-103	–	121	$3587409 = 121^3 + 122^3 = 153^3 + 18^3$
248	-481	121	–	$3587409 = 121^3 + 122^3 = 369^3 + (-360)^3$
37	-192	–	123	$3767491 = 123^3 + 124^3 = 160^3 + (-69)^3$
43	-168	–	163	$8741691 = 163^3 + 164^3 = 206^3 + (-5)^3$
91	-360	163	–	$8741691 = 163^3 + 164^3 = 254^3 + (-197)^3$
819	-1208	197	–	$15407765 = 197^3 + 198^3 = 1016^3 + (-1011)^3$
57	-128	–	235	$26122131 = 235^3 + 236^3 = 292^3 + 107^3$
184	-597	235	–	$26122131 = 235^3 + 236^3 = 419^3 + (-362)^3$
77	-208	–	301	$54814509 = 301^3 + 302^3 = 378^3 + 93^3$
120	-629	393	–	$121861441 = 393^3 + 394^3 = 513^3 + (-236)^3$
120	-629	–	411	$139361059 = 411^3 + 412^3 = 531^3 + (-218)^3$
393	-1178	438	–	$168632191 = 438^3 + 439^3 = 831^3 + (-740)^3$
152	-793	481	–	$223264809 = 481^3 + 482^3 = 633^3 + (-312)^3$
128	-511	–	490	$236019771 = 490^3 + 491^3 = 618^3 + (-21)^3$
3225	-4274	528	–	$295233841 = 528^3 + 529^3 = 3753^3 + (-3746)^3$
148	-687	–	562	$355957875 = 562^3 + 563^3 = 710^3 + (-125)^3$
2258	-3367	562	–	$355957875 = 562^3 + 563^3 = 2820^3 + (-2805)^3$
512	-1591	607	–	$448404255 = 607^3 + 608^3 = 1119^3 + (-984)^3$
777	-1952	633	–	$508476241 = 633^3 + 634^3 = 1410^3 + (-1319)^3$
190	-999	–	640	$525518721 = 640^3 + 641^3 = 830^3 + (-359)^3$
442	-1767	804	–	$1041378589 = 804^3 + 805^3 = 1246^3 + (-963)^3$

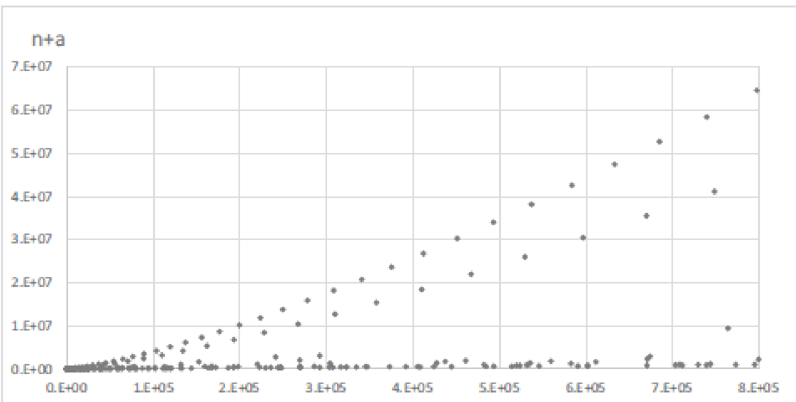

Figure 1. Plot $n + a$ vs n

Table 2. Values of n, $n+a$, $n+b$, $\eta = 2n+a+b$ for first and second families

i	First family				Second family			
	n	$n+a$	$n+b$	η	n	$n+a$	$n+b$	η
1	3	6	-5	1	4	6	-3	3
2	46	151	-148	3	121	369	-360	9
3	197	1016	-1011	5	562	2820	-2805	15
4	528	3753	-3746	7	1543	10815	-10794	21
5	1111	10090	-10081	9	3280	29538	-29511	27
6	2018	22331	-22320	11	5989	65901	-65868	33
7	3321	43356	-43343	13	9886	128544	-128505	39
8	5092	76621	-76606	15	15187	227835	-227790	45
9	7403	126158	-126141	17	22108	375870	-375819	51
10	10326	196575	-196556	19	30865	586473	-586416	57
11	13933	293056	-293035	21	41674	875196	-875133	63
12	18296	421361	-421338	23	54751	1259319	-1259250	69
13	23487	587826	-587801	25	70312	1757850	-1757775	75
14	29578	799363	-799336	27	88573	2391525	-2391444	81
15	36641	1063460	-1063431	29	109750	3182808	-3182721	87
16	44748	1388181	-1388150	31	134059	4155891	-4155798	93
17	53971	1782166	-1782133	33	161716	5336694	-5336595	99
18	64382	2254631	-2254596	35	192937	6752865	-6752760	105
19	76053	2815368	-2815331	37	227938	8433780	-8433669	111
20	89056	3474745	-3474706	39	266935	10410543	-10410426	117

with $\kappa = 72$ and 216 and $\lambda = 576\,(i-2)$ and $1728\,(i-2)$ for respectively the first and second families, and the first three values of n_i, $(n+a)_i$ and $(n+b)_i$ from Table 2.

3.2. Parametric relations

We see from Table 2 that the fourth term $n+b$ of (8) is negative and is decreasing regularly with increasing n. So, let us pose $n+b = -(n+a)+\beta$, yielding from (8)

$$N = n^3 + (n+1)^3 = (n+a)^3 - (n+a-\beta)^3 \tag{15}$$

For specific relations between a and n, one obtains two infinite families of solutions as shown in the following two theorems, giving parametric solutions for n, N, $n+a$ and $n+b$.

Theorem 1. *For $\forall i \in \mathbb{Z}_0^+$, $\exists n, a, \beta \in \mathbb{Z}_0^+$, such that*

$$a = (\beta - 1)\,n + \beta^2 + \beta + 1 \tag{16}$$

and an infinite family of solutions of (15) exists for β odd,

$$\beta = 2i - 1 \tag{17}$$

yielding

$$n = \frac{(2i-1)\left(3\,(2i-1)^2+4\right)-1}{2} \tag{18}$$

$$N = \frac{(2i-1)\left(3\,(2i-1)^2+4\right)\left((2i-1)^2\left(3\,(2i-1)^2+4\right)^2+3\right)}{4} \tag{19}$$

$$n+a = \frac{3\,(2i-1)^2\left((2i-1)^2+2\right)+2i+1}{2} \tag{20}$$

$$n+b = -\frac{3\,(2i-1)^2\left((2i-1)^2+2\right)-2i+3}{2} \tag{21}$$

Proof. Let $n, a, \beta, i \in \mathbb{Z}_0^+$, and let a, n and β satisfy (16). Relation (15) yields then the third degree equation

$$n^3 + (n+1)^3 - \left(\beta n + \beta^2 + \beta + 1\right)^3 + \left(\beta n + \beta^2 + 1\right)^3 = 0 \tag{22}$$

that simplifies immediately in the product of a linear and a quadratic relations

$$\left(2n - \beta\left(3\beta^2 + 4\right) + 1\right)\left(n^2 + (2\beta+1)\,n + \beta^2 + \beta + 1\right) = 0 \tag{23}$$

As the discriminant of the right quadratic polynomial is always negative, the quadratic equation yields two complex solutions of no interest here. The right linear equation yield the only real solution

$$n = \frac{\beta\left(3\beta^2 + 4\right) + 1}{2} \tag{24}$$

As n must be integer, β cannot be even and must be odd, $\beta = 2i - 1$, yielding (18) to (21). □

Theorem 2. *For* $\forall i \in \mathbb{Z}_0^+$, $\exists n, a, \beta \in \mathbb{Z}_0^+$, *such that*

$$a = \frac{(\beta - 3)\,n + 2\beta}{3} \tag{25}$$

and an infinite family of solutions of (15) exists for $\beta \equiv 3 \mathrm{mod} 6$,

$$\beta = 3\,(2i - 1) \tag{26}$$

yielding

$$n = \frac{9\,(2i-1)^3 - 1}{2} \tag{27}$$

$$N = \frac{27\,(2i-1)^3\left(27\,(2i-1)^6 + 1\right)}{4} \tag{28}$$

$$n + a = \frac{3\,(2i-1)\left(3\,(2i-1)^3 + 1\right)}{2} \tag{29}$$

$$n + b = -\frac{3\,(2i-1)\left(3\,(2i-1)^3 - 1\right)}{2} \tag{30}$$

Proof. Let $n, a, \beta, i \in \mathbb{Z}_0^+$, and let a, n and β satisfy (25). Relation (15) yields then the third degree equation

$$n^3 + (n+1)^3 - \left(\frac{\beta}{3}\,(n+2)\right)^3 + \left(\frac{\beta}{3}\,(n-1)\right)^3 = 0 \tag{31}$$

that simplifies immediately in the product of a linear and a quadratic relations

$$\frac{\left(6n - \beta^3 + 3\right)\left(n^2 + n + 1\right)}{3} = 0 \tag{32}$$

The right quadratic equation yields two complex solutions of no interest here. The right linear equation yield the only real solution

$$n = \frac{\beta^3 - 3}{6} \tag{33}$$

As n must be integer, β must 3mod6, $\beta = 3\,(2i-1)$, yielding (27) to (30). □

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