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# On the semilocal convergence analysis of a seventh order four step method for solving nonlinear equations

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**Abstract:** We provide a semi-local convergence analysis of a seventh order four step method for solving nonlinear problems. Using majorizing sequences and under conditions on the first derivative, we provide sufficient convergence criteria, error bounds on the distances involved and uniqueness. Earlier convergence results have used the eighth derivative not on this method to show convergence. Hence, limiting its applicability.

**Keywords:** Banach space; convergence order; Iterative method.

**MSC:** 49M15, 65J15, 65G99.

## 1. Introduction and Preliminaries

In this study we are interested in finding an approximation for the solution $\bar{x}$ of the equation

$$F : D \subset X \to Y \quad F(\overline{x}) = 0 \tag{1}$$

where $X$ and $Y$ are Banach spaces and $D$ is an open subset of $X$. Seventh order method defined for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned}
\overline{y}_n &= \overline{x}_n - \Omega F'(\overline{x}_n)^{-1} F(\overline{x}_n) \\
\overline{z}_n &= \overline{x}_n - F'(\overline{y}_n)^{-1} F(\overline{x}_n), \\
\overline{w}_n &= \overline{z}_n - \left(2F'(\overline{y}_n)^{-1} - F'(\overline{x}_n)^{-1}\right) F(\overline{z}_n) \\
\overline{x}_{n+1} &= \overline{w}_n - \left(2F'(\overline{y}_n)^{-1} - F'(\overline{x}_n)^{-1}\right) F(\overline{w}_n)
\end{aligned} \tag{2}$$

is considered for approximating $\bar{x}$.

In this paper we study the semi-local convergence. Moreover, we use condition only on the first derivative appearing on (2). Hence, we extend its applicability. The local convergence of this method was shown [1] using conditions reaching the fifth derivative which is not on (2).

But these restrictions limit the applicability of the method (2) although it may converge.

For example: Let $X = Y = \mathbb{R}$, $D = [-0.5, 1.5]$. Define $\Psi$ on $D$ by

$$\Psi(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if \ t \neq 0 \\ 0 & if \ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\Psi'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\Psi'''(t)$ is not bounded on $D$, so the analysis in [1] cannot guarantee convergence. In this paper we examine the more interesting semi-local case using conditions only on the first derivative which is on method (2). Hence, we extend the applicability of this method.

The analysis is given in Section 2 and the examples in Section 3.

## 2. Convergence

Let $K_0, K, K_1$ and $\delta$ be positive parameters. Define scalar sequences by
$x_0 = 0, y_0 = \delta$

$$\begin{aligned} z_n &= y_n + \left(\frac{K_1 K(y_n - x_n)}{(1-K_0 x_n)(1-K_0 y_n)} + \left|\frac{\Omega - 1}{\Omega}\right|\right)(y_n - x_n) \\ w_n &= z_n + \left(1 + \frac{K(y_n - x_n)}{1-K_0 x_n}\right)\frac{p_n}{1-K_0 y_n} \\ x_{n+1} &= w_n + \left(1 + \frac{K(y_n - x_n)}{1-K_0 x_n}\right)\frac{q_n}{1-K_0 y_n} \\ y_{n+1} &= x_{n+1} + \frac{K(x_{n+1}-x_n)^2 + 2K_1(x_{n+1}-y_n) + 2K_1|1-\frac{1}{\Omega}|(y_n - x_n)}{2(1-K_0 x_{n+1})}, \end{aligned} \tag{1}$$

where

$$p_n = K_1\left(z_n - y_n + \left(1 - \frac{1}{|\Omega|}\right)(y_n - x_n)\right)$$

and

$$q_n = K_1\left(w_n - z_n + z_n - y_n + \left(1 - \frac{1}{|\Omega|}\right)(y_n - x_n)\right).$$

Next, a sufficient convergence criterion is presented for these sequences.

**Lemma 1.** *Suppose*

$$K_0\, y_n < 1, K_0\, x_{n+1} < 1 \tag{2}$$

*for each* $n = 0, 1, 2, \cdots$ *. Then, the following assertions hold*

$$x_n \le y_n \le z_n \le w_n \le x_{n+1} \le y_{n+1} < \frac{1}{K_0} \tag{3}$$

*and*

$$\lim_{n\to\infty} y_n = y^\star \in \left[0, \frac{1}{K_0}\right]. \tag{4}$$

**Proof.** Using definition (1) and condition (2) we deduce that (3). So, sequence $\{y_n\}$ is non decreasing and bounded from above by $\frac{1}{K_0}$. Hence, it converges to its unique least upper bound $y^\star$.

□

The semilocal convergence of method (2) uses conditions $(H)$ : Suppose:

$(H_1)$ There exists $\overline{x}_0 \in D, \delta \ge 0$ such that $F'(\overline{x}_0)^{-1} \in \mathcal{L}(Y, X)$ and $\|F'(\overline{x}_0)^{-1} F(\overline{x}_0)\||\Omega| \le \delta$.
$(H_2)$ $\|F'(\overline{x}_0)^{-1}(F'(\overline{v}) - F'(\overline{x}_0))\| \le K_0\|\overline{v} - \overline{x}_0\|$ for each $\overline{v} \in D$ for some $K_0 > 0$.

Consider $D_1 = U\left(\overline{x}_0, \frac{1}{K_0}\right) \cap D$.

$(H_3)$ $\|F'(\overline{x}_0)^{-1}(F'(\overline{v}) - F'(\overline{w}))\| \le K\|\overline{v} - \overline{w}\|$ and $\|F'(\overline{x}_0)^{-1} F'(\overline{v})\| \le K_1$ for all $\overline{v} \in D_0$ and $\overline{w} = \overline{v} - F'(\overline{v})^{-1} F(\overline{v})$.
$(H_4)$ Conditions of Lemma 1 hold,

and

$(H_5)$ $U[\overline{x}_0, y^\star] \subset D$.

Next, we present the semilocal convergence result for method (2).

**Theorem 1.** *Suppose conditions $(H)$ hold. Then, iterates $\overline{x}_n, \overline{y}_n, \overline{z}_n, \overline{w}_n, \overline{x}_{n+1}$ are well defined, belong in $U[\overline{x}_0, y^\star]$ and converge to a solution $\overline{x}^\star \in U[x_0, y^\star]$ of equation $F(x) = 0$. Moreover, the following error estimate holds*

$$\|\overline{x}^\star - \overline{x}_m\| \leq y^\star - x_m. \tag{5}$$

**Proof.** Mathematical induction on $m$ is utilized to show assertions

$$\begin{aligned} \|\overline{y}_m - \overline{x}_m\| &\leq y_m - x_m, & (6)\\ \|\overline{z}_m - \overline{y}_m\| &\leq z_m - y_m, \|\overline{w}_m - \overline{z}_m\| \leq w_m - z_m & (7)\\ &\textit{and} \\ \|\overline{x}_{m+1} - \overline{w}_m\| &\leq x_{m+1} - w_m. & (8)\end{aligned}$$

In view of condition $(H_1)$, we have

$$\|\overline{y}_0 - \overline{x}_0\| \leq \delta = y_0 - x_0 \leq y^\star,$$

so $\overline{y}_0 \in U[\overline{x}_0, y^\star]$ and (6) holds for $m = 0$.

Consider $b \in U(\overline{x}_0, y^\star)$. Then, by condition $(H_2)$, we get

$$\|F'(\overline{x}_0)^{-1}(F'(b) - F(\overline{x}_0))\| \leq K_0\|b - \overline{x}_0\| \leq K_0 y^\star < 1. \tag{9}$$

By (9) and a lemma on linear operators with inverses attributed to Banach [2–10] it follows $F'(b)^{-1} \in \mathcal{L}(Y, X)$ and

$$\|F'(b)^{-1} F'(\overline{x}_0)\| \leq \frac{1}{1 - K_0\|b - \overline{x}_0\|}. \tag{10}$$

Iterates $\overline{z}_0, \overline{w}_0, \overline{x}$ are well defined by the second substep of method (2) and (10) for $b = y_0$, since $\overline{y}_0 \in U(\overline{x}_0, y^\star)$. Suppose estimates $(6) - (8)$ hold for all values of $m$ smaller or equal to $n$. Then, we obtain by method (2) and the induction hypotheses in turn that

$$\begin{aligned} \overline{z}_m &= \overline{x}_m - \Omega F'(\overline{x}_m)^{-1}F(\overline{x}_m) + \Omega F'(\overline{x}_m)^{-1}F(\overline{x}_m) - F'(\overline{y}_m)^{-1}F(\overline{x}_m) \\ &= \overline{y}_m + F'(\overline{x}_m)^{-1}(F'(\overline{y}_m) - F'(\overline{x}_m))\, F'(\overline{y}_m)^{-1}F'(\overline{y}_m)^{-1}F(\overline{x}_m) \\ &\quad + |\Omega - 1| F'(\overline{x}_m)^{-1}F(\overline{x}_m) \\ \|\overline{z}_m - \overline{y}_m\| &\leq \|F'(\overline{x}_m)^{-1} F'(\overline{x}_0)\| \|F'(\overline{x}_0)^{-1}(F'(\overline{y}_m) - F'(\overline{x}_m))\| \\ &\quad \times \|F'(\overline{y}_m)^{-1} F'(\overline{x}_0)\| \|F'(\overline{x}_0)^{-1}F(\overline{x}_m)\| \\ &\quad + |\Omega - 1| \|F'(x_m)^{-1} F(\overline{x}_m)\| \\ &\leq \frac{K_1 K\|\overline{y}_m - \overline{x}_m\|^2}{(1 - K_0\|\overline{x}_m - \overline{x}_0\|)(1 - K_0\|\overline{y}_m - \overline{x}_0\|)} + |1 - \frac{1}{\Omega}| \|\overline{y}_m - \overline{x}_m\| \\ &\leq z_m - y_m, \end{aligned}$$

and

$$\|\overline{z}_m - x_0\| \leq \|\overline{z}_m - \overline{y}_m\| + \|\overline{y}_m - x_0\| \leq z_m - y_m + y_m - x_0 = z_m \leq y^\star.$$

So $\overline{z}_m \in U[\overline{x}_0, y^\star]$ and (6) hold.

Define

$$\begin{aligned} A_m &= (F(\overline{z}_m) - F(\overline{y}_m)) + (F(\overline{y}_m) - F(\overline{x}_m)) + F(\overline{x}_m) \\ &= \int_0^1 F'(\overline{y}_m + \theta(\overline{z}_m - \overline{y}_m)) d\theta (\overline{z}_m - \overline{y}_m) \\ &\quad + \int_0^1 F'(\overline{x}_m + \theta(\overline{y}_m - \overline{x}_m)) d\theta (\overline{y}_m - \overline{x}_m) \\ &\quad - \frac{1}{\Omega} F'(\overline{x}_m)(\overline{y}_m - \overline{x}_m). \end{aligned} \tag{11}$$

Then, by induction hypotheses, $(H_3)$ and (11), we get

$$\begin{aligned} \|F'(\overline{x}_0)^{-1} A_m\| &\leq K_1\left(\|\overline{z}_m - \overline{y}_m\| + \|\overline{y}_m - \overline{x}_m\| + \frac{1}{|\Omega|}\|\overline{y}_m - \overline{x}_m\|\right) \\ &\leq K_1\left(z_m - y_m + y_m - x_m + \frac{1}{|\Omega|}(y_m - x_m)\right) = p_m. \end{aligned} \tag{12}$$

Then, by the third substep of method (2) we can write

$$\overline{w}_m - \overline{z}_m = -F'(\overline{y}_m)^{-1}A_m - F'(\overline{y}_m)^{-1}(F'(\overline{x}_m) - F'(\overline{y}_m))F'(\overline{x}_m)^{-1}A_m. \tag{13}$$

In view of (1), (10), (12) and (13), we have in turn that

$$\begin{aligned} \|\overline{w}_m - \overline{z}_m\| &\leq \frac{p_m}{1 - K_0\|\overline{y}_m - \overline{x}_0\|} + \frac{K\|\overline{y}_m - \overline{x}_m\|p_m}{(1 - K_0\|\overline{y}_m - \overline{x}_0\|)(1 - K_0\|\overline{x}_m - \overline{x}_0\|)} \\ &\leq w_m - z_m, \end{aligned}$$

and

$$\begin{aligned} \|\overline{w}_m - \overline{x}_0\| &\leq \|\overline{w}_m - \overline{z}_m\| + \|\overline{z}_m - \overline{y}_m\| + \|\overline{y}_m - \overline{x}_0\| \\ &\leq w_m - z_m + z_m - y_m + y_m - x_0 = w_m \leq y^\star, \end{aligned}$$

so $\overline{w}_m \in U[\overline{x}_0, y^\star]$ and (7) holds.

Define

$$\begin{aligned} B_m &= (F(\overline{w}_m) - F(\overline{z}_m)) + (F(\overline{z}_m) - F(\overline{y}_m)) + (F(\overline{y}_m) - F(\overline{x}_m)) + F(\overline{x}_m) \\ &= \int_0^1 F'(\overline{z}_m + \theta(\overline{w}_m - \overline{z}_m))d\theta(\overline{w}_m - \overline{z}_m) + \int_0^1 F'(\overline{y}_m + \theta(\overline{z}_m - \overline{y}_m))d\theta \\ &\quad + \int_0^1 F'(\overline{x}_m + \theta(\overline{y}_m - \overline{x}_m))d\theta(\overline{y}_m - \overline{x}_m) - \frac{1}{\Omega}F'(\overline{x})(\overline{y_m} - \overline{x_m}). \end{aligned} \tag{14}$$

So

$$\begin{aligned} \|F'(\overline{x}_0)^{-1}B_m\| &\leq K_1\left(\|\overline{w}_m - \overline{z}_m\| + \|\overline{z}_m - \overline{y}_m\| + \|\overline{y}_m - \overline{x}_m\| + \frac{1}{|\Omega|}\|\overline{y}_m - \overline{x}_m\|\right). \\ &\leq K_1\left(w_m - z_m + z_m - y_m + \left(1 + \frac{1}{|\Omega|}\right)(y_m - x_m)\right) = q_m. \end{aligned} \tag{15}$$

By the third substep of method (2), we can write

$$\overline{x}_{m+1} - \overline{w}_m = -F'(\overline{y}_m)^{-1}B_m - F'(\overline{y}_m)^{-1}(F'(\overline{x}_m) - F'(\overline{y}_m))\, F'(\overline{x}_m)^{-1}. \tag{16}$$

Hence we get

$$\begin{aligned} \|\overline{x}_{m+1} - \overline{w}_m\| &\leq \frac{q_k}{1 - K_0\|\overline{y}_m - \overline{x}_0\|} + \frac{Kq_m\|\overline{y}_m - \overline{x}_m\|}{(1 - K_0\|\overline{y}_m - \overline{x}_0\|)(1 - K_0\|\overline{x}_m - \overline{x}_0\|)} \\ &\leq x_{m+1} - w_m, \end{aligned}$$

and

$$\begin{aligned} \|\overline{x}_{m+1} - \overline{x}_0\| &\leq \|\overline{x}_{m+1} - \overline{w}_m\| + \|\overline{w}_m - \overline{z}_m\| + \|\overline{z}_m - \overline{y}_m\| + \|\overline{y}_m - \overline{x}_0\| \\ &\leq x_{m+1} - w_m + w_m - z_m + z_m - y_m + y_m - x_0 = x_{m+1} \leq y^\star, \end{aligned}$$

so $\bar{x}_{m+1} \in U[\bar{x}_0, y^\star]$ and (8) holds. We can write in turn by the first substep of method (2)

$$\begin{aligned}
F(\bar{x}_{m+1}) &= F(\bar{x}_{m+1}) - F(\bar{x}_m) - \frac{1}{\Omega}F'(x_m)(\bar{y}_m - \bar{x}_m) \\
&= (F(\bar{x}_{m+1}) - F(\bar{x}_m) - F'(\bar{x}_m)(\bar{x}_{m+1}) - \bar{x}_m)) \\
&\quad + F'(\bar{x}_m)(\bar{x}_{m+1} - \bar{x}_m) - \frac{1}{\Omega}F'(\bar{x}_m)(\bar{y}_m - \bar{x}_m) \\
&= (F(\bar{x}_{m+1}) - F(\bar{x}_m) - F'(\bar{x}_m)(\bar{x}_{m+1}) - \bar{x}_m)) \\
&\quad + F'(\bar{x}_m)(\bar{x}_{m+1} - \bar{x}_m) + \left(1 - \frac{1}{\Omega}\right)F'(\bar{x}_m)(\bar{y}_m - \bar{x}_m) \\
\|F'(\bar{x}_0)F(x_{m+1})\| &\leq \frac{K}{2}\|\bar{x}_{m+1} - \bar{x}_m\|^2 + K_1\|\bar{x}_{m+1} - \bar{y}_m\| \\
&\quad + |1 - \frac{1}{\Omega}|K_1\|\bar{y}_m - x_m\| \\
&\leq \frac{K}{2}(x_{m+1} - x_m)^2 + K_1(x_{m+1} - y_m) \\
&\quad + |1 - \frac{1}{\Omega}|K_1(y_m - x_m), \qquad (17)
\end{aligned}$$

so

$$\begin{aligned}
\|\bar{y}_{m+1} - \bar{x}_{m+1}\| &\leq \|F'(\bar{x}_{m+1})^{-1}F'(\bar{x}_0)\|\|F'(\bar{x}_0)^{-1}F(\bar{x}_{m+1})\| \\
&\leq \frac{\|F'(\bar{x}_0)^{-1}F(\bar{x}_{m+1})\|}{1 - K_0\|\bar{x}_{m+1} - x_0\|} \leq \frac{\|F'(\bar{x}_0)^{-1}F(\bar{x}_{m+1})\|}{1 - K_0 x_{m+1}} \leq y_{m+1} - x_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{y}_{m+1} - x_0\| &\leq \|\bar{y}_{m+1} - \bar{x}_{m+1}\| + \|\bar{x}_{m+1} - \bar{x}_0\| \\
&\leq y_{m+1} - x_{m+1} + x_{m+1} - x_0 = y_{m+1} \leq y^\star,
\end{aligned}$$

so $\bar{y}_{m+1} \in U[x_0, y^\star]$ and (6) hold. By letting $m \to \infty$ in 17 and using the continuity of $F$ we deduce $F(\bar{x}^\star) = 0$. Finally, to show (2.5), let $i$ be an integer. We can write

$$\|\bar{x}_{m+i} - \bar{x}_m\| \leq x_{m+i} - x_m. \qquad (18)$$

Then, by letting $i \to \infty$, we conclude (5). □

**Proposition 1.** *Suppose that there exists a simple solution $x^* \in U(\bar{x}_0, \rho_0) \subset D$ of equation $F(x) = 0$, and (H3) holds. Set $D_2 = U(x^*, \rho) \cap D$. Moreover, suppose there exist $\rho \geq \rho_0$ such that $\frac{K_0}{2}(\rho_0 + \rho) < 1$. Then, the element $x^*$ is the only solution of equation $F(x) = 0$ in the region $D_2$.*

**Proof.** Consider $\tilde{x} \in D_2$ with $F(\tilde{x}) = 0$. Set $Q = \int_0^1 F'(x^* + \theta(\tilde{x} - x^*))d\theta$. Then, by (H2)

$$\|F'(\bar{x}_0)^{-1}(Q - F'(x^*))\| \leq \ell_0 \int_0^1 [\theta\|\tilde{x} - \bar{x}_0\| + (1-\theta)\|x^* - \bar{x}_0\|]d\theta \leq \frac{\ell_0}{2}(\rho_0 + \rho) < 1.$$

Hence, $\tilde{x} = x^*$ is implied by the inverse of $Q$ and the approximation $Q(\tilde{x} - x^*) = F(\tilde{x}) - F(x^*) = 0 - 0 = 0$. □

**Remark 1.** (1) Condition $(H_3)$ can be replaced by stronger
$(H_3)'$ $\|F'(\bar{x}_0)^{-1}(F'(\bar{v}) - F'(\bar{w}))\| \leq \tilde{K}\|\bar{v} - \bar{w}\|$ for each $\bar{v}, \bar{w} \in D_1$.
or even stronger
$(H_3)''$ $\|F'(\bar{x}_0)^{-1}(F'(\bar{v}) - F'(\bar{w}))\| \leq \tilde{\tilde{K}}\|\bar{v} - \bar{w}\|$ for each $\bar{v}, \bar{w} \in D$.
Notice however that since

$$D_1 \subseteq D, \qquad (19)$$

we have

$$K \leq \tilde{K} \leq \tilde{\tilde{K}} \quad \textit{and} \quad K_0 \leq \tilde{\tilde{K}}. \qquad (20)$$

Similar observations can be made for the second condition in $(H_3)$.

(2) Condition $(H_5)$ can be replaced by

$(H_5)'$ $U\left[x_0, \frac{1}{K_0}\right]$, since $\frac{1}{K_0}$ is obviously in closed form.

(3) Lipschitz constants can be smaller if we define $S = U\left(\bar{y}_0, \frac{1}{K_0} - \delta\right)$ provided that $K_0\,\delta < 1$. Moreover, suppose that $S \subset D$, then we have $S \subset D_1$.

Hence, the Lipschitz constants on $S$ are at least as tight. Notice that we are still using initial data, since $\bar{y}_0 = \bar{x}_0 - \Omega F'(\bar{x}_0)^{-1} F(\bar{x}_0)$.

**Example 1.** Defined the real function $f$ on $D = B[x_0, 1-w]$, $x_0 = 1$, $w \in (0,1)$ by

$$f(s) = s^3 - w.$$

Then, the definitions are satisfied for $\Omega = 1$, $\delta = \frac{1-w}{3}$, $K_0 = 3-w$, $K_1 = 2$, $K = 2(1+\frac{1}{1-w})$. Then for $w = 0.98$, we have

**Table 1.** Sequence (1) and condition (2)

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $x_{n+1}$ | 0.0092 | 0.0162 | 0.0205 | 0.0224 | 0.0228 | 0.0228 |
| $y_n$ | 0.0067 | 0.0145 | 0.0197 | 0.0222 | 0.0227 | 0.0228 |
| $K_0 y_n$ | 0.0067 | 0.0145 | 0.0197 | 0.0222 | 0.0227 | 0.0228 |
| $K_0 x_{n+1}$ | 0.0186 | 0.0327 | 0.0415 | 0.0452 | 0.0460 | 0.0460 |

Hence, the conditions of Lemma 1 hold.

## 3. Conclusion

The technique of recurrent functions has been utilized to extend the application of method (2). The convergence uses conditions on the derivative of the method and not the eighth derivative as in earlier studies. The technique is very general rendering it useful to extend the usage of other iterative methods [11–20].

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