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Some new results on w Up-algebras

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Abstract: The concept of weak UP-algebras (shortly wUP-algebra) is an extension of the notion of UP-algebras introduced in 2021 by Iampan and Romano. In this report, an effective extension of a (weak) UP-algebra to a wUP-algebra is created. In addition to the previous one, the concept of atoms in wUP-algebras is introduced and their important properties are registered. Finally, the concept of wUP-filters in wUP-algebras was introduced and its connections with other substructures in wUP-algebras were analyzed.

Keywords: UP-algebra, weak UP-algebra (shortly wUP-algebra), extension of a (weak) UP-algebra to a wUP-algebra, atoms in wUP-algebras, wUP-filters.

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1. Introduction and Preliminaries

In 1996, K. Iséki introduced the concept of BCI-algebras as algebraic structures intertwined with specific logics. Subsequently, in 1984, Y. Komori introduced another variant of these algebras, now termed BCC-algebras, to address certain challenges within the domain of BCK-algebras. BCK-algebras are closely associated with BCK logic, while BCC-algebras serve as an algebraic model for BCC-logic. An extension of BCC-algebras is known as weak BCC-algebras, also referred to as BZ-algebras.

The notion of UP-algebras was pioneered by A. Iampan in 2017, as documented in [1]. Building upon this groundwork, the concept of wUP-algebras was further delineated in [2] by A. Iampan and D. A. Romano. This extension of UP-algebras involves the relaxation of one of its axioms, thus forming a distinctive algebraic structure.

This paper constitutes a continuation of the inquiry into wUP-algebras initiated in [2]. It commences by devising an efficient methodology for extending a UP-algebra to a wUP-algebra, followed by a systematic enlargement of the latter to accommodate an additional element. Subsequently, the notion of atoms within wUP-algebras is introduced, elucidating key properties inherent in this phenomenon. Lastly, the concept of wUP-filters in wUP-algebras is introduced, accompanied by an exploration of its interconnections with other substructures within wUP-algebras.

2. Preliminaries

In this section, preliminary terms and processes with them (assertions) necessary for easier understanding of the presented material in Section 3, which is the main part of this report, are stated.

The concept of KU-algebras was introduced in 2009 in the paper [3] written by C. Prabpayak and U. Leerawat.

Definition 1. ([3]) An algebra $A =: (A, \cdot, 0)$ of type $(2, 0)$ is called a KU-algebra if it satisfies the following conditions:

(KU-1) $(\forall x, y, z \in A)((y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0)$,
(KU-2) $(\forall x \in A)(0 \cdot x = x)$,
(KU-3) $(\forall x \in A)(x \cdot 0 = 0)$,
(KU-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

The concept of UP-algebras was introduced in 2017 in the paper [1] by A. Iampan as a generalization of the notion of KU-algebras.

Definition 2. ([1]) An algebra $A =: (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following conditions:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$,
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

In [1] it was shown that every KU-algebra is a UP-algebra and that the converse need not hold.

Example 1. Let $A = \{0, 1, 2, 3, 4\}$ and define binary operation $\cdot$ as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	2
4	0	2	2	4	0

Then $(A, \cdot, 1)$ is a UP-algebra but not a KU-algebra.

The concept of weak UP-algebra was introduced in 2021 by A. Iampan and D. A. Romano in the paper [2].

Definition 3. ([2], Definition 2.2) An algebra $A =: (A, \cdot, 0)$ of type $(2, 0)$ is called a weak UP-algebra if it satisfies the following axioms:

(wUP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(wUP-2) $(\forall x \in A)(0 \cdot x = x)$, and
(wUP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

We denote this axiom system by **wUP** and we denote this type of algebraic structure by wUP-algebra.

A non-empty subset S of a wUP-algebra A is called a wUP-subalgebra of A if the following holds
(S) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x \cdot y \in S)$.

Example 2. Let $A = \{0, 1, 2\}$ and the binary operation be defined as

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	2	2	0

Routine calculations show that A is a wUP-algebra but not a UP-algebra because $2 \cdot 0 = 2 \neq 0$.

Example 3. Let $A = \{0, a, b, c, d, e\}$ and the binary operation be defined as

$\cdot$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	b	d	e
b	0	a	0	a	d	d
c	0	0	0	0	d	d
d	d	d	d	d	0	b
e	d	d	d	d	0	0

Routine calculations show that A is a wUP-algebra but not a UP-algebra because $e \cdot 0 = d \neq 0$ for example.

Remark 1. If a wUP-algebra $(A, \cdot, 0)$ satisfies also (UP-3), then it is a UP-algebra. Therefore, the concept of wUP-algebras is a generalization of the concept of UP-algebras since the axiom (UP-3) is independent of the other axioms in **wUP**.

Similarly as in UP-algebras, in any wUP-algebra A we can introduce a natural relation '$\leqslant$' putting

$$(\forall\, x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

It is not difficult to see that in wUP-algebras the following claims are valid:

Lemma 1 ([2], Lemma 2.1)**.** *In a wUP-algebra A, the following properties hold:*

(1) $(\forall\, x \in A)(x \leqslant x)$,
(2) $(\forall\, x, y, z \in A)((x \leqslant y \ \wedge y \leqslant z) \implies x \leqslant z)$,
(3) $(\forall\, x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$, *and*
(4) $(\forall\, x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$.

The map $\varphi : A \longrightarrow A$, defined by

(5) $(\forall\, x \in A)(\varphi(x) =: x \cdot 0)$,

was formally introduced in Dudek and Thomys, 1990 for BCH-algebras, but, in fact, different properties of this map were used in Dudek and Thomys ([4]), Dudek and Thomys ([5]), and Romano ([6]) to characterizations of special subclasses of weak BCC-algebras, and Romano ([7]) in describing the properties of JU-algebras. Some important features of this mapping in wUP-algebras are given in the following proposition.

For $x = 0$ we have $\varphi(0) = 0$ according to (1).

Proposition 1 ([2], Proposition 2.1)**.** *Let A be a JU-algebra. Then*

(6) $(\forall x, y \in A)(y \cdot x \leqslant \varphi(x \cdot y))$,
(7) $(\forall x \in A)(x \leqslant \varphi^2(x))$,
(8) $(\forall x, y \in A)(\varphi(y) \leqslant (x \cdot y) \cdot \varphi(x))$,
(9) $(\forall x, y \in A)(x \leqslant y \implies \varphi(y) \leqslant \varphi(x))$,
(10) $(\forall x, y \in A)(y \leqslant \varphi(x) \cdot (x \cdot y))$,
(11) $(\forall x \in A)(\varphi^3(x) = \varphi(x))$,
(12) $(\forall x, y \in A)(\varphi(x) = (y \cdot x) \cdot \varphi(y))$,
(12a) $(\forall x, t \in A)(y \leqslant x \implies \varphi(x) = \varphi(y))$,
(13) $(\forall x, y \in A)(\varphi^2(y) = \varphi(x) \cdot \varphi(y \cdot x))$,
(14) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi^2(x) \cdot \varphi^2(y))$, *and*
(15) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi(y \cdot x))$.

It is easy to deduce the properties of the subset $Ker(\varphi) = \{x \in A : \varphi(x) = 0\}$.

Lemma 2. *Let A be a wUP-algebra. Then the set $Ker(\varphi)$ is a subalgebra in A.*

Proof. Let $x, y \in A$ be such that $x \in Ker(\varphi)$ and $y \in Ker(\varphi)$. This means $\varphi(x) = 0$ and $\varphi(y) = 0$. Then $\varphi^2(x) = \varphi(0) = 0$ and $\varphi^2(y) = \varphi(0) = 0$, Thus $\varphi(x \cdot y) = \varphi^2(y) \cdot \varphi^2(x) = 0 \cdot 0 = 0$ according to (15). Hence, $x \cdot y \in Ker(\varphi)$. $\square$

Remark 2. According to the result of the previous lemma, we conclude that every wUP-algebra has at least one UP-subalgebra.

Definition 4. A subset J of a wUP-algebra A is called a wUP-ideal of A if

(J1) $0 \in J$, and
(J2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \implies x \cdot z \in J)$.

Example 4. Let A be as in Example 2. The subsets $J_1 = \{0, 1\}$ and $J_2 = \{0, 2\}$ are not-trivial wUP-ideals in A.

Proposition 2 ([2], Proposition 3.1, Corillary 3.1)**.** *Let J be a wUP-ideal of a wUP-algebra A. Then*

(J3) $(\forall y, z \in A)((y \cdot z \in J \wedge y \in J) \implies z \in J)$.
(J4) $(\forall x, z \in J)((\varphi(x) \in J \wedge y \in J) \implies x \cdot y \in J)$.
(J5) $(\forall x, y \in A)((x \cdot \varphi(y) \in J \wedge y \in J) \implies \varphi(x) \in J)$.
(J6) $(\forall y, z \in A)((y \leqslant z \in J \wedge y \in J) \implies z \in J)$.

3. The main results

This section is the main part of this report, It consists of three subsections. In the first subsection, an effective extension of the UP-algebra to a wUP-algebra is created. Also, in this subsection, an effective extension of wUP-algebra to wUP-algebra is designed by adding one element to the first one. The second subsection deals with the concept of atoms in the class of wUP-algebras. In the third subsection, the concept of wUP-filter in wUP-algebras is introduced and some of its basic properties are registered.

3.1. Extension of a (weak) UP-algebra to a wUP-algebra

The following theorem shows how one can design an effective extension of the UP-algebra to the wUP-algebra

Theorem 1. *Any UP-algebra can be extended to a wUP-algebra.*

Proof. Let $(A, \cdot, 0)$ be a UP-algebra and let $a \notin A$. Then $(A \cup \{a\}, *, 0)$ with the operation $*$, defined by

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \text{ and } y \in A, \\ a & \text{for } x \in A \text{ and } y = a, \\ a & \text{for } x = a \text{ and } y \in A, \\ 0 & \text{for } x = a \text{ and } y = a, \end{cases}$$

is a wUP-algebra containing $(A, \cdot, 0)$ since $a * 1 = a \neq 1$. The proof can be demonstrated by direct verification if one, two or all three variables are replaced by the letter a in the **UP** axiom system.

If $x = a$, then we have $(y \cdot z) \cdot ((a \cdot y) \cdot (x \cdot z)) = (y \cdot z) \cdot (a \cdot a) = (y \cdot z) \cdot 0 = 0$.
If $y = a$, then we have $(a \cdot z) \cdot ((x \cdot a) \cdot (x \cdot z)) = a \cdot (a \cdot (x \cdot z)) = a \cdot a = 0$.
If $z = a$, then we have $(y \cdot a) \cdot ((x \cdot y) \cdot (x \cdot a)) = a \cdot ((x \cdot y) \cdot a) = a \cdot a = 0$.
If $x = a$ and $y = a$, then we have $(a \cdot z) \cdot ((a \cdot a) \cdot (a \cdot z)) = a \cdot (0 \cdot a) = a \cdot a = 0$.
If $x = a$ and $z = a$, then we have $(y \cdot a) \cdot ((a \cdot y) \cdot (a \cdot a)) = a \cdot (a \cdot 0) = a \cdot a = 0$.
If $y = a$ and $z = a$, then we have $(a \cdot a) \cdot ((x \cdot a) \cdot (x \cdot a)) = 0 \cdot (a \cdot a) = 0 \cdot 0 = 0$. □

Example 5. Let A be as in Example 1 and let $a \notin A$. Then $B = A \cup \{a\}$ is a wUP-algebra, based on the operation $*$ determined as follows

$*$	0	1	2	3	4	a
0	0	1	2	3	4	a
1	0	0	0	0	0	a
2	0	2	0	0	0	a
3	0	2	2	0	2	a
4	0	2	2	4	0	a
a	a	a	a	a	a	0

and it is not a UP-algebra because $a \cdot 0 = a \neq 0$ is valid. The order in this wUP-algebra is as follows $\leqslant = \{(0,0), (1,1), (2,2), (3,3), (4,4), (a,a), (1,0), (1,2), (1,3), (1,4), (2,0), (2,3), (2,4), (3,0), (4,0)\}$

We can extend the given wUP-algebra by one element to a new wUP-algebra as shown by the following theorem.

Theorem 2. *Any wUP-algebra can be extended to a wUP-algebra containing one element more.*

Proof. Let $(A, \cdot, 0)$ be a wUP-algebra and let $a \notin A$. It is not difficult to see that $A \cup \{a\}$ with the operation $*$, defined by

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ \varphi(x) & \text{for } x \in A \setminus \{0\} \ \wedge \ y = a, \\ y & \text{for } x = a \ \wedge \ y \in A, \\ a & \text{for } x = 0 \ \wedge \ y = a, \\ 0 & \text{for } x = a \ \wedge \ y = a, \end{cases}$$

is a wUP-algebra containing $(A, \cdot, 0)$. The proof can be demonstrated by direct verification if one, two or all three variables are replaced by the letter a in the JU axiom system. For illustration, we show some of those procedures.

If $x = a$, then we have $(y \cdot z) \cdot ((a \cdot y) \cdot (a \cdot z)) = (y \cdot z) \cdot (y \cdot z) = 0$ according to (1).
If $y = a$, then we have $(a \cdot z) \cdot ((x \cdot a) \cdot (x \cdot z)) = z \cdot (\varphi(x) \cdot (x \cdot z)) = 0$ by (10).
If $z = a$, then we have $(y \cdot a) \cdot ((x \cdot y) \cdot (x \cdot a)) = \varphi(y) \cdot ((x \cdot y) \cdot \varphi(x)) = 0$ by (8).
If $x = a$ and $y = a$, then we have $(a \cdot z) \cdot ((a \cdot a) \cdot (a \cdot z)) = z \cdot (0 \cdot z) = z \cdot z = 0$.
If $x = a$ and $z = a$, then we have $(y \cdot a) \cdot ((a \cdot y) \cdot (a \cdot a)) = \varphi(y) \cdot (y \cdot 0) = \varphi(y) \cdot \varphi(y) = 0$.
If $y = a$ and $z = a$, then we have $(a \cdot a) \cdot ((x \cdot a) \cdot (x \cdot a)) = 0 \cdot (\varphi(x) \cdot \varphi(x)) = 0$. $\square$

Example 6. Let A be as in the Example 2. Let us put $B = A \cup \{a\}$ and define the operation $*$ on B in the following way

$\cdot$	0	1	2	a
0	0	1	2	a
1	0	0	2	0
2	2	2	0	2
a	0	1	2	0

Then $(B, *, 0)$ is a wUP-algebra. The order in this wUP-algebra is as follows $\leqslant = \{(0,0),(1,1),(2,2),(a,a),(1,0),(1,a),(a,0)\}$.

Example 7. Let $A = \{0, a, b, c, d, e\}$ be as in Example 3. Let us put $B = A \cup \{f\}$ and the binary operation be defined as

$\cdot$	0	a	b	c	d	e	f
0	0	a	b	c	d	e	f
a	0	0	b	b	d	e	0
b	0	a	0	a	d	d	0
c	0	0	0	0	d	d	0
d	d	d	e	e	0	b	d
e	d	d	d	d	0	0	d
f	0	a	b	c	d	e	0

Routine calculations show that B is a wUP-algebra but not a UP-algebra because $e \cdot 0 = d \neq 0$.

3.2. Atoms in wUP-algebras

The concept of atoms in weak BCC-algebras was studied by Dudek, Zhang and Wang. in [8]. We introduce and analyze the concept of atoms in wUP-algebras in this subsection.

Definition 5. Let A be a wUP-algebra. An element $a \in A$ such that $a \neq 0$ is called an atom of a UP-algebra A if the following holds

(A) $(\forall x \in A)(a \leqslant x \implies (x = 0 \vee x = a))$.

The set of all atoms of A is denoted by $L(A)$.

In the following two statements, sufficient conditions are given for the recognition of atoms in a wUP-algebra.

Proposition 3. *Let A be a wUP-algebra and $a \in A$ such that $0 \neq a$. Then a is an atom in A if the set $\{0, a\}$ is a wUP-ideal in A.*

Proof. Let the subset $\{0, a\}$ be a wUP-ideal in a wUP-algebra A. Then holds

$$(\forall x \in A)(a \leqslant x \wedge a \in \{0, a\}) \implies x \in \{0, a\})$$

by (J6). This means $x = 0$ or $x = a$. $\square$

Proposition 4. *Every isolated element in a wUP-algebra A, i.e. the element that is not comparable to any other element in A, is an atom in A.*

Proof. Let A be a wUP-algebra and let the element $(0 \neq)b \in A$ not be comparable with any other element in A. This means that $x \cdot b \neq 0$ and $b \cdot x \neq 0$ are valid. Therefore, for b it holds $(b \neq 0 \wedge x \neq b) \implies b \nleqslant x$. Since the obtained implication is the contraposition of implication (A), we conclude that b is an atom in A. $\square$

Example 8. Let $A = \{0, 1, 2, a\}$ be as in Example 6. Subsets $J_1 = \{0, 1\}$, $J_2 = \{0, 2\}$ and $J_3 = \{0, a\}$ are non-trivial ideals in A. Therefore, the elements 1, 2 and a are atoms in the wUP-algebra A according to Proposition 3.

Example 9. Let $A = \{0, 1, 2, 3, 4, a\}$ be as in Example 5. The subsets $J_1 = \{0, 3\}$ and $J_2 = \{0, 4\}$ are non-trivial wUP-ideals in A. Therefore, according to Proposition 3, elements 3 and 4 are atoms in A. The subset $K = \{0, a\}$ is not a wUP-ideal in A because, for example, $4 \cdot (a \cdot 2) = 4 \cdot a = a \in K$ and $a \in K$ but $4 \cdot 2 = 2 \notin K$. However, the element a is an atom in A, according to Proposition 4, because it is an isolated element in A.

The previous example shows that the converse of Proposition 3 not need be valid. Therefore, the hypothesis in Proposition 3 is not a necessary condition for the recognition of atoms in a wUP-algebra. Likewise, the previous example shows that the converse of Proposition 4 does not have to hold because there are atoms in a wUP-algebra A that are not isolated elements.

Also, it is not difficult to conclude that:

Proposition 5. *For every atom in a wUP-algebra A holds*

$$\varphi^2(a) = 0 \vee \varphi^2(a) = a.$$

Proof. According to (7), for each $x \in A$ the following is valid $x \leqslant \varphi^2(x)$. In particular, if a is an atom in A, we have $a \leqslant \varphi^2(a)$. Thus $\varphi^2(a) = 0 \vee \varphi^2(a) = a$ in accordance with (A). $\square$

It is quite expected to ask the question whether the converse of the previous proposition is also valid. Following the determination in the article [8], we introduce the following subset: For any wUP-algebra A we consider following subset

$$G(A) = \{x \in A : \varphi^2(x) = 0 \vee \varphi^2(x) = x\}.$$

Theorem 3. *Let A be a wUP-algebra and B be a non-empty subset of A. Then, for the subset $\varphi^2(B) = \{\varphi^2(b) : b \in B\}$ holds $\varphi^2(B) \subseteq G(A)$ and $\varphi^2(A) = G(A)$.*

Proof. For $c \in \varphi^2(B)$ there exists $b \in B$ such that $c = \varphi^2(b)$. Then $\varphi^2(c) = \varphi^4(b) = \varphi^2(b) = c$, by (7). So, $c \in G(A)$. Thus $\varphi^2(B) \subseteq G(A)$. Consequently $\varphi^2(A) \subseteq G(A)$. The inclusion $G(A) \subseteq \varphi^2(A)$ is obvious. $\square$

Theorem 4. *Let A be a wUP-algebra. Then $L(A) \subseteq G(A) = \varphi(A)$.*

Proof. Let $a \in A$ be an atom in A. Them $\varphi^2(a) = 0$ or $\varphi^2(a) = a$ in accordance with Proposition 5. This shows that $a \in G(A)$. Obviously $G(A) \subseteq \varphi(A)$. Thus $L(A) \subseteq G(A) \subseteq \varphi(A)$.

Conversely, for any $a \in \varphi(A)$ there exists $y \in A$ such that $a = \varphi(y)$. Hence $\varphi^2(a) = \varphi^3(y) = \varphi(y) = a$ by (7). This means $\varphi(A) \subseteq G(A)$. $\square$

Remark 3. In the paper [9], written by M. A. Chaudhry et al., the concept of WUP-algebras is determined using axioms

(WUP-1) $\equiv$ (wUP-1) $\equiv$ (UP1),

(WUP-2) $\equiv$ (wUP-2) $\equiv$ (UP-2),

(WUP-3) $(\forall x, y \in A)(\varphi^2(x) \cdot y = 0 \implies \varphi^2(x) = y)$,

(WUP-4) $\equiv$ (wUP-4) $\equiv$ (UP-4).

We denote this axiomatic system by **WUP**. Specially, in this algebra, it is valid

$$(\forall x \in A)(\varphi^2(x) \cdot x = 0 \implies \varphi^2(x) = x).$$

Indeed, from formula (WUP-3) together with (7), we get $\varphi^2(x) = x$ according to (wUP-4). Therefore, the axiomatic system **wUP** is an non-trivial extension of the axiomatic system **WUP**. In addition to the previous one, for each atom a in a WUP-algebra A, holds $\varphi^2(a) = a$. On the other hand, if $\varphi^2(a) \neq 0$, we would have $\varphi(a) = \varphi^2(a) \cdot 0 \neq 0$ according to the contraposition of the axiom (WUP-3).

Let A be a wUP-algebra. Let $H(A)$ denote the set $\{x \in A : \varphi^2(x) = x\}$. This set is not empty because $0 \in H(A)$.

Proposition 6. *The subset $H(A)$ of a wUP-algebra A is a wUP-subalgebra in A.*

Proof. It is clear that $0 \in H(A)$. Let $x, y \in H(A)$ be arbitrary elements. This means $\varphi^2(x) = x$ and $\varphi^2(y) = y$. Then $\varphi^2(x \cdot y) = \varphi^2(x) \cdot \varphi^2(y) = x \cdot y$ according to (14). Hence, $x \cdot y \in H(A)$. $\square$

It is obvious that $H(A) \subseteq G(A)$ holds, but we do not know whether $H(A) \subseteq L(A)$ holds.

The following proposition gives some of the basic properties of the set $H(A)$.

Proposition 7. *Let A be a wUP-algebra. Then holds:*

(16) $a \in H(A) \implies (\forall x \in A)(\varphi(a \cdot x) = x \cdot a)$.

(17) $A \cdot H(A) \subseteq H(A)$.

(18) $H(A) \subseteq L(A)$.

Proof. Let A be a wUP-algebra and $a \in H(A)$ be arbitrary element. This means $\varphi^2(a) = a$. For an arbitrary element $x \in A$ we have

$\varphi(a \cdot x) = \varphi^2(x \cdot a) = \varphi^2(x) \cdot \varphi^2(a) = \varphi^2(x \cdot a) \leqslant x \cdot a$.

On the other hand, according to (6), we have $x \cdot a \leqslant \varphi(a \cdot x)$. Hence $x \cdot a = \varphi(a \cdot x)$. This proves the implication (16).

Let $a \in H(A)$ and $x \in A$ be arbitrary elements. Then $x \cdot a = \varphi(a \cdot x)$ by (16). Since, according to (15), $\varphi(a \cdot x) = \varphi^2(x \cdot a)$ holds, we conclude that $x \cdot a \in H(A)$. This proves (17).

Let $a \in A$ be an arbitrary element such that $a \in H(A)$. This means $\varphi^2(a) = a$. Let us take $x \in A$ such that $a \leqslant x$. From here, applying (12a) twice, we get $a = \varphi^2(a) = \varphi^2(x) \geqslant x$ with respect to (7). Thus $a = x$, which proved that a is an atom in A. So, $a \in L(A)$. $\square$

Example 10. Let $A = \{0, a, b, c, d, e\}$ be as in Example 3. In this wUP-algebra the order relation is given as follows

$$\leqslant \, = \{(0,0), (a,0), (a,a), (b,0), (b,b), (c,0), (c,a), (c,b), (c,c), (d,d), (e,d), (e,e)\}.$$

Elements a, b and d are atoms because they satisfy the condition (A). Besides, we have $\varphi^2(a) = 0$, $\varphi^2(b) = 0$ and $\varphi^2(d) = d$. So, $d \in H(A)$. Here it should be noted that $(\forall x \in A)(x \cdot d \in H(A))$ holds for the aroma d, while this is not the case with the atoms a and b because, for example, we have $a \cdot b = b \notin H(A)$ and $b \cdot a = a \notin H(A)$.

The previous example shows that $H(A) \neq L(A)$ in the general case.

For any wUP-algebra A we consider the subset $T(A) = \{x \in A : x \leqslant \varphi(x)\}$. $T(A)$ is not a wUP-subalgebra in general case, except for the case where $T(A) = A$. A wUP-algebra A with this property is called a T-type wUP-algebra.

The following proposition gives a property of the wUP-algebra in this case:

Proposition 8. *A wUP-algebra A is T-type if and only if the following formula $(\forall x \in A)(\varphi^2(x) = \varphi(x))$ holds.*

Proof. In this wUP-algebra holds $(\forall x \in A)(\varphi^2(x) = \varphi(x))$, by (12a). On the other hand, if $\varphi^2(x) = \varphi(x)$, then $x \cdot \varphi(x) = x \cdot \varphi^2(x) = 0$, by (7). So, a wUP-algebra A is T-type if and only if the following formula $(\forall x \in A)(\varphi^2(x) = \varphi(x))$ is valid formula. □

3.3. Filters in wUP-algebras

In the paper [2], written by A. Uampan and D. A. Romano, the concept of wUP-ideals as well as various types of ideals in wUP-algebras were analyzed. In this subsection, the concept of wUP-filter is introduced in a wUP-algebra and the problems faced by this newly introduced concept in wUP-algebras are reviewed.

The following definition introduces the concept of wUP-filters in wUP-algebras.

Definition 6. Let A be a wUP-algebra. A subset F of A is a wUP-filter in A if it satisfies the following conditions:

(F1) $0 \in F$,

(F2) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \Longrightarrow x \in F)$.

By $\mathcal{F}(A)$ we denote the family of all wUP-filters in A.

Example 11. Every regular wUP-ideal in a wUP-algebra A ([2], Definition 3.6) is a wUP-filter in A.

Lemma 3. *Let F be a wUP-filter in a wUP-algebra A. Then holds*

(F5) $(\forall x, y \in A)((x \leqslant y \wedge y \in F) \Longrightarrow x \in F)$.

Proof. Let $x, y \in A$ be such that $x \leqslant y$ and $y \in F$. Then $x \cdot y = 0 \in F$ by (F1). Thus $x \in F$ according (F2). □

It is obvious that in every wUP-algebra A, the set A is a wUP-filter in A. So, the family $\mathcal{F}(A)$ is not empty.

Theorem 5. *The family $\mathcal{F}(A)$ forms a complete lattice.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of wUP-filters in a wUP-algebra A. It is clear that $0 \in \bigcap_{i \in I} F_i$ is valid. Let $x, y \in A$ be such that $x \cdot y \in \bigcap_{i \in I} F_i$ and $y \in \bigcap_{i \in I} F_i$. Then $x \cdot y \in F_i$ and $y \in F_i$ for any $i \in I$. Thus $x \in F_i \subseteq \bigcap_{i \in I} F_i$ by (F2).

Let $\mathcal{X}$ be a family of wUP-filters in A containing $\bigcup_{i \in I} F_i$. Then $\cap \mathcal{X}$ is a wUP-filter in A that contains $\bigcup_{inI} F_i$ according to the first part of this proof.

If we put $\sqcap_{i \in I} F_{i \in I} = \bigcap_{i \in I} F_i$ and $\sqcup_{i \in I} F_{i \in I} = \cap \mathcal{X}$, then $(F(A), \sqcup, \sqcap)$ is complete lattice. □

It is obvious that for any wUP-filter in a wUP-algebra A holds

Lemma 4. *Let F be a wUP-filter in a wUP-algebra A. Then*

(F3) $(\forall x \in A)(\varphi(x) \in F \Longrightarrow x \in F)$.

Proof. Let $x \in A$ be such that $\varphi(x) = x \cdot 0 \in F$. Hence $x \in F$ by (F2) since $0 \in F$ according to (F1). □

As an important consequence of this lemma, we have the following statement that describes in detail the property of wUP-filters in wUP-algebras.

Corollary 6. *For every wUP-filter F in a wUP-algebra A holds $Ker\varphi \subseteq F$.*

Proof. Let F be a wUP-filter in a wUP-algerba A and let $x \in Ker\varphi$. Then $\varphi(x) = 0 \in F$. This $x \in F$ by (F3). Hence $Ker\varphi \subseteq F$. □

It can be immediately concluded that:

Proposition 9. *The only wUP-filter in the UP-algebra A is the algebra A itself.*

Proof. Let F be a wUP-filter in a UP-algebra A and let $x \in A$ be an arbitrary element. Then $x \cdot 0 = 0 \in F$ and $0 \in F$ implies $x \in F$ by (F2). Hence $A = F$. □

It is justified to ask the question about the existence of a wUP-filter F in a wUP-algebra A for which $F \neq Ker\varphi$ holds.

Also it is valid:

Corollary 7. *Let F be a wUP-filter in a wUP-algebra A. Then*
(F4) $(\forall x \in A)(\varphi^2(x) \in F \implies x \in F)$.

Example 12. Let $A = \{0,1,2\}$ as in Example 2. The subset $F = \{0,1\}$ is a wUP-filter in A. The subset $K = \{0,2\}$ is not a wUP-filter in A because, for example, we have $1 \cdot 2 = 2$ and $2 \in K$ but $1 \notin K$.

Example 13. Let $A = \{0,a,b,c,d,e\}$ as in Example 3. The subset $F = \{0,a,b,c\}$ is a wUP-filter in A.

Example 14. Let $A = \{0,a,b,c,d,e,f\}$ be as in Ecample 7. Subset $F = \{0,a,b,c,f\}$ is a wUP-filter in A. The subset $K = \{0,b\}$ is not a wUP-filter in A because, for example, we have $a \cdot b = b \in K$ and $b \in K$ but $a \notin K$. The subset $M = \{0,d,e\}$ is also not a wUP-filter in A because, for example, we have $a \cdot 0 = a \in M$ and $0 \in M$ but $a \notin M$. Second, the element f is atom in A because it satisfies the condition (A). Element d is also an atom in A because the contraposition of formula (A) holds to it: $x \neq d \wedge x \neq 0 \implies d \nleqslant x$. Besides, $\varphi^2(f) = 0$ and $\varphi^2(d) = d$ hold for these atoms.

4. Final comments

Comparing the properties of atoms in wUP-algebras, determined here, and the properties of atoms in other logical algebras, one gets the impression that there are possibilities for further and deeper research of this concept in wUP-algebras.

It is fully justified to register the following special cases:

- Describe the wUP-algebra when $L(A) = A$; and, secondly
- Describe the wUP-algebra when $H(A) = A$.

Although this second option is associated with WUP-algebra, it could be interesting to find the differences between these two algebras, if they exist.

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