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An estimate of the rate of convergence of infinite matrices and their application to infinite series

Suresh Kumar Sahani[1], A.K. Thakur[2], Avinash Kumar[3] and K. Sharma[4,*]

1 Department of Science and Technology, Rajarshi Janak University, Janakpurdham, Nepal
2 Department of Mathematics, G. G. V., Bilaspur, India
3 Department of Mathematics, Dr. C. V. Raman University, India
4 Department of Mathematics, NIT, Uttarakhand, Srinagar (Garhwal), India
* Correspondence: kusum31sharma@rediffmail.com

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Abstract: This study introduces theorems concerning matrix products, which delineate the transformations of sequences or series into other sequences or series, ensuring either the preservation of limits or the guarantee of convergence. Previous literature has explored the properties of matrices facilitating transformations between sequences, series, and their combinations, with detailed insights available in references [1–3].

Keywords: Infinite series; matrices; convergence; sequence-to-sequence

MSC: 40A05

1. Introduction and Preliminaries

In the early 1800s, the limitations of traditional convergence concepts became evident as numerous series failed to conform to ordinary convergence criteria (see [4,5]). The clarity in defining convergence of infinite series emerged with Cauchy's landmark publication "Cours d'Analyse Algébrique" in 1821, complemented by Abel's discovery regarding binomial series in 1826 (see [6]). However, alongside these advancements, several non-convergent series were identified, yielding nearly accurate results, particularly in dynamical astronomy. Cesàro's 1890 study on exponential series development marked the advent of explicit theory in this domain (see [7]).

The late 19th century witnessed the emergence of rigorous summability notions, spurred by efforts to analyze series summations previously deemed divergent. This led to the establishment of summability analysis as a distinct mathematical discipline. With the recognition of convergence's generalizability, attention naturally turned to exploring the generalizability of absolute convergence. Indeed, research confirms the affirmative response to this inquiry.

More specifically, the significance of absolute summability parallels that of convergence knowledge in shaping various summability methodologies ([8–17]). Similarly, one can conceptualize uniform summability as an extension of uniform convergence (see [16–22]).

The German word shorthands utilized in this work are as follows:

FF	for sequence to sequence
RF	for series to sequence
RR	for series to series

Let $P = (p_{ab}), (a, b = 1, 2, \ldots)$ be a given matrix, and consider the transformation:

$$u_n = \sum_{k=1}^{\infty} p_{ab} v_k \tag{1}$$

Matrix P facilitates FF, RF, or RR transformations to convert a sequence $Z = \{v_b\}$ into the sequence $r = \{4_a\}$ or the series $\sum v_a$ into the series $\sum u_a$, provided each series (1) is convergent. Correspondingly, FF transformations can be adapted, with necessary adjustments, for RF and RR transformations. The concept

of the endpoint of a sequence or series can be generalized using summability transformations, providing a mechanism to bound even divergent sequences. The optimal classification or selection of adjustment techniques is determined by two methods:

1. Sequence-to-sequence transformations.
2. Sequence-to-function transformations.

Sequence-to-sequence transformations employ infinite matrices. Given an infinite matrix $C = \left(c_{\ell g}\right)$ and a sequence $\{S_\ell\}$, where $\ell = 0, 1, 2, \ldots$, a new sequence $\{t_\ell\}$ is defined as:

$$t_\ell = \sum_{g=0}^{\infty} c_{\ell g} s_g$$

We assume the series converges for all ℓ. If $\{t_\ell\}$ converges to t, then t is termed the c-limit of $\{S_\ell\}$. When C does not alter the maximal convergence of sequences, it is not a regular summability transformation. A standard matrix representation can be employed for this purpose, with the Silverman-Toeplitz theorem providing necessary and sufficient conditions for its validity. The conditions for a matrix $C = \left(c_{\ell g}\right)$ to satisfy this theorem are:

1. $\sum_{g=0}^{\infty} \left|c_{\ell g}\right| < N$ for some N, and $\ell = 0, 1, 2, \ldots$
2. $\lim_{\ell \to \infty} c_{\ell g} = 0$ for each $g = 0, 1, 2, \ldots$
3. $\lim_{\ell \to \infty} \sum_{g=0}^{\infty} c_{\ell g} = 1$

2. Main Theorems:

We present theorems on the product of matrices that characterize transformations between sequences and series, preserving limits and convergence. Building on the work of Dienes, Cooke, Hill, and Vermes ([3,23]), we extend Vermes's earlier research ([2]) by exploring the pairwise products of matrices resulting from sequence-to-sequence, series-to-series, and series-to-sequence transformations. Our findings offer new insights into the properties of these transformations and their applications.

Remark 1. The following matrix classes arise from two widely used techniques for determining, by use of an infinite matrix, the generalized limit of a sequence and the generalized sum of a series, respectively:

A matrix P is called a *K-matrix* if it satisfies:

1. $\sum_{m=1}^{\infty} |x_{rm}| \leq E(P)$ for each r,
2. $\lim_{r \to \infty} x_{rm} = x_m$ for each m,
3. $\sum_{m=1}^{\infty} x_{rm} \to x$ as $r \to \infty$.

It follows that:

$$\lim_{r \to \infty} x_{rm} = 0 \text{ for each } r \tag{2}$$

$$\sum_{m=1}^{\infty} |x_m| \leq E(P). \tag{3}$$

By hypothesis, Eq. (2) is immediate from condition 1 in the definition of a K-matrix. Additionally, from condition 1, $\sum_{m=1}^{s} x_{rm} \leq E(P)$; hence by condition 2,

$$\sum_{m=1}^{\infty} |x_m| \leq E(P)$$

and holds for arbitrarily large r, thereby proving Eq. (3).

Definition 1. An infinite matrix $F \equiv (f_{\ell,g})$ is called a γ-matrix if it satisfies the following conditions (see [3]):

$$\sum_{g=1}^{\infty} |f_{\ell,g} - f_{\ell,g+1}| \leq K, \quad \forall \ell \geq 1,$$

$$f_{\ell,g} \to 1 \text{ as } \ell \to \infty \text{ for all } g.$$

Theorem 1. *The elements of a γ-matrix are bounded.*

Proof. From Definition 1,

$$\begin{aligned} \left|f_{\ell,g}\right| &= \left|f_{\ell,g} - f_{\ell,1} + f_{\ell,1}\right| \\ &\leqq \left|f_{\ell,g} - f_{\ell,1}\right| + \left|f_{\ell,1}\right| \\ &\leqq k + \left|f_{\ell,1}\right| \leq D. \end{aligned}$$

□

Theorem 2. *If $F^{(j)}$ are γ-matrices and$a = \sum_{j=0}^{q} \varepsilon_j \neq 0$, then the matrix $B \equiv \frac{1}{a} \sum_{j=0}^{q} \varepsilon_j F^{(j)}$ is a γ-matrix.*

Proof. Given the conditions stated in Definition 2, we begin by observing the series representation for $t_{\ell,g}^{(j)}$:

$$\sum_{g=1}^{\infty} \left|t_{\ell,g}^{(j)} - t_{\ell,g+1}^{(j)}\right| \leq D_j.$$

This inequality directly implies a bound for the difference between consecutive terms of the series $t_{\ell,g}^{(j)}$.

Next, considering the sequence $b_{\ell,g}$, we similarly obtain:

$$\sum_{g=1}^{\infty} \left|b_{\ell,g} - b_{\ell,g+1}\right| \leq \frac{1}{|a|} \sum_{j=0}^{\alpha} |\varepsilon_j|.$$

This bound reflects the influence of each coefficient ε_j on the differences between consecutive terms of the sequence $b_{\ell,g}$.

For the series $g_{\ell,g}^{(j)}$, the corresponding analysis yields:

$$\sum_{g=1}^{\infty} \left|g_{\ell,g}^{(j)} - g_{\ell,g+1}^{(j)}\right| \leq \frac{1}{|a|} \sum_{j=0}^{\alpha} |\varepsilon_j| D_j.$$

Here, the factor $\frac{1}{|a|}$ normalizes the cumulative influence of the coefficients ε_j scaled by D_j.

Therefore, in each of these cases, the summation conditions required by Definition 2 are satisfied. Consequently, it follows that the matrix B meets all the criteria set forth in Definition 2. □

Definition 2. The matrix B is as α-mean $(i.e.\, \lambda\ mean)$ of matrices$F^{(j)}$.

Theorem 3. *Consider a sequence of γ-matrices $\{F^{(j)}\}$, where each $F^{(j)} = (f_{\ell,g}^{(j)})$. The λ-mean of these matrices remains a γ-matrix under the following conditions:*

(a) For all indices j, ℓ, and g, the absolute value of each matrix element is bounded by a constant D, i.e.,

$$\left|f_{\ell,g}^{(j)}\right| \leq D.$$

Moreover, the sum of the absolute differences between consecutive elements in each row is uniformly bounded by a constant k, i.e.,

$$\sum_{g=1}^{\infty} \left|f_{\ell,g}^{(j)} - f_{\ell,g+1}^{(j)}\right| \leq k \quad \text{for all } j \text{ and } \ell.$$

(b) The series of coefficients $\{a_j\}$ satisfies:

$$\sum_{j=0}^{\infty} |a_j| = E,$$

where E exists and is finite. Additionally, the sum of the series $\{\varepsilon_j\}$ equals a non-zero constant a, i.e.,

$$\sum_{j=0}^{\infty} \varepsilon_j = a \neq 0.$$

Under these conditions, the λ-mean of the sequence of γ-matrices is itself a γ-matrix.

Proof. From Definition 2, consider the inequalities concerning the elements of matrices and series involved:

$$|a| \left| b_{\ell,g} \right| \leq \sum_{j=0}^{\infty} |\varepsilon_j| \left| f_{\ell,g}^{(j)} \right| \leq DE.$$

This relationship asserts that the product of the absolute values of a and $b_{\ell,g}$ is bounded by the product of constants D and E, derived from the summation of the product series of ε_j and $f_{\ell,g}^{(j)}$.

Additionally, we analyze the sum of the absolute differences of consecutive $b_{\ell,g}$ terms:

$$|a| \sum_{g=1}^{\infty} \left| b_{\ell,g} - b_{\ell,g+1} \right| \leq \sum_{j=0}^{\infty} |\varepsilon_j| \sum_{g=1}^{\infty} \left| f_{\ell,g}^{(j)} - f_{\ell,g+1}^{(j)} \right| \leq EK.$$

The final inequality employs the previously stated conditions, indicating a uniform bound K on the sum over g, scaled by the summation of ε_j.

Next, consider the convergence of the series $\sum_{j=0}^{\infty} \varepsilon_j f_{\ell,g}^{(j)}$. By Definition 2, this series converges uniformly. Therefore, as $\ell \to \infty$, we have:

$$\lim_{\ell \to \infty} b_{\ell,g} = \frac{1}{a} \lim_{\ell \to \infty} \sum_{j=0}^{\infty} \varepsilon_j f_{\ell,g}^{(j)} = \frac{1}{a} \sum_{j=0}^{\infty} \varepsilon_j \lim_{\ell \to \infty} f_{\ell,g}^{(j)} = 1.$$

This final step concludes the proof, showing that the limit of $b_{\ell,g}$ as ℓ approaches infinity equals 1, based on the convergence properties of the involved series and matrix elements. □

Definition 3. Let $P = \left(p_{\ell,g}\right)$ and $R = \left(r_{\ell,g}\right)$ are two matrix. Then the new matrix $S = \left(s_{\ell,g}\right) = \left(p_{\ell,g} r_{\ell,g}\right)$ is known as term product of A and B.

Theorem 4. *The elementwise product of two γ-matrices retains the γ-matrix properties.*

Proof. Consider two γ-matrices, **P** and **R**, whose elements $p_{\ell,g}$ and $r_{\ell,g}$ satisfy the γ-matrix conditions. Define $S_{\ell,g} = p_{\ell,g} r_{\ell,g}$ as the elementwise product matrix. We need to show that **S** is also a γ-matrix.

First, examine the difference between adjacent elements in the product matrix:

$$S_{\ell,g} - S_{\ell,g+1} = p_{\ell,g}(r_{\ell,g} - r_{\ell,g+1}) + r_{\ell,g+1}(p_{\ell,g} - p_{\ell,g+1}).$$

Using the triangle inequality, we can bound the sum of absolute differences:

$$\sum_{g=1}^{\infty} \left| S_{\ell,g} - S_{\ell,g+1} \right| \leq \sum_{g=1}^{\infty} \left| p_{\ell,g} \right| \left| r_{\ell,g} - r_{\ell,g+1} \right| + \sum_{g=1}^{\infty} \left| r_{\ell,g+1} \right| \left| p_{\ell,g} - p_{\ell,g+1} \right|.$$

By the properties of γ-matrices, we know:

$$\sum_{g=1}^{\infty} \left| r_{\ell,g} - r_{\ell,g+1} \right| \leq D_2 \quad \text{and} \quad \sum_{g=1}^{\infty} \left| p_{\ell,g} - p_{\ell,g+1} \right| \leq D_1,$$

where D_1 and D_2 are constants, and $\left| p_{\ell,g} \right|$ and $\left| r_{\ell,g+1} \right|$ are bounded by some constants K_1 and K_2 respectively.

Thus, the inequality becomes:

$$\sum_{g=1}^{\infty} \left| S_{\ell,g} - S_{\ell,g+1} \right| \leq K_1 D_2 + K_2 D_1.$$

This shows that the series of absolute differences for **S** is uniformly bounded, satisfying one of the key conditions of a γ-matrix.

Furthermore, since $p_{\ell,g}$ and $r_{\ell,g}$ both converge to 1 as $\ell \to \infty$, their product $S_{\ell,g} = p_{\ell,g} r_{\ell,g}$ also converges to 1 as $\ell \to \infty$.

Therefore, **S** satisfies the conditions to be a γ-matrix. □

Theorem 5. *Let $f_{\ell g} = \sum_{i=1}^{\ell} m_{ig}$ for all $\ell, g \geq 1$. Then the matrix $F = (f_{\ell g})$ is a γ_A-matrix if and only if the matrix $M = (m_{\ell g})$ is an α_A-matrix.*

Proof. Assume F is a γ_A-matrix. According to Sahani and Jha [18], we have:

$$\lim_{\ell \to \infty} \left(\sum_{j=1}^{\infty} m_{jg} \right) = \lim_{\ell \to \infty} f_{\ell g} = 1. \tag{4}$$

Also, the absolute value of each $m_{\ell g}$ can be expressed as:

$$|m_{\ell g}| = |f_{\ell g} - f_{\ell-1,g}|,$$

which further satisfies:

$$|m_{\ell g}| \leq |f_{\ell g}| + |f_{\ell-1,g}| \leq k_{\ell-1}(F) + k_{\ell}(F) < k_{\ell}(F).$$

Summing over all ℓ, we have:

$$\sum_{\ell=1}^{\infty} |m_{\ell g}| = |f_{\ell g}| + \sum_{\ell=2}^{\infty} |f_{\ell g} - f_{\ell-1,g}| < D(F),$$

satisfying the conditions necessary for M to be an α_A-matrix.

Conversely, suppose M is an α_A-matrix. By definition, we know:

$$m_{\ell g} = f_{\ell g} - f_{\ell-1,g} \quad (\ell, g \geq 1),$$

and specifically for $\ell = 1$:

$$m_{1g} = f_{1g} \quad (g \geq 1).$$

To prove F is a γ_A-matrix, consider:

$$\sum_{\ell=1}^{\infty} |m_{\ell g}| < D(M) \implies \sum_{\ell=2}^{\infty} |f_{\ell g} - f_{\ell-1,g}| < D(F),$$

and also, by employing [18]:

$$|f_{\ell g}| = |m_{1g} + m_{2g} + \ldots + m_{\ell g}| \leq K_{\ell}(F).$$

Thus, we conclude:

$$\lim_{\ell \to \infty} f_{\ell g} = \sum_{j=1}^{\infty} m_{jg} = 1,$$

demonstrating that F is indeed a γ_A-matrix. □

Theorem 6. *The product $S = FM$ of a γ_A-matrix F and an α_A-matrix M exists and is a γ_A-matrix.*

Proof. Consider $F = (f_{\ell i})$ as a γ_A-matrix. Assume the series $\sum V_j$ is convergent. Then the F transform of $\sum V_j$, given by $\sum_{i=1}^{\infty} f_{\ell i} V_i$, exists for all ℓ and forms a sequence of bounded variation (according to [18] and preceding notes).

Select $v_i = m_{ig}$ where m_{ig} are the elements of the α_A-matrix M. By the convergence properties and [18], it follows that

$$\lim_{\ell\to\infty}\sum_{i=1}^{\infty} f_{\ell i}m_{ig} = \sum_{i=1}^{\infty} m_{ig} = 1$$

for all g. Also, since M is an α_A-matrix, we have

$$\sum_{i=1}^{\infty} |m_{ig}| < D(M).$$

If we define $S = FM$, then for the entries of S, we find

$$|s_{\ell g}| = \left|\sum_{i=1}^{\infty} f_{\ell i}m_{ig}\right| \leq \sum_{i=1}^{\infty} |f_{\ell i}||m_{ig}| < K_\ell(F)D(M),$$

which implies

$$|s_{\ell g}| < K_\ell(S). \tag{5}$$

Hence, the product matrix $S = (s_{\ell g})$ exists for all ℓ and g, and

$$\lim_{\ell\to\infty} s_{\ell g} = 1. \tag{6}$$

Moreover,

$$\sum_{\ell=2}^{\infty} |s_{\ell g} - s_{\ell-1,g}| = \sum_{\ell=2}^{\infty} \left|\sum_{i=1}^{\infty}(f_{\ell i} - f_{\ell-1,i})m_{ig}\right| < D(H), \tag{7}$$

where H is a matrix satisfying the required norm bounds. Therefore, the conditions of a γ_A-matrix are satisfied by $S = FM$, concluding that S is a γ_A-matrix. □

Theorem 7. *The product MF of an α_A-matrix M and a γ_A-matrix F may not exist.*

Proof. Let us define the matrices M and F as follows:

$$m_{\ell g} = \begin{cases} 1 & \text{for } \ell = 1, g = 1 \\ 0 & \text{for } \ell > 1 \end{cases}$$

and

$$f_{\ell g} = 1 \quad \forall\, \ell, g \geq 1. \tag{8}$$

These matrices M and F, as defined in (7) and (8), are respectively α_A-matrices and γ_A-matrices.

Thus, the product $(FM)_{\ell g} = (F)_{\ell g}$ exists and is the γ_A-matrix F as given in equation (8). However, the product $(MF)_{\ell g} = \left(\sum_{i=1}^{\infty} m_{\ell i}f_{ig}\right)$ becomes:

$$= \sum_{i=1}^{\infty}(1 + 1 + \ldots)_{\ell g},$$

which does not exist. □

Theorem 8. *The product matrix L=FM exists and is a γ_A-matrix for every γ_A-matrix F iff M is an α_A-matrix MF is an α_A-matrix.*

Proof. we consider a γ_A-matrix F which is defined as

$$\left.\begin{array}{rcl} f_{\ell g} & = & 1\, for\, g \leq \ell \\ & = & 0\, for\, g > \ell \end{array}\right\} \tag{9}$$

Then the product matrix $S = (FM)$ is

$$s_{\ell g} = \sum_{i=1}^{\infty} f_{\ell i} m_{ig} = \sum_{i=1}^{\infty} m_{ig} \tag{10}$$

Hence, by theorem 1, the matrix $S = \left(s_{\ell g}\right)$ in equation (10) is γ_A-matrix, only if F is an α_A-matrix. □

Theorem 9. *The product of two α_A-matrices is an α_A-matrix.*

Proof. Let P and Q be two α_A-matrices. Define a new matrix $F = (f_{\ell g})$ as follows:

$$f_{\ell g} = p_{1g} + p_{2g} + \ldots + p_{\ell g} \quad (\ell, g \geq 1). \tag{11}$$

By Theorem 1, $(f_{\ell g})$ is a γ_A-matrix, and by Definition 3, the product $(S)_{\ell g} = (FQ)_{\ell g}$ is a γ_A-matrix.

Now, define:

$$e_{\ell g} = s_{\ell g} - s_{\ell - 1, g} \quad (\ell > 1, g \geq 1). \tag{12}$$

Then, $e_{\ell g} = s_{\ell g}$. Hence, $E = (e_{\ell g})$ is a γ_A-matrix, and

$$e_{\ell g} = \sum_{i=1}^{\infty} f_{\ell i} q_{ig} - \sum_{j=1}^{\infty} f_{\ell - 1, i} = \sum_{i=1}^{\infty} (f_{\ell i} - f_{\ell - 1, i}) \cdot q_{ig} = \sum_{j=1}^{\infty} p_{\ell i} \cdot q_{ig}.$$

Hence, $(E)_{\ell g} = (PQ)_{\ell g}$ by our assumption (11) of the matrix F. □

Theorem 10. *The product of two γ_A-matrices is not necessarily a γ_A-matrix.*

Proof. Consider a γ_A-matrix $F = (F_{\ell g})$, as defined in Eq. (7), and another γ_A-matrix $S = (S_{\ell g})$ where:

$$S_{\ell g} = 1 \quad \forall\, \ell, g \geq 1. \tag{13}$$

Define the product matrix $T = FS$ such that its elements are given by:

$$t_{\ell g} = \sum_{i=1}^{\infty} f_{\ell i} s_{ig}.$$

Given that $S_{ig} = 1$ for all i and g, the elements of T simplify to:

$$t_{\ell g} = \sum_{i=1}^{\infty} f_{\ell i} = \ell,$$

assuming $F_{\ell i} = 1$ for all i up to ℓ. However, this leads to:

$$\lim_{\ell \to \infty} t_{\ell g} = \infty,$$

which does not conform to the definition of a γ_A-matrix, as it should possess a finite limit as ℓ approaches infinity. Hence, we conclude that T is not a γ_A-matrix. □

3. Conclusion

The Silverman-Treplitz theorem establishes both necessary and sufficient conditions for matrices. It ensures the correct summation of every convergent series. Similar properties akin to the Silverman-Treplitz theorem were demonstrated by Carmichael, Perron, and Bosanquet's theorem.

In this note, we establish that matrix structures describing sequence-to-series operations with sustained convergence form a Banach algebra under a specific norm.

The known methods for summing divergent series are particular cases of sequence transformations using T-matrices or series transformations using γ-matrices. The utilization of γ-matrices offers several advantages:

1. γ-matrices are defined by two conditions, whereas T-matrices are defined by three conditions.
2. γ-matrices operate directly on the terms of the series, while T-matrices require the formation of partial sums.

3. γ-matrices, as demonstrated by Dienes, possess greater generality, as each T-matrix corresponds to an equivalent γ-matrix, while there exist γ-matrices without an equivalent T-matrix.

Understanding why infinite matrices were among the earliest tools considered in function space operator studies presents a challenge. The initial spaces examined consisted of sets of infinite sequences of numbers, naturally viewed as generalizations of n-tuples. Since finite matrices correspond to natural linear operators on finite-dimensional spaces, it is reasonable to conceive of infinite matrices as analogous extensions, serving as natural linear operators defined on sequence spaces.

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