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Article

Fixed point theorems concerning mappings that exhibit asymptotic regularity in *b*-metric spaces

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Abstract: In this article, we establish fixed point outcomes for mappings that are asymptotically regular within the context of *b*-metric spaces. These findings broaden and enhance the familiar outcomes found in existing literature. Additionally, we present corollaries to demonstrate that our results are more encompassing compared to the established findings in the literature.

Keywords: fixed point, asymptotically regular map, *b*-metric spaces.

MSC: 47H10, 54H25.

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1. Introduction

Recently, numerous authors have explored various extensions of asymptotically regular sequences and maps, as discussed in [1]. The origin of the term of asymptotic regularity is credited to Browder *et al.* [2]. In this context, we present some adaptations of established results concerning specific asymptotically regular mappings.

The notion of *b*-metric has been independently introduced by various researchers under different names, such as quasi-metric and generalized metric, as documented in [3–5]. Several results within such spaces have been established (refer to [6–9]). Although classical metric is continuous its generalization *b*-metric is not continuous, [10].

Definition 1. Suppose $\mathcal{M} \neq \emptyset$ is any set.

A map $\rho : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is said to be a *b*-metric if $\forall x, y, z \in \mathcal{M}$ the following generalized condition is satisfied together with all other conditions of a metric.

$$\rho(x,z) \leq s\,[\rho(x,y) + \rho(y,z)], s \geq 1. \tag{1}$$

2. Results

Theorem 1. *Consider a complete b-metric space $(\mathcal{M}, \rho)$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a continuous asymptotically regular mapping. Suppose there exist constants $0 \leq M < 1$ and $0 \leq K < \infty$ having the condition,*

$$\rho(\mathcal{T}x, \mathcal{T}y) \leq M\rho(x,y) + K[\rho(x,\mathcal{T}x) + \rho(y,\mathcal{T}y)] \ \ \forall x, y \in \mathcal{M}. \tag{2}$$

Under these conditions, $\mathcal{T}$ possesses a unique fixed point $p \in \mathcal{M}$. Also, for every $x \in \mathcal{M}$, the sequence $\{\mathcal{T}^n x\} \to p$.

Proof. For an arbitrarily chosen $p \in \mathcal{M}$, define a sequence $\{x_n\}$ such that $x_{n+1} = \mathcal{T}x_n = \mathcal{T}^{n+1}x_0, \ (n = 1, 2, ...)$ Since ρ satisfies inequality (1), for any n and $k > 0$, we obtain

$$\begin{aligned} \rho(x_{n+k}, x_n) &\leq s[\rho(x_{n+k}, x_{n+k+1}) + \rho(x_{n+k+1}, x_n)] \\ &\leq s[\rho(x_{n+k}, x_{n+k+1}) + s\rho(x_{n+k+1}, x_{n+1}) + s\rho(x_{n+1}, x_n)] \end{aligned}$$

$$\begin{aligned} &= s\rho(x_{n+k}, x_{n+k+1}) + s^2\rho(x_{n+k+1}, x_{n+1}) + s^2\rho(x_{n+1}, x_n) \\ &\leq s\rho(x_{n+k}, x_{n+k+1}) + s^2\rho(x_{n+1}, x_n) \\ &\quad + s^2[M\rho(x_{n+k}, x_n) + K\{\rho(x_n, \mathcal{T}x_n) + \rho(x_{n+k}, \mathcal{T}x_{n+k})\}], \end{aligned}$$

therefore,

$$(1 - s^2M)\rho(x_{n+k}, x_n) \leq s^2K[\rho(x_n, \mathcal{T}x_n) + \rho(x_{n+k}, \mathcal{T}x_{n+k}] \to 0$$

as $n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$. So, it converges to a point $p \in \mathcal{M}$. Since $\mathcal{T}$ is continuous and $x_{n+1} = \mathcal{T}x_n$,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}(\lim_{n\to\infty} x_n) = \mathcal{T}p,$$

it follows that $\mathcal{T}p = p$.

Now, let q be another fixed point of $\mathcal{T}$. Then

$$\begin{aligned} 0 < \rho(p, q) &= \rho(\mathcal{T}p, \mathcal{T}q) \leq M\rho(p, q) + K.[\rho(\mathcal{T}p, p), \rho(\mathcal{T}q, q)] \\ &= M\rho(p, q) < \rho(p, q), \end{aligned}$$

so, for $0 \leq M < 1$

$$(1 - M)\rho(p, q) = 0 \ n \to \infty$$

Since $\rho(p, q) = 0$, it follows from the b-metric property that $p = q$. Namely, $\mathcal{T}$ is convergent to a fixed point p. □

Note 1. If we choose $M = 0$ in this theorem, then the condition (1) changes into the Kannan type inequality

$$\rho(\mathcal{T}x, \mathcal{T}y) \leq K[\rho(x, \mathcal{T}x) + \rho(y, \mathcal{T}y)], \tag{3}$$

for all $x, y \in \mathcal{M}$.

Theorem 2. *Let $(\mathcal{M}, \rho)$ be a complete b-metric space with $s \geq 1$ and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be an asymptotically regular map satisfying the condition,*

$$\begin{aligned} \rho(\mathcal{T}x, \mathcal{T}y) &\leq f_1(x, y)\rho(x, y) + f_2(x, y)\varphi(\min\{\rho(x, \mathcal{T}x), \rho(y, \mathcal{T}y)\}) \\ &\quad + f_3\rho(x, y)[\rho(x, \mathcal{T}y) + \rho(y, \mathcal{T}x)], \end{aligned}$$

$\forall x, y \in X$, $f_1 + 2f_3 < 1$, $sf_1(x, y) < 1$, $sf_3(x, y) < 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ where $f_i = f_i(x, y)$ positive valued functions. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.

Proof. Let $x_n = \mathcal{T}^n x_0 = \mathcal{T}x_{n-1}$. Denote $\varrho_n = \rho(x_n, x_{n+1})$.

$$\begin{aligned} \rho(x_n, x_m) &\leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{m+1}) + \rho(x_{m+1}, x_m)] \\ &= s[\varrho_n + \rho(\mathcal{T}x_n, \mathcal{T}x_m) + \varrho_m] \\ &\leq s(\varrho_n + \varrho_m) + s[f_1\rho(x_n, x_m) + f_2\varphi(\min\{\rho(x_n, \mathcal{T}x_n), \rho(x_m, \mathcal{T}x_m)\}) \\ &\quad + f_3\{\rho(x_n, \mathcal{T}x_m) + \rho(x_m, \mathcal{T}x_n)\}] \\ &= s\varrho_n + s\varrho_m + sf_1\rho(x_n, x_m) + sf_2\varphi(\min\{\rho(x_n, \mathcal{T}x_n), \rho(x_m, \mathcal{T}x_m)\}) \\ &\quad + sf_3\{\rho(x_n, \mathcal{T}x_m) + \rho(x_m, \mathcal{T}x_n)\} \\ &\leq s(\varrho_n + \varrho_m) + sf_1\rho(x_n, x_m) + sf_2\varphi(\min\{\varrho_n, \varrho_m\}) + sf_3(\varrho_n + \varrho_m). \end{aligned}$$

Similarly, we have

$$(1 - sf_1)\rho(x_n, x_m) \leq s(1 + f_3)(\varrho_n + \varrho_m) + sf_2\varphi(\min\{\varrho_n, \varrho_m\}).$$

Since $\mathcal{T}$ is asymptotically regular, for $n > m$ with $n, m \to \infty$, $\varrho_n \to 0, \varrho_m \to 0$, $\min\{\varrho_n + s\varrho_m\} \to 0$. Therefore, we get,

$$(1 - sf_1) lim\, \rho(x_n, x_m) \leq 0,\ (1 - sf_1 > 0)\,.$$

So $\{x_n\}$ is a Cauchy sequence. Because of the completeness of $\mathcal{M}$, $\exists u \in \mathcal{M}$ s.t.

$$\lim_{n\to\infty}(x_n) = u\,.$$

Now, let's see that this point of convergence $u \in \mathcal{M}$ is a fixed point of $\mathcal{T}$. Suppose $\rho(u, \mathcal{T}u) \neq 0$. Then

$$\begin{aligned}\rho(u,\mathcal{T}u) &\leq s[\rho(u,\mathcal{T}x_n) + \rho(\mathcal{T}x_n,\mathcal{T}u)]\\ &\leq s[\rho(u,x_{n+1}) + f_1\rho(x_n,u) + f_2\varphi(\min\{\rho(x_n,\mathcal{T}x_n),\rho(u,\mathcal{T}u)\})\\ &\quad +f_3(\rho(x_n,\mathcal{T}u) + \rho(u,\mathcal{T}x_n))],\end{aligned}$$

so,

$$\begin{aligned}\rho(u,\mathcal{T}u) &\leq s[\rho(u,x_{n+1}) + f_1\rho(x_n,u) + f_2\varphi[\min\{\varrho_n,\rho(u,\mathcal{T}u)\}]\\ &\quad +f_3(\rho(x_n,\mathcal{T}u) + \rho(u,x_{n+1}))]\\ &= \rho(u,x_{n+1})(s+sf_3) + sf_2\varphi(\min\{\varrho,\rho(u,\mathcal{T}u)\})\\ &\quad +sf_1\rho(x_n,u) + sf_3\rho(u,\mathcal{T}u),\end{aligned}$$

and we obtain

$$(1 - sf_3)\rho(u,\mathcal{T}u) \leq 0\ .$$

Thus $\rho(u, \mathcal{T}u) = 0$, and so $\mathcal{T}u = u$. Therefore, $u \in \mathcal{M}$ is a fixed point of $\mathcal{T}$. Now, let's see the uniqueness. Suppose u and v are points satisfying $\mathcal{T}u = u$ and $\mathcal{T}v = v$. Therefore,

$$\begin{aligned}\rho(u,v) = \rho(\mathcal{T}u,\mathcal{T}v) &\leq f_1\rho(u,v) + f_2\varphi(\min\rho(u,\mathcal{T}u),\rho(v,\mathcal{T}v)) + f_3[\rho(u,\mathcal{T}v) + \rho(v,\mathcal{T}u)]\\ &= f_1\rho(u,v) + f_2\varphi(0) + f_3[\rho(u,v) + \rho(v,u)]\\ &= (f_1 + 2f_3)\rho(u,v)\end{aligned}$$

or $(1 - f_1 - 2f_3)\rho(u,v) \leq 0 \implies \rho(u,v) = 0 \implies u = v.$ □

We can express the following theorems as a consequence of this theorem for b-metric spaces, which are proven for classical metric spaces.

Corollary 1. *Suppose $(\mathcal{M}, \rho)$ satisfies the same conditions of the Theorem 2, with $s \geq 1$ and $p + 2r < 1$, $ps < 1$, $rs < 1$ and $\varphi(0) = 0$, $\forall x, y \in \mathcal{M}$.*

Let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a map having the conditions respectively,

- $\rho(\mathcal{T}x,\mathcal{T}y) \leq p.\ \rho(x,y) + r[\rho(x,\mathcal{T}y) + \rho(y,\mathcal{T}x)]$.
- $\rho(\mathcal{T}x,\mathcal{T}y) \leq p.\ \rho(x,y) + \alpha(\min\{\rho(x,\mathcal{T}x),\rho(y,\mathcal{T}y)\})^2 + r[\rho(x,\mathcal{T}y) + \rho(y,\mathcal{T}x)]$.

If $\mathcal{T}$ is asymptotically regular in $\mathcal{M}$, then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.

Proof.
- Choose $f_1 = \varphi$, $f_2 = 0$, $f_3 = r$, in Theorem 2.
- Choose $f_1(x,y) = p$,$f_2(x,y) = \alpha$, $f_3(x,y) = r$ and $\varphi(t) = t^2$ in Theorem 2.

□

3. Conclusion

In this article, we establish fixed point outcomes for operators that are asymptotically regular within the context of b-metric spaces. These findings broaden and enhance the familiar outcomes found in existing literature. Additionally, we present corollaries to demonstrate that our results are more encompassing compared to the established findings in the literature.

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