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Mathematical simulation of *À LA* Einstein loss of the non-separable character of a function for two particles at the quantum-classical boundary

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Abstract: In this article, we present mathematical simulations of non-separable functions (those that would "correspond" to two entangled quantum particles) that lose this character only as a result of approaching the quantum-classical frontier. No mathematical representation of the action of deteriorating agents of quantum entanglement was included in the simulation. Such loss manifests itself both from the point of view of position space and momentum space. For certain limits, compatible with the space considered, the non-separable functions defined here transform into separable functions or cancel each other out at this boundary, thus erasing the (mathematical representation of) the quantum characteristic with no equivalent in the classical world. These simulations do not concern the loss of a physical property or characteristic, but rather the loss of a mathematical characteristic of a function for two quantum particles. The "ghostly action at a distance", colloquially expressed by Prof. A. Einstein, has a "spatially limited and non-instantaneous action" as it's opposite, which mathematically takes place in simulations of non-separable quantum functions, as shown here.

Keywords: mathematical simulation, non-separable function, quantum-classical boundary.

MSC: 34C60.

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1. Introduction

Within a mathematical context to be defined, in addition to what is usually assumed in quantum mechanics [1–18], we will consider non-separable functions that fundamentally present an outstanding property: their "self-collapse", that is, they themselves, under certain mathematical conditions, progressively cancel each other out or give way to separable functions. A function with these characteristics would be of interest in quantum physics, even in the case where it is not a solution to the Schroedinger equation, as it would serve to establish certain mathematical simulations.

The term simulation, which is widely used in the computational context, as in the recent discovery of gravitational waves by LIGO[1], can also be used in other scenarios. There is discussion, for example, of a physical simulation of quantum entanglement in NMR quantum computing [1], or of a mathematical

1 The signals recorded by LIGO, subsequently processed, were compared with a bank of computer simulations of all possible signals that would be generated in the process of formation of a black hole from the collision and merger of two smaller black holes, whose masses would be identified by criterion of the best coincidence with the recorded signal [3].

simulation of the Zeeman effect of a magnetic field [2]. In concrete terms, by mathematical simulation of a specific physical effect, we mean a construction that must characterize the effect considered independently of any mathematical modeling of the physical causes that generated them.

Specifically, we consider a mathematical simulation for the loss of the non-separable character of a function, the same one that refers to (but does not necessarily correspond with) a physical system in a context of proximity to the quantum-classical boundary, including in the absence of any process or physical environment that could generate such a loss. The variables of the functions that mathematically simulate this loss would correspond to those of the particles that move away from each other.

Our physical system of reference, more specifically, is formed by two stable quantum particles[2] that are in interaction[3] until a certain instant, and then move freely, at low energies, initially within a region of empty space whose characteristic linear dimension is comparable to the Compton wavelength of the interacting particles, so that quantum mechanics completely describes the dynamics of this system.

What would be the physical mechanism whose correct mathematical description would be compatible with the loss of the non-separable character of the function for two particles simply because the particles move far enough away to reach a macroscopic context, even in the complete absence of an environment? Here we only know how to mathematically simulate this in terms of functions that cancel each other out with the progressive increase in separation between the "particles", which will be shown in the following sections.

To achieve the objective of mathematically simulating the behavior (the progressive erasure) of a non-separable function for two quantum particles corresponding to a context of gradual approximation to the quantum-classical boundary, we consider functions with the following characteristics: (i) it must have its mathematical structure compatible with the progressive loss of its non-separable character when approaching the quantum-classical boundary, including in the absence of an environment, (ii) must be compatible with the preservation of the non-separable (or separable) character of the total function regardless of the space in which the functions[4] are expressed, which corresponds to a compatibility criterion in physical vs. mathematics correspondence, and (iii) it follows from the above that there will be no need to consider functions normalized to unity.

In dealing with the case of a simulation, no need to require that the functions considered are solutions of the Schroedinger equation. The functions in this simulation will only show spatial dependence, with both the speed with which the particles move away and the time intervals involved being irrelevant here. Note that the situation considered here does not correspond to a stationary context, as we will have a change in the character of the function when the distance between the particles reaches macroscopic dimensions; for example, 1 millimeter or even smaller.

In this article, we partially address the problem of simulating the transition from quantum to classical behavior through the passage from a non-separable function of a two-particle system to a function (or situation) in which this character is nullified without the intervention of an external agent.

1.1. Function must preserve its non-separable (or separable) character regardless of the space in which it is expressed

The mathematical fact that the character of a function that represents the state of a compound quantum system is separable or non-separable in its spatial variables is due to the physical fact that this system is formed by particles that do not interact, or by correlated particles who have already interacted, respectively. It follows, as a matter of consistency between the mathematical representation of some physical characteristic of the system considered, that the corresponding function must preserve its character regardless of the space in which this function is written. Through this argument, we can fix an expression for a parameter λ to be later introduced in some functions. A completely different issue not considered here is the possible preservation or not of the character of a quantum state during the temporal evolution of the physical system considered.

Before proceeding, since from the following subsection we will consider functions for two particles, both in the 1-dimensional, 2-dimensional and 3-dimensional cases, we give below the notation that will be used:

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[2] Those that, unlike unstable ones, do not decay into other quantum particles.
[3] Whereby the function that characterizes the two particles can be of the non-separable type.
[4] Here we only consider the position and momentum spaces.

• 1−dimensional case: x represents the variable in terms of which the value of a function for a particle in the position space is written and p is the corresponding value for the function in the momentum space. Likewise, with the corresponding adaptations in their meanings, we will use y and q for the second particle.

• 2−dimensional case: x_1 and x_2 represent the variables in terms of which the value of functions associated with a particle in the space of positions are written, where p_1, and p_2 are the corresponding values for the function in the space of moments. Likewise, with the corresponding adaptations in their meanings, we will use y_1, y_2, and q_1, q_2 for the second particle.

• 3−dimensional case: we have a simple extension of the cases above, writing sometimes $\vec{x}$, $\vec{p}$ for the first particle and $\vec{y}$, $\vec{q}$ for the second particle.

1.1.1. Simple examples of non-separable and separable functions in position and momentum spaces

Let ϕ be a function whose value is defined by the x coordinate (1−dimensional case). To express the value of the function ϕ in the momentum space, in terms of a variable p, we have to apply the Fourier transform ($\hat{\mathcal{F}}$) [4] on the function ϕ, then: $\hat{\mathcal{F}}\phi \equiv \phi^F$ and we write,

$$\phi^F(p) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx\, \phi(x) e^{-i(p/\hbar)x}, \tag{1}$$

with inverse Fourier transform [4], in the momentum space,

$$\phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{dp}{\hbar}\, \phi^F(p) e^{+i(p/\hbar)x}. \tag{2}$$

The Einstein-Podolsky-Rosen function [5] (3−dimensional case) is non-separable in position space. The Ψ_S function in (3) is separable.

$$\Psi_{EPR}(\vec{x},\vec{y}) = \sum_n \psi_n(\vec{x})\, u_n(\vec{y}),$$
$$\Psi_S(\vec{x},\vec{y}) = \sum_n \sum_m \psi_n(\vec{x})\, u_m(\vec{y}), \tag{3}$$

The non-separable character of the Ψ_{EPR} function remains when moving to momentum space. The separable function (3) maintains this character in momentum space. Functions that do not preserve their non-separable (or separable) character when moving to a distinct space are not of interest in physics (nor in our simulation).

2. Mathematical development

The following sections define some non-separable functions in the position space. We sought to identify those that could simulate, for a two-particle system, the loss of this character at the quantum-classical boundary by calculating appropriate limits.

2.1. A function defined in the 3−dimensional position space

Consider a function whose value depends on six position variables in 3−dimensional space: x_1, x_2, x_3, or $\vec{x}$, and y_1, y_2, y_3, or $\vec{y}$, which is written as,

$$\varphi(\vec{x},\ \vec{y}) = e^{-\,|\,\vec{x}\,-\,\vec{y}\,|^2} = e^{-\,|\vec{x}|^2\,-\,|\vec{y}|^2\,+\,2\,\vec{x}\cdot\vec{y}}, \tag{4}$$

where two characteristics are visibly noted: the function φ is non-integrable and non-separable. In particular, the second characteristic is written,

$$\varphi(\vec{x},\ \vec{y}) \neq F(\vec{x})\, G(\vec{y}). \tag{5}$$

Here we do not intend to consider function (4) as an initial condition ($t = 0$), since, in the context of this mathematical simulation, we are interested in the limit of function (4), and other functions, when[5] $|\vec{x} - \vec{y}| \to \infty$, intending to identify suitable non-separable functions that collapse only as a result of approaching the

[5] Or an equivalent limit in momentum space.

quantum-classical boundary[6]; that is, when we consider distances between "particles" (whose wave functions are simulated here) that correspond to macroscopic dimensions, regardless of how quickly this separation occurs. It follows that the finite time intervals involved are irrelevant.

The form of expression (4) suggests writing expressions (which are not separable) for each pair of variables x_j and y_j, with $j = 1, 2, 3$, as follows,

$$\varphi_j(x_j, y_j) \equiv e^{-\,(x_j - y_j)^2}, \tag{6}$$

so that expression (4) can be rewritten as,

$$\varphi(\vec{x},\ \vec{y}) = \varphi_1(x_1,\ y_1)\ \varphi_2(x_2,\ y_2)\ \varphi_3(x_3,\ y_3). \tag{7}$$

2.1.1. Limit of φ when $|\vec{x} - \vec{y}| \to \infty$

Let's take the limit $|\vec{x} - \vec{y}| \to \infty$ of function (4); in this case we have,

$$L_1 = \lim_{|\vec{x}-\vec{y}|\to\infty} e^{-|\vec{x}-\vec{y}|^2} = 0, \tag{8}$$

Thus, after the limit, the non-separable character of the function φ was lost.

2.2. Extension of the function φ to a new function, $\tilde{\varphi}$, which incorporates a parameter λ

Let us consider a variant of function (4) that results from placing a parameter λ, initially free, in such a way that a new function, $\tilde{\varphi}$, is generated, as follows:

$$\tilde{\varphi}(\vec{x},\ \vec{y};\ \lambda) = e^{-\,|\,\vec{x}\,-\,\vec{y}\,|^2/\lambda}. \tag{9}$$

At this point, anticipating the analysis, it seems reasonable to expect that there are non-separable functions that, depending on a parameter [6], could not preserve their inseparable character by simply moving to another space, for example, to the momentum space, from according to the values (in particular, the limit values) of this parameter. Here we will discard this type of function.

2.2.1. Calculation of limits of the function $\tilde{\varphi}$

(I) Function (9) is explicitly non-separable, but in the limit $|\vec{x} - \vec{y}| \to \infty$, with λ constant (or when λ does not increase the magnitude of its value), we obtain the value zero:

$$L_2 \equiv \lim_{|\vec{x}-\vec{y}|\to\infty} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{|\vec{x}-\vec{y}|\to\infty} e^{-\,|\,\vec{x}-\vec{y}\,|^2/\lambda} = 0. \tag{10}$$

that is, after the limit, the non-separable character of the function $\tilde{\varphi}$ was lost.
(II) This time, by expressly taking the limit $\lambda \to 0$ of the function $\tilde{\varphi}$, keeping $|\vec{x} - \vec{y}|$ *finite* ($|\vec{x} - \vec{y}|$ must be distinct from zero), we find the same result as before:

$$L_3 \equiv \lim_{\lambda\to 0} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{\lambda\to 0} e^{-\,|\,\vec{x}\,-\,\vec{y}\,|^2/\lambda} = 0. \tag{11}$$

Limits (10) and (11), for functions of type (9), are equivalent. That equivalence, extended as a type of correspondence, will be important when we take the functions considered from the position space to the momentum space, where we will not have the possibility of directly considering the distance between the particles (meaning when $|\vec{x} - \vec{y}| \to \infty$), but we will be able to calculate limits of its equivalent in the momentum space. But, what is this equivalent?
(III) By taking the limit $\lambda \to \infty$, keeping $|\vec{x} - \vec{y}|$ *finite*, we find,

$$L_4 \equiv \lim_{\lambda\to\infty} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{\lambda\to\infty} e^{-\,|\,\vec{x}\,-\,\vec{y}\,|^2/\lambda} = 1. \tag{12}$$

thus, the non-separable character of the original function was lost.

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[6] Regardless of the presence of an environment or any agent degrading the non-separable character of the function.

2.3. Defining a function Ψ for "two particles" in the 1-dimensional case

As a second extension of the function in (4) we consider[7] the case of a function with two coordinate variables x and y, which, in the context of our simulation, would correspond to two particles moving in a 1-dimensional space, separately.

$$\Psi(x,y) = exp\Big\{-\frac{(x-y)^2}{\lambda}\Big\}exp\Big\{-\frac{x^2}{2}\Big\}exp\Big\{-\frac{y^2}{2}\Big\}, \tag{13}$$

or,

$$\Psi(x,y) = exp\Big\{-\Big(\frac{1}{2}+\frac{1}{\lambda}\Big)x^2\Big\}exp\Big\{-\Big(\frac{1}{2}+\frac{1}{\lambda}\Big)y^2\Big\}exp\Big\{\frac{2}{\lambda}xy\Big\}. \tag{14}$$

explicitly manifesting the non-separable character of the function Ψ. Note that the function in (14) is not normalized to unity, and there will be no need to normalize it.

2.3.1. Calculation of limits of the function Ψ

(I) Let us determine the limit of $\Psi(x,y)$ for $\lambda \to \infty$,

$$\Phi(x,y) \equiv \lim_{\lambda\to\infty} \Psi(x,y) = exp\Big\{-\frac{1}{2}x^2\Big\}exp\Big\{-\frac{1}{2}y^2\Big\}, \tag{15}$$

As expected, a separable function is obtained from a non-separable function due to how Ψ depends on x, y and λ.
(II) Let us now calculate the limit of $\Psi(x,y)$ for $\lambda \to 0$. Note that this limit of (14), can be written as follows,

$$L_5 \equiv \lim_{\lambda\to 0}\left\{exp\left(-\left\{\frac{1}{2}(x^2+y^2) - \frac{2}{\lambda}xy + \frac{1}{\lambda}(x^2+y^2)\right\}\right)\right\},$$

which is rewritten as,

$$L_5 \equiv exp\left\{-\frac{1}{2}(x^2+y^2)\right\}\lim_{\lambda\to 0}\left(\left\{exp\left\{-\frac{1}{\lambda}(x-y)^2\right\}\right)\right) = 0, \tag{16}$$

as long as $(x-y)$ remains finite (must be distinct from zero). Here too the non-separable character of the original function was lost.
(III) Keeping the value of λ fixed, this time we take the $\lim|x-y| \to \infty$ of the function Ψ, that is,

$$L_6 \equiv \lim_{|x-y|\to\infty} \Psi(x,y),$$

$$L_6 \equiv \lim_{|x-y|\to\infty}\left(exp\left\{-\frac{1}{2}(x^2+y^2)\right\}\right)\lim_{|x-y|\to\infty}\left(exp\left\{-\frac{1}{\lambda}(x-y)^2\right\}\right) = 0, \tag{17}$$

losing, in the limit process, the non-separable character of the initial function.

2.4. Fourier transform of the function φ

Next, considering (4), (6) and (7), we write (the value of) the Fourier transform of φ, here denoted as φ^F,

$$\varphi^F(\vec{p},\,\vec{q}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3\vec{x}d^3\vec{y}\;\varphi_1(x_1,y_1)\;\varphi_2(x_2,y_2)\;\varphi_3(x_3,y_3)\;e^{-\mathbf{i}\left((\vec{p}/\hbar)\cdot\vec{x}+(\vec{q}/\hbar)\cdot\vec{y}\right)}, \tag{18}$$

or, equivalently, as,

$$\varphi^F(\vec{p},\,\vec{q}) = \varphi_1^F(p_1,q_1)\;\varphi_2^F(p_2,q_2)\;\varphi_3^F(p_3,q_3), \tag{19}$$

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[7] Function suggested by Prof. Javier Garcia, UAB, Spain.

where φ_j^F is the double Fourier transformed function of the function φ_j (in relation to the variables x_j and y_j); that is,

$$\varphi_j^F(p_j, q_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_j dy_j \; \varphi_j(x_j, y_j)\; e^{-\mathbf{i}\left((p_j/\hbar)x_j + (q_j/\hbar)y_j\right)}, \tag{20}$$

or also,

$$\varphi_j^F(p_j, q_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx_j \; e^{-\mathbf{i}(p_j/\hbar)x_j}\, \mathcal{K}_j(x_j, q_j), \tag{21}$$

being that:

$$\mathcal{K}_j(x_j, q_j) = \int_{-\infty}^{\infty} dy_j \; e^{-(x_j - y_j)^2 - \mathbf{i}(q_j/\hbar)y_j}. \tag{22}$$

In (22), we will complete squares with respect to y_j, so we write,

$$e^{-\left\{(x_j-y_j)^2 + \mathbf{i}(q_j/\hbar)y_j\right\}} = e^{-\left\{y_j + \left(-x_j + \mathbf{i}(q_j/2\hbar)\right)\right\}^2} e^{\left(-x_j + \mathbf{i}(q_j/2\hbar)\right)^2} e^{-x_j^2}. \tag{23}$$

We simplify expression (23) by writing $\beta \equiv -x_j + \mathbf{i}(q_j/2\hbar)$, then we have,

$$e^{-\left\{(x_j-y_j)^2 + \mathbf{i}(q_j/\hbar)y_j\right\}} = e^{-(y_j+\beta)^2} e^{-(q_j^2/4\hbar^2) - \mathbf{i}(x_j q_j/\hbar)} \tag{24}$$

Therefore, the integral in (22) is rewritten as,

$$\mathcal{K}_j(x_j, q_j) = e^{-(q_j^2/4\hbar^2)\, -\, \mathbf{i}(x_j q_j/\hbar)} \int_{-\infty}^{+\infty} dy_j \; e^{-(y_j+\beta)^2}, \tag{25}$$

Using an interesting and non-trivial result[8] (see Appendix, or [7]):

$$\int_{-\infty}^{+\infty} e^{-(y+\alpha)^2} dy = \sqrt{\pi}\; e^{-4ab\mathbf{i}}, \tag{26}$$

with $\alpha = a + \mathbf{i}b$, a complex number, we have that the integral in (25) is equal to,

$$\int_{-\infty}^{+\infty} e^{-(y_j+\beta)^2} dy_j = \sqrt{\pi}\; e^{\mathbf{i}(2x_j q_j/\hbar)}, \tag{27}$$

Thus, from (27) in (25), we arrive at the partial result,

$$\mathcal{K}_j(x_j, q_j) = \sqrt{\pi}\; e^{-\left(q_j/2\hbar\right)^2 + \mathbf{i}(x_j q_j/\hbar)}. \tag{28}$$

Substituting (28) in expression (21) we have,

$$\varphi_j^F(p_j, q_j) = \sqrt{\pi}\; e^{-\left(\frac{q_j}{2\hbar}\right)^2} \frac{1}{2\pi}\int_{-\infty}^{\infty} dx_j \; e^{-\mathbf{i}\left((p_j - q_j)/\hbar\right)x_j}, \tag{29}$$

and considering the following result,

$$\delta(p - p') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{-\mathbf{i}\left((p-p')/\hbar\right)x}, \tag{30}$$

can be written,

$$\varphi_j^F(p_j, q_j) = \sqrt{\pi}\; e^{-(1/2\hbar)^2 q_j^2}\; \delta(p_j - q_j), \tag{31}$$

Therefore, substituting (31) in (19), for $j = 1, 2, 3$, we have the result,

$$\varphi^F(\vec{p}, \vec{q}) = \pi^{3/2}\; e^{-(1/2\hbar)^2 |\vec{q}|^2} \delta(\vec{p} - \vec{q}), \tag{32}$$

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8 We have to note that it would not be correct to make a variable change in the exponent of the integrand in (26) as in the case when α is a real number; here α is a complex number

which demonstrates that the function φ^F preserves the non-separable character of the function φ, as the Dirac delta[9] $\delta(\vec{p}-\vec{q})$ is non-separable [8]. This, in a complementary way, is compatible with the fact that φ, in (4), is a non-integrable function.

2.5. Fourier transform of the function $\tilde{\varphi}$ dependent on λ

One can adapt what was developed for φ^F in section 2.4, to the function $\tilde{\varphi}^F$, the Fourier transform of the function given in (9), which, the difference of φ, in (4), includes a parameter λ. In this way, we obtain the Fourier transform of $\tilde{\varphi}$, which we write directly,

$$\tilde{\varphi}^F(\vec{p},\vec{q}) = \lambda^{3/2}(\pi)^{3/2}\, e^{-\lambda(1/2\hbar)^2|\vec{q}|^2}\delta(\vec{p}-\vec{q}), \tag{33}$$

the one that preserves the non-separable character of the function $\tilde{\varphi}$ in the space of positions. Note that in the momentum space, it is only possible to take the limit of the function $\tilde{\varphi}^F$ when $\lambda \to 0$ or when $\lambda \to \infty$, with the need to identify which of these limits could correspond (in the simulation sense) to the limit of $\tilde{\varphi}$, for $|\vec{x}-\vec{y}| \to \infty$, in the space of positions.

2.5.1. Calculation of limits of the function $\tilde{\varphi}^F$ for $\lambda \to 0$ and $\lambda \to \infty$

(I). Let us determine the limit of $\tilde{\varphi}^F$ for $\lambda \to 0$. In this case, we have[10],

$$L_7 \equiv \lim_{\lambda\to 0} \tilde{\varphi}^F(\vec{p},\vec{q}) = 0, \tag{34}$$

In (34), the non-separable character of the function $\tilde{\varphi}$ was lost.
(II) Let us determine the limit of $\tilde{\varphi}^F$ for $\lambda \to \infty$,

$$L_8 \equiv \lim_{\lambda\to\infty} \tilde{\varphi}^F(\vec{p},\vec{q}) = (\pi)^{3/2}\,\delta(\vec{p}-\vec{q})\left(\lim_{\lambda\to\infty}\lambda^{3/2}\right)\left(\lim_{\lambda\to\infty} e^{-\lambda(1/2\hbar)^2|\vec{q}|^2}\right) = 0, \tag{35}$$

because the growth of the factor $\lambda^{3/2}$ is slow in relation to the rapid decrease of the exponential function $e^{-\lambda(1/2\hbar)^2|\vec{q}|^2}$, which dominates in the limit $\lambda \to \infty$. In both cases, in expressions (34) and (35), the non-separable character of the function $\tilde{\varphi}$ was lost due to the transition from the position space to the momentum space.

2.6. Fourier (double) transform of $\Psi(x,y)$ dependent on λ

To express the value of the function $\Psi(x,y)$, given in (13), in the momentum space, that is, to obtain $\Psi^F(p,q)$, we have to apply the Fourier transform on the function Ψ in relation to each of its variables; so we write [4],

$$\Psi^F(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, \Psi(x,y)\, e^{-\mathbf{i}(p/\hbar)x}\, e^{-\mathbf{i}(q/\hbar)y}, \tag{36}$$

and for the (double) inverse Fourier transform [4],

$$\Psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp}{\hbar}\frac{dq}{\hbar}\, \Psi^F(p,q)\, e^{+\mathbf{i}(p/\hbar)x}\, e^{+\mathbf{i}(q/\hbar)y}. \tag{37}$$

Substituting (14) into (36) we have,

$$\Psi^F(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, e^{-\alpha x^2-\alpha y^2+\beta xy-\mathbf{i}(p/\hbar)x-\mathbf{i}(q/\hbar)y}, \tag{38}$$

being:

$$\alpha = \frac{\lambda+2}{2\lambda}, \qquad \beta = \frac{2}{\lambda}. \tag{39}$$

It is,

$$J \equiv -\alpha x^2 - \alpha y^2 + \beta xy - \mathbf{i}(p/\hbar)x - \mathbf{i}(q/\hbar)y, \tag{40}$$

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[9] Also called the Lanczos-Dirac delta [8].
[10] Note that we could not commute the limit and Fourier integral operations (i.e., reverse the order of application of these operations) without first verifying that the integrand function converges uniformly.

and through simple manipulations we arrive at the expression,

$$J \equiv -\alpha\left(x-\left(\frac{\beta y-\mathbf{i}(p/\hbar)}{2\alpha}\right)\right)^2+\alpha\left(\frac{\beta y-\mathbf{i}(p/\hbar)}{2\alpha}\right)^2-\left(\alpha y^2+\mathbf{i}y(q/\hbar)\right), \tag{41}$$

so that the value of the function Ψ^F, given in (38), is written as,

$$\Psi^F(p,q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} exp\left(-\alpha\left(x-\left(\frac{\beta y-\mathbf{i}(\frac{p}{\hbar})}{2\alpha}\right)\right)^2\right)dx\right\}\times$$

$$\times\, exp\left\{\alpha\left(\frac{\beta y-\mathbf{i}(\frac{p}{\hbar})}{2\alpha}\right)^2-\left(\alpha y^2+\mathbf{i}\left(\frac{q}{\hbar}\right)y\right)\right\}dy. \tag{42}$$

In expression (42), the integral between braces is easily calculated using the result given in (82), in the Appendix,

$$\int_{-\infty}^{\infty} exp\left\{-\alpha\left(x-\left(\frac{\beta y-\mathbf{i}(\frac{p}{\hbar})}{2\alpha}\right)\right)^2\right\}dx=\left(\frac{2\lambda\pi}{\lambda+2}\right)^{1/2} exp\left\{\mathbf{i}\left(\frac{\beta yp}{\hbar\alpha}\right)\right\},$$

or also, using (39),

$$\int_{-\infty}^{\infty} exp\left\{-\alpha\left(x-\left(\frac{\beta y-\mathbf{i}(\frac{p}{\hbar})}{2\alpha}\right)\right)^2\right\}dx=\left(\frac{2\lambda\pi}{\lambda+2}\right)^{1/2} exp\left\{\mathbf{i}\left(\frac{4py}{(\lambda+2)\hbar}\right)\right\}, \tag{43}$$

so that, after substituting (43) into (42) and simplifying, we have,

$$\Psi^F(p,q)=\frac{1}{2\pi}\left(\frac{2\lambda\pi}{\lambda+2}\right)^{1/2}\int_{-\infty}^{\infty} exp\left\{\frac{1}{4\alpha}\left(\beta^2y^2+2\mathbf{i}\beta\left(\frac{p}{\hbar}\right)y-\left(\frac{p}{\hbar}\right)^2-4\alpha^2y^2-4\mathbf{i}\alpha\left(\frac{q}{\hbar}\right)y\right)\right\}dy. \tag{44}$$

Or also, after doing some simple calculations, we find the following equivalent expression,

$$\Psi^F(p,q)=\frac{1}{2\pi}\left(\frac{2\lambda\pi}{\lambda+2}\right)^{1/2} exp\left\{\frac{-\left(b^2+a(\frac{p}{\hbar})^2\right)}{4a\alpha}\right\}\int_{-\infty}^{\infty} exp\left\{\frac{a}{4\alpha}\left(y+\frac{b}{a}\right)^2\right\}dy, \tag{45}$$

with the parameters "a" and "b" being defined by the expressions,

$$a\equiv\beta^2-4\alpha^2, \qquad b\equiv\mathbf{i}\left(\beta\left(\frac{p}{\hbar}\right)-2\alpha\left(\frac{q}{\hbar}\right)\right). \tag{46}$$

On the other hand, the integral in (45) is calculated directly,

$$\int_{-\infty}^{\infty} exp\left\{\frac{a}{4\alpha}\left(y+\frac{b}{a}\right)^2\right\}dy=\sqrt{\frac{\pi}{(-a/4\alpha)}}=\left(\frac{2(\lambda+2)\pi}{(\lambda+4)}\right)^{1/2}, \tag{47}$$

Where we have used a particular case of result (82), in the Appendix, having previously verified, through (39) and (46), that "$a/(4\alpha)$" is negative and that "b/a" is purely imaginary. Expression (45), using (46) and (47), would be rewritten as,

$$\Psi^F(p,q)=\left(\frac{\lambda}{\lambda+4}\right)^{1/2} exp\left\{-\frac{\left(-\left\{\beta(\frac{p}{\hbar})-2\alpha(\frac{q}{\hbar})\right\}^2+\left\{\beta^2-4\alpha^2\right\}(\frac{p}{\hbar})^2\right)}{4\left\{\beta^2-4\alpha^2\right\}\left\{(\lambda+2)/(2\lambda)\right\}}\right\} \tag{48}$$

and the previous exponential, after some algebraic manipulations, can be rewritten as,

$$exp\left\{\frac{\frac{1}{\lambda^2}\left(2\left(\frac{p}{\hbar}\right)+\left(\lambda+2\right)\left(\frac{q}{\hbar}\right)\right)^2-\frac{1}{\lambda^2}\left(4-\left(\lambda+2\right)^2\right)\left(\frac{p}{\hbar}\right)^2}{2\left(\frac{4-(\lambda+2)^2}{\lambda^2}\right)\left(\frac{\lambda+2}{\lambda}\right)}\right\} \tag{49}$$

and a few more steps, we reach exponential,

$$exp\left\{-\left(\frac{(\lambda+2)(p^2+q^2)-4pq}{2(\lambda+4)\hbar^2}\right)\right\} \tag{50}$$

Finally, we arrive at result[11],

$$\Psi^F(p,q)=\left(\frac{\lambda}{\lambda+4}\right)^{1/2}exp\left\{-\left(\frac{(\lambda+2)(p^2+q^2)}{2(\lambda+4)\hbar^2}\right)\right\}exp\left\{\frac{2pq}{(\lambda+4)\hbar^2}\right\}. \tag{51}$$

We see a non-separable factor in the transformed function; that is, the character of the function Ψ was preserved in the transition from position space to momentum space.

2.6.1. Calculation of limits with the function $\Psi^F(p,q)$

(I) The limit of Ψ^F, for $\lambda \to \infty$, is calculated directly from expression (51),

$$\phi(p,q)=\lim_{\lambda\to\infty}\Psi^F(p,q)=e^{-(p^2+q^2)/2\hbar^2}, \tag{52}$$

losing the characteristic non-separability of Ψ^F when generating a separable function. (II) On the other hand, the limit of Ψ^F, for $\lambda \to 0$, can also be calculated from (51):

$$L_9 \equiv \lim_{\lambda\to 0}\Psi^F(p,q)=0, \tag{53}$$

because the limit of the coefficient in (51) nullifies the finite limits of the factors in the center and right in the same expression, losing the non-separable character of the function Ψ^F.

3. Discussion

Within the context of this work, as explained in the introduction, we have introduced the functions φ, in (4), $\tilde{\varphi}$, in (9), and Ψ, in (13), the last two being dependent on a free parameter λ. The limits $|\vec{x}-\vec{y}| \to \infty$, $\lambda \to 0$ and $\lambda \to \infty$ were calculated for the three functions defined in the position space. Subsequently, the corresponding functions in the momentum space were determined (via the Fourier transform): $\tilde{\varphi}^F$, in section 2.5, and Ψ^F, in (51), and their limits were calculated for $\lambda \to 0$ and $\lambda \to \infty$.

Functions that would be suitable to carry out the intended simulation should preserve the non-separable character of the original function after moving to momentum space (because, in the simulation, this would correspond to a physical characteristic of the system considered). Furthermore, any pair of corresponding limits of $\tilde{\varphi}$ and $\tilde{\varphi}^F$, or of Ψ and Ψ^F, for example in relation to $\lambda \to 0$, which do not coincide with each other, should be disregarded for the purpose of our simulation. Based on this, we must discard the function φ, as we would not have a limit equivalent to that of $|\vec{x}-\vec{y}| \to \infty$, for φ^F, in momentum space.

The limit $\lambda \to 0$, applied separately to the functions Ψ and Ψ^F, has the same effect on both: generating the null value, "erasing" the non-separability of the original function, as can be seen in expressions (16) and (53). The same happens with the functions $\tilde{\varphi}$ and $\tilde{\varphi}^F$, expressions (11) and (34). Furthermore, the limits of the function $\tilde{\varphi}$ when $|\vec{x}-\vec{y}| \to \infty$ and $\lambda \to 0$, separately, given in (10) and (11), respectively, coincide. The same happens with the limits of the function Ψ when $|x-y| \to \infty$ and $\lambda \to 0$, separately, given in (17) and (16), respectively.

Based on the results described above, the limits of the non-separable function Ψ, for $\lambda \to 0$ and $|x-y| \to \infty$ are equivalent in the position space. The same can be said for the non-separable function $\tilde{\varphi}$ for $\lambda \to 0$ and

[11] Verified by Prof. Javier Garcia, UAB, Spain, using the "Mathematica" software.

$|\vec{x}-\vec{y}| \to \infty$. The process of approaching the quantum-classical boundary in position space should have an equivalent in momentum space (for adequate functions). In the space of positions, such an approximation to the boundary is direct, just take the limit: $|\vec{x}-\vec{y}| \to \infty$. The equivalent of this in momentum space, as will be shown in the following paragraph, consists of taking the limit $\lambda \to 0$, for a specifically defined λ.

Table 1. We present all the calculated limits

Limits	**Equation**
$L_1 = \lim_{\|\vec{x}-\vec{y}\|\to\infty} e^{-\|\vec{x}-\vec{y}\|^2} = 0\,.$	(8)
$L_2 \equiv \lim_{\|\vec{x}-\vec{y}\|\to\infty} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{\|\vec{x}-\vec{y}\|\to\infty} e^{-\mid \vec{x}-\vec{y}\mid^2/\lambda} = 0.$	(10)
$L_3 \equiv \lim_{\lambda\to 0} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{\lambda\to 0} e^{-\mid \vec{x}-\vec{y}\mid^2/\lambda} = 0.$	(11)
$L_4 \equiv \lim_{\lambda\to \infty} \tilde{\varphi}(\vec{x},\vec{y}) = \lim_{\lambda\to \infty} e^{-\mid \vec{x}-\vec{y}\mid^2/\lambda} = 1.$	(12)
$\Phi(x,y) \equiv \lim_{\lambda\to\infty} \Psi(x,y) = exp\Big\{-\frac{1}{2}x^2\Big\} exp\Big\{-\frac{1}{2}y^2\Big\}.$	(15)
$L_5 \equiv exp\Big\{-\frac{1}{2}(x^2+y^2)\Big\} \lim_{\lambda\to 0}\Big(\Big\{exp\Big\{-\frac{1}{\lambda}(x-y)^2\Big\}\Big) = 0.$	(16)
$L_6 \equiv \lim_{\|x-y\|\to\infty}\Big(exp\Big\{-\frac{1}{2}(x^2+y^2)\Big\}\Big)\times$ $\times \lim_{\|x-y\|\to\infty}\Big(exp\Big\{-\frac{1}{\lambda}(x-y)^2\Big\}\Big) = 0.$	(17)
$L_7 \equiv \lim_{\lambda\to 0} \tilde{\varphi}^F(\vec{p},\vec{q}) = 0.$	(34)
$L_8 \equiv \lim_{\lambda\to\infty} \tilde{\varphi}^F(\vec{p},\vec{q}) = (\pi)^{3/2}\,\delta(\vec{p}-\vec{q}) \times \Big(\lim_{\lambda\to\infty}\lambda^{3/2}\Big)\times$ $\times\Big(\lim_{\lambda\to\infty} e^{-\lambda(1/2\hbar)^2\|\vec{q}\|^2}\Big) = 0.$	(35)
$\phi(p,q) \equiv \lim_{\lambda\to\infty} \Psi^F(p,q) = e^{-(p^2+q^2)/2\hbar^2}.$	(52)
$L_9 \equiv \lim_{\lambda\to 0} \Psi^F(p,q) = 0.$	(53)

Regardless of the above, note that the mass parameter m (and not the mass itself) of a quantum particle, in the limit $m \to \infty$, can be interpreted as corresponding to a classical particle, to a classical context. So, if we attribute:

$$\lambda \equiv \lambda^2_{Compton} = (h/mc)^2.$$

that is, if we identify λ with the expression given by the square of the Compton wavelength of the quantum particles considered, we have:

$$m \;\to\; \infty \qquad \text{implies:} \qquad \lambda \equiv \lambda^2_{Compton} \;\to\; 0,$$

which can be taken as the equivalent in momentum space of the limit $|x-y| \to \infty$ in position space. Note also the compatibility with the classical limit of quantum mechanics,

$$h \;\to\; 0 \qquad \text{implies:} \qquad \lambda \equiv \lambda^2_{Compton} \;\to\; 0.$$

This definition of λ resolve an ambiguity in the choice of the limit $\lambda \to 0$ or $\lambda \to \infty$, as the momentum space equivalent of the limit $|\vec{x}-\vec{y}| \to \infty$, for the non-separable function Ψ, given in (13), since, in both cases, the non-separable character of the function is lost.

4. Results

The limits $\lambda \equiv \lambda^2_{Compton} \to 0$ (for functions that are defined in momentum space) and $|x-y| \to \infty$ (considered in position space), are compatible with a macroscopic context, approaching the quantum-classical boundary. Under these considerations, the function given in (9),

$$\tilde{\varphi}(\vec{x},\, \vec{y}) = e^{-\,|\,\vec{x}\,-\,\vec{y}\,|^2/\lambda_C^2}, \tag{54}$$

and by the effect of the limits in (10), (11) and (34), with $\lambda_C = \lambda_{Compton}$, it effectively simulates the transition from a non-separable function to one that loses this character at the quantum-classical boundary, both from the point of view from position space, as well as from momentum space. Similarly, the function given in (13),

$$\Psi(x,y) = exp\Big\{-\frac{(x-y)^2}{\lambda_C^2}\Big\}exp\Big\{-\frac{x^2}{2}\Big\}exp\Big\{-\frac{y^2}{2}\Big\}, \tag{55}$$

and due to the limits in (16), (17) and (49), with $\lambda_C = \lambda_{Compton}$, it also effectively simulates the same transition on the quantum-classical frontier.

5. Conclusions

After the considerations and development presented, two functions were identified, given in (54) and (55), which mathematically simulate the loss of the non-separable character of these in the "quantum-classical boundary", both from the point of view of the space of positions and, also momentum space. Note that this loss has no relation to any possible mathematical representation of degrading agents of usual quantum entanglement.

The simulation considered here does not concern the loss of a physical property or characteristic, but the loss of a mathematical characteristic of functions (which would be) for two particles that move away from each other. The "ghostly action at a distance", colloquially expressed by Prof. A. Einstein refers to the instantaneous and non-local collapse of the wave function of a quantum particle when observed [9] (or, in this case, when the particles reach the quantum-classical boundary), has as its opposite an "action not instantaneously and spatially limited". The spatial part of this "action" can be mathematically implemented in simulations of non-separable functions, as we have shown.

6. Appendix. Complex translation in the Gaussian Integral

A memorable integral in Calculus is the Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\,. \tag{56}$$

It is also a well-known fact, and of elementary verification, that the above integral is invariant due to translations, that is,

$$\int_{-\infty}^{\infty} e^{-(x+c)^2} dx = \sqrt{\pi}\,,\ \ \forall\, c \in \mathbb{R}\,. \tag{57}$$

This raises the question regarding complex translations (for a real integral, as in (57), but with c being a complex number); more precisely, for $\alpha \in \mathbb{C}$, the function $I : \mathbb{C} \longrightarrow \mathbb{C}$, defined by

$$I(\alpha) := \int_{-\infty}^{\infty} e^{-(x+\alpha)^2} dx\,, \tag{58}$$

is it a real constant? (If it is, it will obviously be, $\sqrt{\pi}$, according to (56).) In this Appendix, we will show that the answer to the above question is NO. In fact, we will see that $I(\alpha)$ is constant when restricted to reals and purely imaginary ones, and the constant is equal to $\sqrt{\pi}$ in both cases (for the first case it is obvious, since over reals $I(\alpha)$ is a real integral); in other terms, the integral is invariant only for real translations ($\alpha \in \mathbb{R}$) and for imaginary translations ($\alpha = bi$). All this is the content of the following proposition.

Proposition 1. *For all $\alpha = a + bi \in \mathbb{C}$, we have,*

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-(x+\alpha)^2} dx = \sqrt{\pi} e^{-4ab\mathbf{i}}. \tag{59}$$

Proof 1. The proof of Proposition 1 is a mere application of the Real Analysis theorems on commutation of limits; more specifically, about commutation of limit and integration. And to do this we will employ the results present in [7]. Consider the function

$$\psi(\xi) = \int_0^\infty e^{-x^2} \cos \xi x \, dx. \tag{60}$$

By Propositions A6.5 and A6.6 of [7], ψ is continuous and differentiable, with derivative

$$\psi(\xi) = \int_0^\infty \frac{d}{d\xi} \left(e^{-x^2} \cos \xi x \right) dx = -\int_0^\infty x e^{-x^2} \sin \xi x \, dx \,. \tag{61}$$

Integrating by parts, we obtain that ψ satisfies the ordinary differential equation,

$$\psi' + \frac{\xi}{2} \psi = 0, \tag{62}$$

with the initial condition

$$\psi(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{63}$$

Therefore,

$$\psi(\xi) = \frac{\sqrt{\pi}}{2} e^{-\frac{\xi^2}{4}}. \tag{64}$$

Thus we obtained the following identity

$$\int_0^\infty e^{-x^2} \cos \xi x \, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\xi^2}{4}}. \tag{65}$$

Similarly, we obtain

$$\int_0^\infty e^{-x^2} \sin \xi x \, dx = \frac{e^{-\frac{\xi^2}{4}}}{2} \int_0^\xi e^{\frac{s^2}{4}} ds. \tag{66}$$

Now, to demonstrate Proposition 1, we will calculate the translations ($a \in \mathbb{R}$) in the exponential of the integrals above.

- $$I_1 = dsp \int_0^\infty e^{-(x+a)^2} \cos \xi x \, dx. \tag{67}$$

Now, for $t > 0$ we have

$$\begin{aligned} \int_0^t e^{-(x+a)^2} \cos \xi x \, dx &= \int_a^{a+t} e^{-y^2} \cos \xi (y-a) dy \\ &= \cos a\xi \int_0^{a+t} e^{-y^2} \cos \xi y \, dy - f_1(\xi, a) + \sin a\xi \int_0^{a+t} e^{-y^2} \sin \xi y \, dy - f_2(\xi, a), \end{aligned} \tag{68}$$

where,

$$f_1(\xi, a) = \cos a\xi \int_0^a e^{-y^2} \cos \xi y \, dy \quad \text{e} \quad f_2(\xi, a) = \sin a\xi \int_0^a e^{-y^2} \sin \xi y \, dy.$$

Hence, using (67) and (68), we have

$$\begin{aligned} I_1 &= \int_0^\infty e^{-(x+a)^2} \cos \xi x \, dx \\ &= \lim_{t \to \infty} \int_0^t e^{-(x+a)^2} \cos \xi x \, dx \\ &= \cos a\xi \lim_{t \to \infty} \int_0^{a+t} e^{-y^2} \cos \xi y \, dy + \sin a\xi \lim_{t \to \infty} \int_0^{a+t} e^{-y^2} \sin \xi y \, dy - f_1(\xi, a) - f_2(\xi, a) \\ &= \frac{\sqrt{\pi} \cos a\xi}{2} e^{-\frac{\xi^2}{4}} + \frac{\sin a\xi}{2} e^{-\frac{\xi^2}{4}} \int_0^\xi e^{\frac{s^2}{4}} ds - f_1(\xi, a) - f_2(\xi, a) \end{aligned} \tag{69}$$

where in the last equality we apply (65) and (66). Now let's see

- $$I_2 = dsp \int_{-\infty}^{0} e^{-(x+a)^2} \cos \xi x \, dx. \tag{70}$$

Now,

$$\begin{aligned} I_2 &= \int_{-\infty}^{0} e^{-(x+a)^2} \cos \xi x \, dx = \int_{0}^{\infty} e^{-(y-a)^2} \cos \xi y \, dy \\ &= \frac{\sqrt{\pi} \cos a\xi}{2} e^{-\frac{\xi^2}{4}} - \frac{\sin a\xi}{2} e^{-\frac{\xi^2}{4}} \int_0^{\xi} e^{\frac{s^2}{4}} ds - f_1(\xi, -a) - f_2(\xi, -a), \end{aligned} \tag{71}$$

where in the last equality we apply (69).

Now, since

$$f_1(\xi, -a) = -f_1(\xi, a) \quad \text{e} \quad f_2(\xi, -a) = -f_2(\xi, a), \tag{72}$$

follow that

$$I_2 = f_1(\xi, a) + f_2(\xi, a) + \frac{\sqrt{\pi} \cos a\xi}{2} e^{-\frac{\xi^2}{4}} - \frac{\sin a\xi}{2} e^{-\frac{\xi^2}{4}} \int_0^{\xi} e^{\frac{s^2}{4}} ds \,. \tag{73}$$

And so, from (69) and (73), we have

$$\int_{-\infty}^{\infty} e^{-(x+a)^2} \cos \xi x \, dx = \sqrt{\pi} e^{-\frac{\xi^2}{4}} \cos a\xi. \tag{74}$$

A similar reasoning leads to

$$\int_{-\infty}^{\infty} e^{-(x+a)^2} \sin \xi x \, dx = \sqrt{\pi} e^{-\frac{\xi^2}{4}} \sin a\xi. \tag{75}$$

Now we will finish the proof of Proposition 1. Given $\alpha = a + b\mathbf{i} \in \mathbb{C}$, we have

$$\begin{aligned} I(\alpha) &= \int_{-\infty}^{\infty} e^{-(x+a+b\mathbf{i})^2} dx = e^{b^2} \int_{-\infty}^{\infty} e^{-(x+a)^2 - 2(x+a)b\mathbf{i}} dx \\ &= e^{b^2} \int_{-\infty}^{\infty} e^{-(x+a)^2} \cos 2b(x+a) dx - \mathbf{i} e^{b^2} \int_{-\infty}^{\infty} e^{-(x+a)^2} \sin 2b(x+a) dx. \end{aligned} \tag{76}$$

Let's calculate, separately, each of the integrals, J_1 and J_2, above.

- $$J_1 = \int_{-\infty}^{\infty} e^{-(x+a)^2} \cos 2b(x+a) dx. \tag{77}$$

$$\begin{aligned} J_1 &= \cos 2ab \int_{-\infty}^{\infty} e^{-(x+a)^2} \cos 2bx \, dx - \sin 2ab \int_{-\infty}^{\infty} e^{-(x+a)^2} \sin 2bx \, dx \\ &= \sqrt{\pi} e^{-b^2} \cos 2ab \cos 2ab - \sqrt{\pi} e^{-b^2} \sin 2ab \sin 2ab = \sqrt{\pi} e^{-b^2} \cos 4ab, \end{aligned} \tag{78}$$

where in the second equality we apply (74) and (75).

- $$J_2 = \int_{-\infty}^{\infty} e^{-(x+a)^2} \sin 2b(x+a) dx. \tag{79}$$

$$\begin{aligned} J_2 &= \sin 2ab \int_{-\infty}^{\infty} e^{-(x+a)^2} \cos 2bx \, dx + \cos 2ab \int_{-\infty}^{\infty} e^{-(x+a)^2} \sin 2bx \, dx \\ &= \sqrt{\pi} e^{-b^2} \sin 2ab \cos 2ab + \sqrt{\pi} e^{-b^2} \sin 2ab \cos 2ab = \sqrt{\pi} e^{-b^2} \sin 4ab, \end{aligned} \tag{80}$$

here too, in the second equality above, we apply (74) and (75).
Therefore, from (76)−(80), we have

$$\begin{aligned} I(\alpha) &= e^{b^2} J_1 - \mathbf{i} e^{b^2} J_2 = e^{b^2} \sqrt{\pi} e^{-b^2} \cos 4ab - \mathbf{i} e^{b^2} \sqrt{\pi} e^{-b^2} \sin 4ab \\ &= \sqrt{\pi} \big(\cos 4ab - \mathbf{i} \sin 4ab \big) = \sqrt{\pi} e^{-4ab\mathbf{i}}. \end{aligned} \tag{81}$$

This ends the proof of the proposition. □

In addition, having proven (59), it is straightforward to show that, with σ being a positive real parameter, we have,

$$\int_{-\infty}^{+\infty} e^{-\sigma(x+\alpha)^2} dx = \sqrt{\frac{\pi}{\sigma}}\, e^{-4ab\sigma\mathbf{i}}, \tag{82}$$

for the Gaussian integral dependent on one parameter and with complex translation.

Finally, we would like to mention that several developments in mathematical physics [10]−[19] present aspects that could find application in more sophisticated and formal simulation models.

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