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On Euler sequence spaces

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Abstract: Investigating the sequence spaces e_p^r, $0 \leq p < \infty$, and e_∞^r, is the aim of this work, which is done with some consideration to [1] and [2]. Also, we put forward some elite features of these spaces in terms of their bounded linear operators. To be more specific, we provide a response to the following: which of these spaces contain the properties of the Approximation, the Dunford-Pettis, the Radon-Riesz, and the Hahn-Banach extensions. Our study also examines the rotundity and smoothness of these spaces.

Keywords: approximation property, Dunford-Pettis property, Radon-Riesz property, Hahn-Banach extension property, smoothness, rotundity, Euler sequence spaces.

MSC: Primary 46B50, 46B20; Secondary 46B25, 46B26,46A22, 46A16.

1. Introduction

In the historical development of summability theory, Abel, Cesaro, Riesz, N örlund, Borel, Hölder, Hausdorff and Euler transformations are the first ones that come to mind. One of the most important methods developed to make some divergent sequences convergent is the Euler method, or Euler mean. This method actually consists of a matrix with infinite rows and columns, and it was used by Euler to converge a divergent sequence under this matrix transformation. This matrix, of course, cannot make all divergent sequences convergent.

Euler means E^r, for $|r| < 1$, is defined by the matrix $E^r = \left(e_{nk}^r\right)$ such that

$$e_{nk}^r = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k, & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

More exactly

$$E^r = \begin{bmatrix} r & 0 & 0 & 0 & \cdots \\ 2r(1-r) & r^2 & 0 & 0 & \cdots \\ 3r(1-r)^2 & 3r^2(1-r) & r^3 & 0 & \cdots \\ 4r(1-r)^3 & 6r^2(1-r)^2 & 4r^3(1-r) & r^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We know that this triangle matrix is invertible, that is, $(E^r)^{-1}$ exists. Altay and Başar investigated in [1] and [2], the sequence spaces e_p^r, $0 \leq p < \infty$, and e_∞^r , as the set of all sequences such that E^r-transforms of them are in the spaces ℓ_p and ℓ_∞ respectively; that is

$$e_p^r = \left\{ v = (v_n) \in w : E^r v \in \ell_p \right\},$$

and

$$e_\infty^r = \left\{ v = (v_n) \in w : E^r v \in \ell_\infty \right\}.$$

Thus e_p^r, $0 \le p < \infty$, and e_∞^r are BK-spaces with the norms

$$\|x\|_{e_p^r} = \left(\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^p \right)^{1/p},$$

and

$$\|x\|_{e_\infty^r} = \sup_n \left| \sum_{k=1}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|,$$

respectively. There have been several attempts to introduce sequence spaces using the Euler matrix by several authors and using another matrix combined with Euler matrix, for instance, e_0^r and e_c^r [3] and $e_0^r(B(m))$, $e_c^r(B(m))$, $e_\infty^r(B(m))$ [4]. In a recent study, Meng and Mei, in [5], has been introduced the Euler difference sequence spaces $e_c^r\left(B_v^{(m)}\right)$ and $e_c^r\left(B_v^{(m)}\right)$. Furthermore they examined α, β and γ-duals of these sequence spaces. Additionally, B^m-Riesz Sequence Spaces has been examined in [6]-[10].

This study investigates the Euler sequence spaces e_p^r, $0 \le p < \infty$, and e_∞^r and some of its properties. Here we will try to understand their geometrical structures. Moreover, we will investigate whether some nice properties related to linear operators are present on these spaces. It should be noted that many of the results we will give for these basic classical Euler sequence spaces also hold for other generalizations or versions. To better explain some of the nice properties of e_p^r, $0 \le p < \infty$, and e_∞^r, we first give some well-known definitions and results of Banach spaces. We adopt the necessary definitions and results mainly from the book meggison in this paper. X and Y are assumed to be a Banach spaces. If $T(B)$ is a relatively compact (i.e., $\overline{T(B)}$ is compact) subset of Y when B is a bounded subset of X, then a linear operator T from X to Y is compact. $K(X,Y)$ is the set of all compact linear operators from X to Y, perhaps just $K(X)$ if $X = Y$. The domain of a compact linear operator from a Banach space into a Banach space is closed if and only if the operator has a finite rank, i.e., the domain of the operator is finite-dimensional [11].

Definition 1. [12] A normed space X is *rotund* or strictly convex or strictly normed if $\|tx_1 + (1-t)x_2\| < 1$ whenever x_1 and x_2 are distinct points of the unit sphere S_X and $0 < t < 1$.

Rotundity can be characterized more easily with the subsequent theorem.

Theorem 1. *1) Let X is a normed space. Then X is rotund if and only if* $\left\|\frac{1}{2}(x_1+x_2)\right\| < 1$ *whensoever* x_1 *and* x_2 *are distinct points of* S_X, *see* [11].

2) A normed space is rotund if and only if each of its two-dimensional subspaces is rotund, see [11].

Definition 2. [11] Presume x_0 is an element of the unit sphere S_X of a normed space X. Then x_0 is a point of smoothness of the unit sphere B_X if there is no more than one support hypersurface for B_X that supporting B_X at x_0. The space X is smooth if every point of S_X is a point of smoothness of B_X.

Assume that X is a normed space, that $x \in S_X$ and that $y \in X$. Let

$$G_-(x,y) = \lim_{t \to 0^-} \frac{\|x+ty\| - \|x\|}{t},$$

and

$$G_+(x,y) = \lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}.$$

Then $G_-(x,y)$ and $G_+(x,y)$ are, the left-hand and right-hand *Gateaux derivatives* of the norm at x in the direction y, respectively. The norm is *Gateaux differentiable* at x in the direction y if $G_-(x,y) = G_+(x,y)$, in this case the common value of $G_-(x,y)$ and $G_+(x,y)$ is demonstrated by $G(x,y)$ and is named the Gateaux derivative of the norm at x in the y direction. If the norm is Gateaux differentiable at x in any direction y, then the norm is Gateaux differentiable at x. At last, we simply say that the norm is *Gateaux differentiable* if it is Gateaux differentiable at every point of the unit sphere S_X [11].

Theorem 2. *1. A normed space is smooth if and only if its norm is Gateaux differentiable* [11].

2. A normed space is smooth if and only if each of its two-dimensional subspaces is smooth [11].

Definition 3. [11] Consider X as a normed space. Define a function $\rho_X : (0, \infty) \to [0, \infty)$ by the formula

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : x, y \in S_X\right\},$$

if $X \neq \{0\}$, and by the formula

$$\rho_X(t) = \begin{cases} 0, & \text{if } 0 < t < 1, \\ t-1, & \text{if } t \geq 1, \end{cases}$$

if $X = \{0\}$. Then ρ_X is the *modulus of smoothness* of X. The space X is *uniformly smooth* if $\lim_{t\to 0^+} \rho_X(t)/t = 0$.

Remark 1. The circumstance $\lim_{t\to 0^+} \rho_X(t)/t = 0$ implies that the norm of the space is uniformly Gateaux differentiable, which includes the Frechet differentiability of the norm function at any point in any direction. Thus, uniformly smooth spaces are smooth, but the converse may not be true [11].

Now let us mention about special properties of some Banach spaces.

Definition 4. [13] A Banach space X has the *approximation property* if, for every Banach space Y, the set of finite-rank members of $B(Y, X)$ is dense in $K(Y, X)$.

Proposition 1. *The spaces c_0 and ℓ_p, $1 \leq p < \infty$, have the approximation property* [11].

Now we refer to the reader to [11] for the definition of weak topology and weak convergence in Banach spaces. Let X and Y be two Banach spaces. If $T(B)$ is a relatively weakly compact subset of Y when B is a bounded subset of X, then a linear operator T from X to Y is weakly compact. $K^w(X, Y)$ denotes the collection of all weakly compact linear operators from X to Y, or simply $K^w(X)$ when $X = Y$. It is important to note that a subset U of X is relatively weakly compact which means that $\overline{U}$ is a weakly compact subset of Y.In its weak topology, $\overline{U}$ is a weakly compact subset of Y if and only if $\overline{U}$ is a compact subset of Y. From the Eberlein-Smulian theorem comes the following result [11].

Proposition 2. *The operator T can be defined as a linear operator from a Banach space X to a Banach space Y. Then T is weakly compact if and only if for every bounded sequence (x_n) in X has a subsequence $\left(x_{n_j}\right)_{j=0}^{\infty}$ such that $\left(Tx_{n_j}\right)$ converges weakly.*

Definition 5. Let X and Y are Banach spaces. If $T(K)$ is a compact subset of Y when K is a weakly compact subset of X, then a linear operator T from X to Y is *completely continuous,* or a Dunford-Pettis operator. If any weakly compact linear operator from Banach space X to Banach space Y is completely continuous then Banach space X has the *Dunford-Pettis property* [11].

Proposition 3. [11] *ℓ_1 has the Dunford-Pettis Property.*

Let us now mention another special property of Banach spaces.

Definition 6. [11] If a normed space meets the following criteria, it is referred to as a Radon-Riesz space. A normed space possesses the *Radon-Riesz property*, also known as the *Kadets-Klee property*. Whenever (x_n) is a sequence in the space and x is an element of the space such that $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$.

The Hahn-Banach extension property is an unconventional property of the sequence space ℓ_∞, as demonstrated by R.S. Phillips. More specifically, the following theorem contains this property.

Theorem 3. *(R.S.Phillips, [14]) Assume that $T : Y \to \ell_\infty$ is a bounded linear operator and that Y is a linear subspace of the Banach space X. Subsequently, T can be expanded to a bounded linear operator $S : X \to \ell_\infty$ with a identical norm to T.*

We refer to this theorem as the *Hahn-Banach extension property* of ℓ_∞. This terminology can be used to rewrite the classical Hahn-Banach theorem as "$\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ has the Hahn-Banach extension property".

2. Main Results

A Banach space with uniform smoothness indicates that all the balls in the space have no sharp edges or cliffs. Let us now see that most e_p^r spaces have this nice property.

Theorem 4. *The space e_p^r is uniformly smooth for $1 < p < \infty$.*

Proof. Primarily, let us compute $\|x+ty\|_{e_p^r}$ and $\|x-ty\|_{e_p^r}$.

$$\|x+ty\|_{e_p^r}^p = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \binom{n}{k} (1-r)^{n-k} r^k (x_k + ty_k) \right|^p,$$

and

$$\|x-ty\|_{e_p^r}^p = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \binom{n}{k} (1-r)^{n-k} r^k (x_k - ty_k) \right|^p$$

$\lim_{t\to 0^+} \rho_X(t)/t$ gives the $0/0$ uncertainty in the primary stage, and we can perform the L'Hospital rule here. At that time

$$\lim_{t\to 0^+} \rho_X(t)/t = \lim_{t\to 0^+} \frac{d}{dt}(\rho_X(t)).$$

Let's find out what $\frac{d}{dt}(\rho_X(t))$ is now.The linearity of the derivative and the properties of the supremum lead us to write

$$\frac{d}{dt}(\rho_X(t)) = \sup\left\{ \frac{1}{2}\left(\frac{d}{dt}\|x+ty\| + \frac{d}{dt}\|x-ty\| \right) : x, y \in S_X \right\}.$$

Let us recall

$$(E^r x)_n = \sum_{k=1}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k.$$

Now we have

$$\begin{aligned} \frac{d}{dt}\left(\|x+ty\|^p\right) &= \frac{d}{dt}\left(\|E^r(x+ty)\|_{l_p}^p\right) \\ &= \frac{d}{dt}\sum_{n=1}^{\infty} |(E^r(x+ty))_n|^p \\ &= \sum_{n=1}^{\infty} \frac{d}{dt}|(E^r(x+ty))_n|^p \\ &= \sum_{n=1}^{\infty} \left(p\,|(E^r(x+ty))_n|^{p-1} \frac{d}{dt}|(E^r(x+ty))_n| \right), \end{aligned}$$

and similarly

$$\frac{d}{dt}\left(\|x-ty\|^p\right) = \sum_{n=1}^{\infty} \left(p\,|(E^r(x-ty))_n|^{p-1} \frac{d}{dt}|(E^r(x-ty))_n| \right).$$

In particular,

$$\begin{aligned} \frac{d}{dt}|(E^r(x+ty))_n| &= \begin{cases} \frac{d}{dt}((E^r(x+ty))_n), & \text{if } (E^r(x+ty))_n \geq 0 \\ -\frac{d}{dt}((E^r(x+ty))_n), & \text{if } (E^r(x+ty))_n < 0 \end{cases} \\ &= \begin{cases} (E^r y)_n, & \text{if } (E^r(x+ty))_n \geq 0 \\ -(E^r y)_n, & \text{if } (E^r(x+ty))_n < 0 \end{cases} \end{aligned}$$

and similarly

$$\frac{d}{dt}|(E^r(x-ty))_n| = \begin{cases} -(E^r y)_n, & \text{if } (E^r(x-ty))_n \geq 0, \\ (E^r y)_n, & \text{if } (E^r(x-ty))_n < 0. \end{cases}$$

When we apply $t \to 0^+$ we get the following equations:

$$\lim_{t\to 0^+} \frac{d}{dt}\|x+ty\|^p = \begin{cases} p\sum\limits_{n=1}^{\infty} |(E^r x)_n|^{p-1}(E^r y)_n, & \text{if } (E^r x)_n \geq 0 \\ -p\sum\limits_{n=1}^{\infty} |(E^r x)_n|^{p-1}(E^r y)_n, & \text{if } (E^r x)_n < 0 \end{cases}$$

and

$$\lim_{t\to 0^+} \frac{d}{dt}\|x-ty\|^p = \begin{cases} -p\sum\limits_{n=1}^{\infty} |(E^r x)_n|^{p-1}(E^r y)_n, & \text{if } (E^r x)_n \geq 0, \\ p\sum\limits_{n=1}^{\infty} |(E^r x)_n|^{p-1}(E^r y)_n, & \text{if } (E^r x)_n < 0. \end{cases}$$

We just see that

$$\lim_{t\to 0^+} \frac{d}{dt}\|x+ty\|^p + \lim_{t\to 0^+} \frac{d}{dt}\|x-ty\|^p = 0.$$

Remember that $|a| \leq |a|^p$ for $1 < p < \infty$, and so

$$\lim_{t\to 0^+} \frac{d}{dt}\|x+ty\| + \lim_{t\to 0^+} \frac{d}{dt}\|x-ty\| = 0$$

Eventually we get

$$\lim_{t\to 0^+} \frac{d}{dt}(\rho_X(t)) = 0.$$

The proof is now complete. □

Theorem 5. *e_1^r and e_∞^r are not uniformly smooth.*

We know that all $\ell_p, 1 < p < \infty$, spaces inform us that the unit spheres of them are rotund. e_p^r enjoys this property as ell.

Theorem 6. *For $1 < p < \infty$, all e_p^r spaces are rotund.*

Proof. By the Theorem 1 it is sufficient to prove the rotundity of the space $span\{e_1, e_2\} = Z$ in e_p^r where e_1, e_2 are elements of the unit vector basis of ℓ_p. In other words, we will think about the two-dimensional subspace

$$Z = \left\{(x_1, x_2, 0, 0...) : (x_1, x_2, 0, 0...) \in e_p^r\right\}.$$

Let x and y be arbitrary elements of S_Z and let $x + y = (x_1 + y_1, x_2 + y_2, 0, 0, \cdots)$.

$$\left\|\frac{1}{2}(x+y)\right\|_{e_p^r}^p = \left\|\left(\left(E^r\left(\frac{1}{2}(x+y)\right)\right)_n\right)_{n=1}^{\infty}\right\|_{\ell_p}^p = \left\|\frac{1}{2}\left[((E^r(x))_n)_{n=1}^{\infty} + ((E^r(y))_n)_{n=1}^{\infty}\right]\right\|_{\ell_p}^p.$$

Remember that

$$\|x\|_{e_p^r}^p = \|((E^r(x))_n)_{n=1}^{\infty}\|_{\ell_p}^p = 1 = \|y\|_{e_p^r}^p = \|((E^r(y))_n)_{n=1}^{\infty}\|_{\ell_p}^p$$

and ℓ_p is rotund. Then

$$\left\|\frac{1}{2}\left[((E^r(x))_n)_{n=1}^{\infty} + ((E^r(y))_n)_{n=1}^{\infty}\right]\right\|_{\ell_p}^p < 1.$$

□

In general, finite rank linear operators are easy to study. If any linear operator between normed spaces can be approximated by a set of finite rank operators, then any problem or scientific model can be easily solved with an acceptable error rate. But in order to do this, the Banach space in which the problem is constructed must have the approximation property. Many such fortunate circumstances are allowed by Banach spaces with this characteristic.

Theorem 7. *The Banach space e_p^r has the approximation property for $1 \leq p < \infty$.*

Proof. Assuming T be a compact linear operator from a Banach space Y into e_p^r. We will find a sequence (T_n) of bounded linear operators of finite-rank from Y into e_p^r such that $T_n \to T$ in $B\left(Y, e_p^r\right)$. From the assumption, for any $x \in Y$, $Tx \in e_p^r$ and for any bounded sequence (x_n) in Y, the sequence (Tx_n) has a convergent subsequence $\left(Tx_{n_j}\right)_{j=0}^{\infty}$ in e_p^r. Thus

$$\left\|Tx_{n_i} - Tx_{n_j}\right\|_{e_p^r} = \left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{e_p^r} \to 0 \text{ for } i, j \to \infty.$$

Recalling the definition of the space $\ell_p(E^r)$,

$$\left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{e_p^r} = \left\|(E^r T)\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p},$$

we get the operator $E^r T : Y \to \ell_p$ is well-defined and compact. The matrix transformation E^r is bounded (See [15]) linear operator and so is $E^r T$. Since ℓ_p has the approximation property, there exits a sequence $(A_m)_{m=0}^{\infty}$ of bounded linear operators of finite-rank from Y to ℓ_p such that $\|E^r T - A_m\| \to 0$ as $m \to \infty$. Now the sequence $\left((E^r)^{-1} A_m = T_m\right)_{m=1}^{\infty}$ is the desired sequence of finite-rank from Y to e_p^r. It is easy to see that any $(E^r)^{-1} A_m$ has the finite-rank.and is bounded linear. Further

$$\begin{aligned}
\|T - T_m\| &= \left\|T - (E^r)^{-1} A_m\right\| \\
&= \sup_{\|x\|=1} \left\|\left(T - (E^r)^{-1} A_m\right) x\right\|_{e_p^r} \\
&= \sup_{\|x\|=1} \left\|Tx - \left((E^r)^{-1} A_m\right) x\right\|_{e_p^r} \\
&= \sup_{\|x\|=1} \left\|E^r Tx - E^r\left((E^r)^{-1} A_m\right) x\right\|_{\ell_p} \\
&= \sup_{\|x\|=1} \|(E^r T - A_m) x\|_{\ell_p} \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}$$

This completes the proof. □

Theorem 8. *e_1^r has the Dunford-Pettis Property.*

Proof. Let $T : e_1^r \to Y$ be a weakly compact linear operator and compose T with $(E^r)^{-1}$. It follows that $T(E^r)^{-1}$ is clearly a bounded linear operator from ℓ_1 into Y. It is also just weakly compact. Let's demonstrate this: Assume U is a bounded set in ℓ_1.We know that $(E^r)^{-1}(U)$ is a bounded subset of e_1^r because to the boundedness of the matrix operator $(E^r)^{-1}$. For this reason

$$T\left((E^r)^{-1}(U)\right) = \left(T(E^r)^{-1}\right)(U)$$

is a relatively weakly compact set in Y. Because of this $T(E^r)^{-1} : \ell_1 \to Y$ is weakly compact. Now, we can deduce that $T(E^r)^{-1}$ is entirely continuous because ℓ_1 possesses the Dunford-Pettis Property. Let W be a weakly compact subset of e_1^r. As a result $E^r(W)$ is a weakly compact subset of ℓ_1 [11, Exercise 3.50.], and

$$\left(T(E^r)^{-1}\right) E^r(W) = T(W)$$

is a compact subset of Y. □

We will show that e_∞^r possesses the Hahn-Banach extension property.

Theorem 9. *Let Y be a linear subspace of the Banach space X and suppose $T : Y \to e_\infty^r$ is a bounded linear operator. Then T may be extended to a bounded linear operator $S : X \to e_\infty^r$ having the identical norm as T.*

Proof. For any bounded linear operator $T : Y \to e^r_\infty$, $E^r T \in B(Y, \ell_\infty)$ and from the Theorem 3, ℓ_∞ has the Hahn-Banach extension property. Therefore, we get $E^r T$ may be extended to a bounded linear operator $U : X \to \ell_\infty$ having the identical norm as $E^r T$. We can now think about the operator $(E^r)^{-1} U$. According to conventional operator algebra, the $(E^r)^{-1} U = S$ operator from X to e^r_∞ is a bounded linear operator. Our goal is to demonstrate that S is an extension of T and $\|T\| = \|S\|$. For any $y \in Y$,

$$\begin{aligned} Sy &= \left((E^r)^{-1} U\right) y = (E^r)^{-1} (Uy) \\ &= (E^r)^{-1} (E^r T) y = Ty. \end{aligned}$$

Now

$$\begin{aligned} \|S\| &= \left\|(E^r)^{-1} U\right\| = \left\|(E^r)^{-1} (E^r T)\right\| \\ &= \left\|\left((E^r)^{-1} E^r\right) T\right\| = \|I_{e^r_\infty} T\| = \|T\| \end{aligned}$$

where $I_{e^r_\infty}$ is the identity operator on e^r_∞. □

We see that e^r_2 has an elegance property which we call as the Radon-Riesz Property. we refer the reader to sources [16–18]. Radon-Riesz Property also named as the Kadets-Klee property since their further investigation and application of the topic [19–21].

Theorem 10. *e^r_2 has the Radon-Riesz Property.*

Proof. Let (x_n) be a sequence in e^r_2 and x be an element of e^r_2. Take into account $x_n \xrightarrow{w} x$ and $\|x_n\|_{e^r_2} \to \|x\|_{e^r_2}$. We'll demonstrate that $x_n \to x$. Since we now suppose that $x_n \xrightarrow{w} x$, for each $y \in (e^r_2)^*$ yx we find $yx_n \to yx$. To finish the proof, let's demonstrate that $\|x_n - x\|_{e^r_2} \to 0$:

$$\begin{aligned} \|x_n - x\|^2_{e^r_2} &= \|E^r x_n - E^r x\|^2_{\ell_2} \\ &= \langle E^r x_n - E^r x, E^r x_n - E^r x\rangle_{\ell_2} \\ &= \langle E^r x_n, E^r x_n\rangle_{\ell_2} - \langle E^r x_n, E^r x\rangle_{\ell_2} \\ &\quad - \langle E^r x, E^r x_n\rangle_{\ell_2} + \langle E^r x, E^r x\rangle_{\ell_2} \\ &= \|E^r x_n\|^2_{\ell_2} + \|E^r x\|^2_{\ell_2} - \langle E^r x_n, E^r x\rangle_{\ell_2} - \langle E^r x, E^r x_n\rangle_{\ell_2} \end{aligned}$$

Let $z = E^r x \in \ell_2 = \ell_2^*$ and let us consider $z \circ E^r$ such that $(z \circ E^r) x = \langle E^r x, E^r x\rangle_{\ell_2}$. Then from the properties of the matrix E^r and by the Riesz's Theorem (on ℓ_2) we have $z \circ E^r$ is a continuous linear functional on e^r_2 and

$$(z \circ E^r) x_n = z(E^r x_n) = \langle E^r x_n, E^r x\rangle_{\ell_2}.$$

In accordance with the assumption $x_n \xrightarrow{w} x$ we get

$$\begin{aligned} (z \circ E^r)(x_n) &= \langle E^r x_n, E^r x\rangle_{\ell_2} \\ &\to \langle E^r x, E^r x\rangle_{\ell_2}, \text{ as } n \to \infty, \\ &= (z \circ E^r)(x) \\ &= \|E^r x\|^2_{\ell_2} \end{aligned}$$

Let us now take $z_n = E^r x_n \in \ell_2^* = \ell_2$, for each n, then

$$(z_n \circ E^r) x = z_n(E^r x) = \langle E^r x, E^r x_n\rangle_{\ell_2}.$$

Now, each $z_n \circ E^r$ is a continuous linear functional on e_2^r and again based on the assumption $x_n \overset{w}{\to} x$ we have

$$\begin{aligned}(z_n \circ E^r)(x) &= z_n(E^r x) \\ &= \langle E^r x, E^r x_n \rangle_{\ell_2} \\ &= \overline{\langle E^r x_n, E^r x \rangle_{\ell_2}} \\ &\to \overline{(f \circ E^r)(x)}, \text{ as } n \to \infty, \\ &= \overline{\langle E^r x, E^r x \rangle_{\ell_2}} \\ &= \|E^r x\|_{\ell_2}^2 .\end{aligned}$$

Eventually, by the assumption $\|x_n\|_{e_2^r} \to \|x\|_{e_2^r}$, we obtain

$$\begin{aligned}\|x_n - x\|_{e_2^r}^2 &= \|E^r x_n\|_{\ell_2}^2 + \|E^r x\|_{\ell_2}^2 - \langle E^r x_n, E^r x \rangle_{\ell_2} - \langle E^r x, E^r x_n \rangle_{\ell_2} \\ &\to \|E^r x\|_{\ell_2}^2 + \|E^r x\|_{\ell_2}^2 - \|E^r x\|_{\ell_2}^2 - \|E^r x\|_{\ell_2}^2 \\ &= 0, \text{ as } n \to \infty.\end{aligned}$$

□

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