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On sequential Henstock-Stieltjes integral for interval valued functions

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Abstract: In this paper, we introduce the concept of Sequential Henstock Stieltjes integral for interval valued functions and prove some properties of this integral.

Keywords: sequential Henstock stieltjes integral, interval valued functions, guages.

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1. Introduction and Preliminaries

Henstock and J. Kurzweil independently introduced, in the late 1950s, a Riemann-type integral widely known as the Henstock-Kurzweil integral to overcome the deficiencies of the Lebesgue integral. It is well established that the Henstock integral generalizes both the Riemann and Lebesgue integrals. While the Lebesgue integral is considerably more complex than the Riemann integral, the Henstock integral is more powerful and simpler than the Wiener, Lebesgue, and Feynman integrals [1–15]. Furthermore, the Henstock integral has been shown to be equivalent to the Perron and Denjoy integrals, which recover a continuous function from its derivative.

In 2000, Wu and Gong [14] introduced the notion of Henstock (H) integrals for interval-valued functions and fuzzy number-valued functions, obtaining several properties. Two years earlier, Lim et al. [11] introduced the concept of Henstock-Stieltjes integrals for real-valued functions, which generalized the Henstock (H) integral, and established their properties.

Yoon [15] introduced the interval-valued Henstock-Stieltjes integral and investigated some of its properties. In 2016, Paxton [13] introduced the notion of Sequential Henstock (SH) integrals for real-valued functions as a generalization of the Henstock (H) integral and obtained several properties of this integral. Hamid and Elmuiz [2] presented the concept of Henstock-Stieltjes integrals for interval-valued functions and fuzzy number-valued functions, and established various properties of these integrals.

In this paper, we introduce the concept of Sequential Henstock-Stieltjes (SHS) integrals for interval-valued functions and discuss some properties of this integral.

We let $\mathbb{R}$ denote the set of real numbers, $F(X)$ as a function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a,b]$, X, as non empty interval in $\mathbb{R}$ and $\ll$ as much more smaller.

A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$ while a sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.

Definition 1. ([13]) A function $F : [a,b] \to \mathbb{R}$ is said to be *Henstock integrable* on $[a,b]$, denoted by $F \in H[a,b]$, with Henstock integral α, written as

$$\alpha = (H) \int_{[a,b]} F(x)\, dx,$$

provided that for every $\varepsilon > 0$, there exists a gauge function $\delta(x) > 0$ on $[a,b]$ such that for any $\delta(x)$-fine tagged partition $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ of $[a,b]$, the following inequality holds:

$$\left| \sum_{i=1}^{n} F(t_i)(u_i - u_{i-1}) - \alpha \right| < \varepsilon.$$

Definition 2. ([11]) Let $g : [a, b] \to \mathbb{R}$ be a non-decreasing function. A real-valued function $F : [a, b] \to \mathbb{R}$ is *Henstock-Stieltjes integrable* on $[a, b]$, denoted by $F \in HS[a, b]$, with Henstock-Stieltjes integral α, written as

$$\alpha = (H) \int_{[a,b]} F \, dg,$$

provided that for every $\varepsilon > 0$, there exists a gauge function $\delta(x) > 0$ on $[a, b]$ such that for any $\delta(x)$-fine tagged partition $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ of $[a, b]$, the following inequality holds:

$$\left| \sum_{i=1}^{n} F(t_i) \left[g(u_i) - g(u_{i-1}) \right] - \alpha \right| < \varepsilon.$$

Definition 3. ([13]) A function $F : [a, b] \to \mathbb{R}$ is *Sequential Henstock integrable* on $[a, b]$, denoted by $F \in SH[a, b]$, with Sequential Henstock integral α, written as

$$\alpha = (SH) \int_{[a,b]} F(x) \, dx,$$

provided that for every $\varepsilon > 0$, there exists a sequence of gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x)$-fine tagged partition of $[a, b]$, the following inequality holds:

$$\left| \sum_{i=1}^{m_n} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha \right| < \varepsilon.$$

Definition 4. ([12,15]) Let

$$I_{\mathbb{R}} = \{ I = [I^-, I^+] \mid I \text{ is a closed and bounded interval on the real line } \mathbb{R} \}.$$

For $X, Y \in I_{\mathbb{R}}$, define the following operations and relations:

(i) *Order*: $X \leq Y$ if and only if $Y^- \leq X^-$ and $X^+ \leq Y^+$.
(ii) *Addition*: $X + Y = Z$ if and only if $Z^- = X^- + Y^-$ and $Z^+ = X^+ + Y^+$.
(iii) *Multiplication*: $X \cdot Y = \{ x \cdot y \mid x \in X, y \in Y \}$, where

$$(X \cdot Y)^- = \min\{X^-Y^-, X^-Y^+, X^+Y^-, X^+Y^+\}$$

and

$$(X \cdot Y)^+ = \max\{X^-Y^-, X^-Y^+, X^+Y^-, X^+Y^+\}.$$

The metric distance between intervals X and Y is defined by

$$d(X, Y) = \max \left(|X^- - Y^-|, \ |X^+ - Y^+| \right).$$

Definition 5. ([2]) An interval-valued function $F : [a, b] \to I_{\mathbb{R}}$ is *Henstock integrable* on $[a, b]$, denoted by $F \in IH[a, b]$, with Henstock integral $I_0 \in I_{\mathbb{R}}$, written as

$$I_0 = (IH) \int_{[a,b]} F,$$

provided that for every $\varepsilon > 0$, there exists a gauge function $\delta(x) > 0$ on $[a, b]$ such that for any $\delta(x)$-fine Henstock tagged partition $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ of $[a, b]$, the following inequality holds:

$$d\left(\sum_{i=1}^{n} F(t_i)(u_i - u_{i-1}), I_0 \right) < \varepsilon.$$

Definition 6. ([2]) Let $g : [a, b] \to \mathbb{R}$ be a non-decreasing function. An interval-valued function $F : [a, b] \to I_{\mathbb{R}}$ is *Henstock-Stieltjes integrable* on $[a, b]$ with respect to g, denoted by $F \in IHS[a, b]$, with Henstock-Stieltjes integral $I_0 \in I_{\mathbb{R}}$, written as

$$I_0 = (IHS) \int_{[a,b]} F \, dg,$$

provided that for every $\varepsilon > 0$, there exists a gauge function $\delta(x) > 0$ on $[a,b]$ such that for any $\delta(x)$-fine Henstock tagged partition $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$ of $[a,b]$, the following inequality holds:

$$d\left(\sum_{i=1}^{n} F(t_i)\left[g(u_i) - g(u_{i-1})\right], I_0\right) < \varepsilon.$$

Definition 7. Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function. An interval-valued function $F : [a,b] \to I_{\mathbb{R}}$ is *Sequential Henstock-Stieltjes integrable* on $[a,b]$ with respect to g, denoted by $F \in ISHS[a,b]$, with Sequential Henstock-Stieltjes integral $I_0 \in I_{\mathbb{R}}$, written as

$$I_0 = (ISHS)\int_{[a,b]} F\,dg,$$

provided that for every $\varepsilon > 0$, there exists a sequence of gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x)$-fine Henstock tagged partition of $[a,b]$, the following inequality holds:

$$d\left(\sum_{i=1}^{m_n} F(t_{i_n})\left[g(u_{i_n}) - g(u_{(i-1)_n})\right], I_0\right) < \varepsilon.$$

Remark 1. If g is the identity function, the Sequential Henstock-Stieltjes integral reduces to the definition of the *Interval Sequential Henstock integral* as presented in [14].

2. Main Results

We examine some of the properties of the interval Sequential Henstock integral.

Theorem 1. *Let $g : [a,b] \to \mathbb{R}$ be non decreasing function. If $F \in ISHS[a,b]$, then there exists a unique integral value.*

Proof. Suppose the integral is not unique and let $I_1 = (ISHS)\int_{[a,b]} F$ and $I_2 = (ISHS)\int_{[a,b]} F$ such that $I_1 \neq I_2$. Let $\varepsilon > 0$ then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$ and $\{\delta_n^2(x)\}_{n=1}^{\infty}$ such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a,b]$ and for any $\delta_n^2(x)$-fine tagged partitions P_n^2 of $[a,b]$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_1) < \frac{\varepsilon}{2},$$

and

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_2) < \frac{\varepsilon}{2}.$$

We define a positive function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$. Then, we have

$$\begin{aligned} d(I_1, I_2) &= d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n}), I_1) + \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_2) \\ &\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n}), I_1)) + d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So for all $\varepsilon > 0$, there exists a $\delta_n(x) > 0$ as above, then $I_1 = I_2$. □

Theorem 2. *Let $g : [a,b] \to \mathbb{R}$ be non decreasing function. Then $F \in ISHS[a,b]$ if and only if $F^-, F^+ \in SHS[a,b]$ and*

$$(ISHS)\int_{[a,b]} F dg = [(SHS)\int_{[a,b]} F^- dg, (SHS)\int_{[a,b]} F^+ dg]$$

Proof. Let $F \in ISHS[a,b]$, from Definition 7 there exists a unique interval number $I_0 = [I_0^-, I_0^+]$ in the property, then for any $\varepsilon > 0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$, $n \geq \mu$ on $[a,b] \in \mathbb{R}$ such that for any $\delta_n(x)$-fine partition P_n, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_0) < \varepsilon.$$

Observe that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), I_0)$$
$$= \max(|\sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - I_0^-|, |\sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n}) - I_0^+)|).$$

Since $u_{i_n} - u_{(i-1)_n} \geq 0$, for $1 \leq i_n \leq m_n$, then it follows that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - I_0^-| < \varepsilon,$$
$$|\sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n}) - I_0^+)| < \varepsilon,$$

for every $\delta_n(x)$-tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$. Thus, we obtain $F^+, F^- \in SHS[a,b]$,

$$I_0^- = (SHS)\int_{[a,b]} F^- dg$$

and

$$I_0^+ = (SHS)\int_{[a,b]} F^+ dg.$$

Conversely, let $F^- \in SH[a,b]$. Then there exist a unique $\beta_1 \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$, such that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \beta_1| < \varepsilon,$$

whenever P_n^1 is a $\delta_n^1(x)$-fine tagged partition of $[a,b]$.

Also, Let $F^+ \in SHS[a,b]$. Then there exist a unique $\beta_2 \in \mathbb{R}$ such that $\varepsilon > 0$, there exists a $\{\delta_n^2(x)\}_{n=1}^{\infty}$, such that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n}) - \beta_2)| < \varepsilon,$$

whenever P_n^2 is a $\delta_n^2(x)$-fine tagged partitions of $[a,b]$.

Let $\beta = [\beta_1, \beta_2]$. If $F^- \leq F^+$, then $\beta_1 \leq \beta_2$. We define $\delta_n(x) = \min(\delta_n^1(x), \delta_n^2(x))$ and $I_0 = [\beta_1, \beta_2]$, then if P_n is a $\delta_n(x) - fine$ tagged partition of $[a,b]$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})), \beta) < \varepsilon.$$

Hence, $F : [a,b] \to I_{\mathbb{R}}$ is Sequential Henstock integrable with respect to g on $[a,b]$. This completes the proof. □

Theorem 3. *If $F, G \in ISHS[a,b]$ and $\gamma, \xi \in \mathbb{R}$. Then $\gamma F + \xi G \in ISHS[a,b]$ and*

$$(ISHS)\int_{[a,b]} (\gamma F + \xi G) dg = \gamma (ISHS)\int_{[a,b]} F dg + \xi (ISHS)\int_{[a,b]} G dg.$$

Proof. If $F, G \in ISHS[a,b]$, then $[F^-, F^+]$ and $[G^-, G^+] \in SHS[a,b]$ by Theorem 2.2. Hence, $\gamma F^- + \xi G^-, \gamma F^- + \xi G^+, \gamma F^+ + \xi G^-, \gamma F^+ + \xi G^+ \in SHS[a,b]$.
Case 1. If $\gamma > 0$ and $\xi > 0$, then

$$\begin{aligned}(SHS)\int_{[a,b]}(\gamma F+\xi G)^- dg &= (SHS)\int_{[a,b]}(\gamma F^- + \xi G^-)dg\\ &= \gamma(SHS)\int_{[a,b]} F^- dg + \xi(SHS)\int_{[a,b]} G^- dg\\ &= \gamma((ISHS)\int_{[a,b]} Fdg)^- + \xi((ISHS)\int_{[a,b]} Gdg)^-\\ &= (\gamma(ISHS)\int_{[a,b]} Fdg + \xi(ISHS)\int_{[a,b]} Gdg)^-.\end{aligned}$$

Case 2. If $\gamma < 0$ and $\xi > 0$, then

$$\begin{aligned}(SHS)\int_{[a,b]}(\gamma F+\xi G)^- dg &= (SHS)\int_{[a,b]}(\gamma F^+ + \xi G^+)dg\\ &= \gamma(SHS)\int_{[a,b]} F^+ dg + \xi(SHS)\int_{[a,b]} G^+ dg\\ &= \gamma((ISHS)\int_{[a,b]} Fdg)^+ + \xi((ISHS)\int_{[a,b]} Gdg)^+\\ &= (\gamma(ISHS)\int_{[a,b]} Fdg + \xi(ISHS)\int_{[a,b]} Gdg)^-.\end{aligned}$$

Case 3. If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$\begin{aligned}(ISHS)\int_{[a,b]}(\gamma F+\xi G)^- dg &= (SHS)\int_{[a,b]}(\gamma F^- + \xi G^+)dg\\ &= \gamma(SHS)\int_{[a,b]} F^- dg + \xi(SHS)\int_{[a,b]} G^+ dg\\ &= \gamma((ISHS)\int_{[a,b]} Fdg)^- + \xi((ISHS)\int_{[a,b]} Gdg)^+\\ &= (\gamma(ISHS)\int_{[a,b]} Fdg + \xi(ISHS)\int_{[a,b]} Gdg)^-.\end{aligned}$$

Similarly, for the three cases above, we have

$$(ISHS)\int_{[a,b]}(\gamma F+\xi G)^+ dg = (\gamma(ISHS)\int_{[a,b]} Fdg + \xi(ISHS)\int_{[a,b]} Gdg)^+.$$

Hence, by Theorem 2, $\gamma F, \xi G \in ISHS[a,b]$ and

$$(ISHS)\int_{[a,b]}(\gamma F+\xi G)dg = \gamma(ISHS)\int_{[a,b]} Fdg + \xi(ISHS)\int_{[a,b]} Gdg.$$

This completes the proof. □

Theorem 4. *Let $g : [a,b] \to \mathbb{R}$ be non decreasing function. Let $F, G \in ISHS[a,b]$ and $F(x) \leq G(x)$ nearly everywhere on $[a,b]$, then*

$$(ISHS)\int_{[a,b]} Fdg \leq (ISHS)\int_{[a,b]} Gdg$$

Proof. Let $F(x) \leq G(x)$ nearly everywhere on $[a,b]$ and $F, G \in ISHS[a,b]$, then $F^-, F^+ \in SHS[a,b]$ and $, G^-, G^+ \in SHS[a,b]$ with $F^- \leq F^+$, and $G^- \leq G^+$ nearly everywhere on $[a,b]$.

$$(SHS)\int_{[a,b]} F^-(x)dg \leq (SHS)\int_{[a,b]} G^-(x)dg$$

and

$$(SHS)\int_{[a,b]} F^+ dg \le (SHS)\int_{[a,b]} G^+(x)dg.$$

Hence by Theorem 2, we have

$$(ISHS)\int_{[a,b]} F(x)dg \le (ISHS)\int_{[a,b]} G(x)dg.$$

This completes the proof. □

Theorem 5. *Let $F, G \in ISHS[a,b]$ and $d(F,K)$ is Sequential Henstock Stieltjes(SHS) integrable on $[a,b]$, then*

$$d((ISHS)\int_{[a,b]} Fdg, (ISHS)\int_{[a,b]} Gdg) \le (SHS)\int_{[a,b]} d(F,G)dg.$$

Proof. By metric definition, we have

$$\begin{aligned} & d((ISHS)\int_{[a,b]} Fdg, (ISHS)\int_{[a,b]} Gdg) \\ &= \max(|((SHS)\int_{[a,b]} Fdg)^- - ((SH)\int_{[a,b]} Gdg)^-|, |((SHS)\int_{[a,b]} Fdg)^+ - ((SHS)\int_{[a,b]} Gdg)^+|) \\ &= \max(|(SHS)\int_{[a,b]} (F^- - G^-)dg|, |(SHS)\int_{[a,b]} (F^+ - G^+)dg|) \\ &\le \max((SHS)\int_{[a,b]} |(F^- - G^-)|dg, (SHS)\int_{[a,b]} |(F^+ - G^+)|dg) \\ &\le (SHS)\int_{[a,b]} \max(|(F^- - G^-)|, |(F^+ - G^+)|)dg \\ &\le (SHS)\int_{[a,b]} d(F,G)dg. \end{aligned}$$

This completes the proof. □

Theorem 6. *Let $F \in ISHS[a,c]$ and $F \in ISHS[c,b]$, then $F \in ISHS[a,b]$ and*

$$(ISHS)\int_a^b Fdg = (ISHS)\int_a^c Fdg + (ISHS)\int_c^b Fdg.$$

Proof. If $f \in ISHS[a,c])$ and $f \in ISHS[c,b])$ then by Theorem 2, $f^-, f^+ \in SHS[a,c])$ and $f^-, f^+ \in SHS[c,b])$. Hence, $f^-, f^+ \in SHS[a,b])$ and

$$\begin{aligned} SHS)\int_a^b f^- dg &= (SHS)\int_a^c f^- dg + (SHS)\int_c^b f^- dg \\ &= ((ISHS)\int_a^c fdg + (ISHS)\int_c^b fdg)^-. \end{aligned}$$

Similarly,

$$\begin{aligned} (SHS)\int_a^b f^+ dg &= (SHS)\int_a^c f^+ dg + (SHS)\int_c^b f^+ dg \\ &= ((ISHS)\int_a^c fdg + (ISHS)\int_c^b fdg)^+. \end{aligned}$$

Hence by Theorem 2, $f \in ap\text{-}ISHS[a,b]$ and

$$(ISHS)\int_a^b fdg = (ISHS)\int_a^c fdg + (ISHS)\int_c^b fdg.$$

□

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