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Regularity of random attractors for non-autonomous stochastic strong damped wave equations driven by linear memory and additive noise

Abdelmajid Ali Dafallah[1,*], Qiaozhen MA[2] and Eshag Mohamed Ahmed[3]

1 Faculty of Petroleum and Hydrology Engineering, *Alsalam* University, Almugled, Sudan
2 Faculty of Mathematics and Informatics, *Northwest Normal* University, Lanzhou 730070, P.R. China
3 Faculty of Pure and Applied Sciences, *International* University of Africa, Khartoum, Sudan
* Correspondence: dafallah@yahoo.com

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Abstract: In this paper, the study identified existence regularity of a random attractor for the stochastic dynamical system generated by non-autonomous strongly damping wave equation with linear memory and additive noise defined on $\mathbb{R}^n$. First, to prove the existence of the pullback absorbing set and the pullback asymptotic compactness of the cocycle in a certain parameter region by using tail estimates and the decomposition technique of solutions. Then it proved the existence and uniqueness of a random attractor.

Keywords: Stochastic wave equation, Asymptotic Compactness, Random Dynamical System, random attractor.

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1. Introduction

It is considered the following non-autonomous wave equation with linear memory on an unbounded domain:

$$u_{tt} - k(\infty)\triangle u - \alpha\triangle u_t - \int_0^\infty k'(s)\Delta(u(t) - u(t-s))ds + f(u) = g(x,t) + \varepsilon\sum_{j=1}^m h_j\dot{W}_j(t), \tag{1}$$

with initial data

$$\begin{cases} u(\tau,x) = u_0(\tau,x), \\ u_t(\tau,x) = u_1(\tau,x),\ x\in\ \mathbb{R}^n,\ \tau\in\mathbb{R}, \\ \eta(x,\tau,s) = \eta_0, x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+. \end{cases} \tag{2}$$

Where $\alpha, k(\infty) > 0$ and $k'(s)\leq 0$ for every $s\in\ \mathbb{R}^+$, $-\Delta$ is the Laplacian operator with respect to the variable $x\in\mathbb{R}^n$ with $n=3$, $u=u(t,x)$ is a real function of $x\in\mathbb{R}^n$ and $t\geq\tau$, $\tau\in\mathbb{R}$, ε is a positive constant. The given function $g(x,t)\in Ł^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ is a external force depending on t, $h_j\in H^2(\mathbb{R}^n)$ and $W(t)$ is an independent two sided real-valued wiener processes on probability space.

Following the idea of [1–3], it is introduced a Hilbert " history " space $\mathfrak{R}_\mu = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$ with the inner product and new variable, respectively

$$\begin{cases} (\eta_1,\eta_2)_{\mu,1} = \displaystyle\int_0^\infty \mu(s)(\nabla\eta_1(s),\nabla\eta_2(s))ds, \\ \|\eta\|^2_{\mu,1} = (\eta,\eta)_{\mu,1} = \displaystyle\int_0^\infty \mu(s)(\nabla\eta(s),\nabla\eta(s))ds, \end{cases} \tag{3}$$

whereas

$$\eta(x,t,s) = u(x,t) - u(x,t-s), \eta_t = \frac{\partial}{\partial t}\eta\ ,\ \eta_s = \frac{\partial}{\partial s}\eta.$$

Let $\mu(s) = k'(s)$ and $k(\infty) = \beta$. Thus, the Eq. (1) can be transformed into the following system

$$\begin{cases} u_{tt} - \beta\triangle u - \alpha\triangle u_t - \int_0^\infty \mu(s)\triangle\eta(s)ds + f(u) = g(x,t) + \varepsilon\sum_{j=1}^m h_j\dot{W}_j, \\ \eta_t = -\eta_s + u_t, \end{cases} \tag{4}$$

with initial conditions

$$\begin{cases} u(\tau,x) = u_0(\tau,x), u_t(\tau,x) = u_1(\tau,x), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, \\ \eta_0 = \eta(x,\tau,s) = u_0(x,\tau) - u_0(x,\tau-s), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, s \in\ \mathbb{R}^+. \end{cases} \tag{5}$$

The following conditions are necessary to obtain our main results, which come from [4,5].
(a) Concerning the memory kernel μ, it is assumed the following hypotheses hold:

$$\mu \in \mathbb{C}^1(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+), \mu(s) \geq 0, \mu'(s) \leq 0\,, \forall s \in \mathbb{R}^+, \tag{6}$$

$$\mu'(s) + \sigma\mu(s) \leq 0\,, \forall s \in \mathbb{R}^+ and\ \sigma > 0, \tag{7}$$

and denote

$$m_0 := \int_0^\infty \mu(s)ds < \infty. \tag{8}$$

(b) The nonlinear function $f \in \mathbb{C}^2(\mathbb{R})$ with $f(0) = 0$, and it satisfies the following conditions.

$$|f'(u)| \leq C(1 + |u|^\gamma)\,, \forall\ u \in \mathbb{R}, 0 \leq \gamma \leq 4, \tag{9}$$

$$C_3 u^{\gamma+2} - C_4 u^2 \leq F(u) \leq C_1 u f(u) + C_2 u^2, \forall\ u \in\ \mathbb{R}, 0 \leq \gamma \leq 4, \tag{10}$$

whereas $F(s) = \int_0^s f(r)dr$, C, C_1, C_2, C_3, C_4 are positive constants.
We need the following condition on $g(x,t) \in Ł^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, there exists a positive constant σ such that

$$\int_{-\infty}^\tau e^{\sigma r}\|g(\cdot,r)\|^2 dr < \infty, \ \ \forall\tau \in \mathbb{R} \tag{11}$$

which implies that

$$\lim_{k\to\infty}\int_{-\infty}^\tau\int_{|x|\geq k} e^{\sigma r}|g(x,r)|^2 dxdr = 0, \ \forall\tau \in \mathbb{R} \tag{12}$$

the condition (11) does not require $g(\cdot,t)$ to be bounded in $L^2(\mathbb{R})$ when $|t| \to \infty$.

Random attractors for the stochastic dynamical system were first introduced by Crauel and Flandoli [6,7]. In the decades, the existence of random attractors for the stochastic partial differential equations were investigated by many authors, see [8–23] and reference therein. For instance, Yin and Liu considered the random attractors for the non-autonomous wave equation with additive noise on the unbounded domain in [24]. Zhou and Zhao in [25] studied the existence of random attractors for the non-autonomous wave equation with linear memory and white noise on the bounded domain. However, there are no results on random attractors for the non-autonomous stochastic strongly damped wave equation with memory and additive noise on unbounded domain. Since the equation (1)-(2) is defined an unbounded domain, it brings some extra obstacles to prove the asymptotic compactness of solutions since the Sobolev embedding are not compact on $\mathbb{R}^n$. The difficulty will be overcome by the uniform estimates on the tails of solution. the splitting technique will be combined in [2,5,28] with the idea of uniform estimates of solutions to investigate the existence of a random attractor. When $\varepsilon = 0$ in (4)-(5), the determined damping wave equations with linear memory has been discussed by several authors in [1,4,26,27].

This paper is organized as follows. Next Sections, we recall some preliminaries and properties for general random dynamical system and results on the existence of a pullback random attractor for random dynamical systems. In Section three, we define a continuous random dynamical system for (4)-(5) in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_\mu$. In Section four and five, we decompose the solutions of the random differential equation into two parts in order to obtain the asymptotic compactness of the uniform estimates of solution as t$\to\infty$. Then, the last

Section we prove the existence and uniqueness of a random attractor. Without loss of generality, the letters C and $C_i(i=1,2,...)$ are denoted as some generic positive constants which may change their values from line to line or even in the same line.

2. Preliminaries

In this Section, we recall some basic concepts related to RDS and a random attractor for RDS in [6,7,13,14,16], which are important for getting our main results. Let $(\Omega,\mathcal{F},P)$ be a probability space and (X,d) be a Polish space with the Borel σ-algebra $B(X)$. The distance between $x\in X$ and $B\subseteq X$ is denoted by $d(x,B)$. If $B\subseteq X$ and $C\subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B,C)=\sup_{x\in B} d(x,C)$.

Definition 1. $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta:\ \mathbb{R}\times\Omega\longrightarrow\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t}=\theta_t\circ\theta_s$ for all s,t $\in\mathbb{R}$ and θ_0P = P for all t$\in\mathbb{R}$.

Definition 2. A mapping $\Phi(t,\tau,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$, if for all $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t,s\in\mathbb{R}^+$, the following conditions are satisfied:

i) $\Phi(t,\tau,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is a $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F},\mathcal{B}(\mathbb{R}))$ measurable mapping.

ii) $\Phi(0,\tau,\omega,x)$ is identity on X.

iii) $\Phi(t+s,\tau,\omega,x)=\Phi(t,\tau+s,\theta_s\omega,,x)\circ\Phi(s,\tau,\omega,,x)$.

iv) $\Phi(t,\tau,\omega,x):X\to X$ is continuous.

Definition 3. Let 2^X be the collection of all subsets of X, a set valued mapping $(\tau,\omega)\mapsto\mathcal{D}(t\ \omega):\mathbb{R}\times\Omega\mapsto 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if $\mathcal{D}(t\ \omega)$ is a(usually closed)nonempty subset of X and the mapping $\omega\in\Omega\mapsto d(X,B(\tau,\omega))$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$ -measurable for every fixed $x\in X$ and $\tau\in\mathbb{R}$. Let $B=B(t,\omega)\in\mathcal{D}(t,\omega):\tau\in\mathbb{R},\omega\in\Omega$ is called a random set.

Definition 4. A random bounded set $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for p-a.e $\omega\in\Omega$,

$$\lim_{t\ \mapsto\infty} e^{-\beta t}\, d(B(\theta_{-t}\omega))=0\,,\ \forall\,\beta\ >0\,,$$

where

$$d(B)=\sup_{x\in B}\|x\|_X.$$

Definition 5. Let $\mathcal{D}$ be a collection of random subset of X and $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, then K is called an absorbing set of $\Phi\in\mathcal{D}$ if for all $\tau\in\mathbb{R},\omega\in\Omega$ and $B\in\mathcal{D}$, there exists, $T=T(\tau,\omega,B)>0$ such that.

$$\Phi(t\ ,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega))\subseteq K(\tau,\omega)\ ,\ \forall\ t\ \geq T.$$

Definition 6. Let $\mathcal{D}$ be a collection of random subset of X, the Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for p-a.e $\omega\in\Omega$, $\{\Phi(t_n\ ,\theta_{-t_n}\omega\ ,x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n\mapsto\infty$ and $x_n\in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$.

Definition 7. Let $\mathcal{D}$ be a collection of random subset of X and $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$,then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied:for all $t\in\mathbb{R}^+,\tau\in\mathbb{R}$ and $\omega\in\Omega$:

i) $\mathcal{A}(\tau,\omega)$ is compact, and $\omega\mapsto d(x,\mathcal{A}(\omega))$ is measurable for every $x\in X$.

ii) $\mathcal{A}(\tau,\omega)$ is invariant, that is

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega))=\ \mathcal{A}(\tau+t,\theta_t\omega),\forall\ t\ \geq\tau.$$

iii) $\mathcal{A}(\tau,\omega)$ attracts every set in $\mathcal{D}$, that is for every $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$,

$$\lim_{t\ \mapsto\infty} d_X(\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)),\mathcal{A}(\tau,\omega))=0,$$

where d_X is the Hausdorff semi -distance given by

$$d_X(Y,Z) = \sup_{y\in Y}\inf_{z\in Z} \|y-z\|_X$$

for any $Y \in X$ and $Z \in X$.

Lemma 1. *Let $\mathcal{D}$ be a neighborhood-closed collection of (τ,ω)- parameterized families of nonempty subsets of X and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \digamma, P, (\theta_t)_{t\in\mathbb{R}})$. Then Φ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is pullback $\mathcal{D}$-asymptotically compact in X and Φ has a closed, $\digamma$-measurable pullback $\mathcal{D}$-absorbing set $K \in \mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \mathcal{A}(\tau,\omega)$ is given*

$$\mathcal{A}(\tau,\omega) = \bigcap\nolimits_{r\geq 0}\overline{\bigcup\nolimits_{t\geq r}\Phi(t,\tau-t,\theta_{-t}\omega, K(\tau-t,\theta_{-t}\omega))}\ \tau\in\mathbb{R}\ ,\omega\in\Omega.$$

3. Existence and Uniqueness of Solution

In this Section, the focusing is on the existence of a continuous cocycle for the stochastic wave equation on $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times\mathfrak{R}_\mu$. First, we recall some important results, let $A=-\triangle$, and $D(A) = H^2(\mathbb{R}^n)$. Generally, let $V_{2r} = D(A^r)$, $r\in\mathbb{R}$, is Hilbert space, the injection $V_{r_1}\hookrightarrow V_{r_2}$ is compact if $r_1 > r_2$. Especially, $V_1 \hookrightarrow L^2(\mathbb{R}^n)$, $V_2 \hookrightarrow H^1(\mathbb{R}^n)$, the space $V_{\mu,2r} = \mathfrak{R}_{\mu,2r} = L^2_\mu(\mathbb{R}^+, V_{2r}(\mathbb{R}^n))$. Then, it is defined

$$\begin{cases} (u,v) = \int_{\mathbb{R}^n} uvdx,\ \|u\| = (u,u)^{\frac{1}{2}}\ ,\forall u,v\in L^2(\mathbb{R}^n), \\ ((u,v)) = \int_{\mathbb{R}^n} \nabla u\nabla v dx,\ \|\nabla u\| = ((u,u))^{\frac{1}{2}}\ ,\forall\ u,v\in H^1(\mathbb{R}^n), \\ ((\cdot,\cdot))_{D(A^r)} = (A^r\cdot, A^r\cdot),\ \|\cdot\|_{D(A^r)} = \|A^r\cdot\|, \\ (\eta_1,\eta_2)_{\mu,1} = \int_0^\infty \mu(s)(\nabla\eta_1(s),\nabla\eta_2(s))_{\mathbb{H}^1}ds \\ \|\eta\|^2_{\mu,1} = (\eta,\eta)_{\mu,1} = \int_0^\infty \mu(s)(\nabla\eta(s),\nabla\eta(s))ds. \end{cases} \tag{13}$$

For our purpose, it is introduced a transformation $\xi = u_t + \delta u$, δ is a positive constant, the following equivalent system (4)-(5) could be rewritten like

$$\begin{cases} \xi = u_t+\delta u, \\ \xi_t - \delta\xi + \delta^2 u + \alpha A\xi - \alpha\delta Au + \beta Au + \int_0^\infty \mu(s)A\eta(s)ds + f(u) = g(x,t) + \varepsilon\sum_{j=1}^m h_j\dot{W}_j, \\ \eta_t+\eta_s = u_t, \end{cases} \tag{14}$$

with initial data

$$\begin{cases} u(\tau,x) = u_0(\tau,x), \\ u_t(\tau,x) = u_1(\tau,x), x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, \\ \eta_0 = \eta(x,\tau,s) = u_0(x,\tau) - u_0(x,\tau-s), x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+. \end{cases} \tag{15}$$

To study the dynamical behavior of problem(14)-(15), it is needed convert the stochastic system into deterministic one with a random parameter. We introduce an Ornstein-Uhlenbeck process driven by the Brownian motion, which satisfies the following differential equation

$$dz_j + \delta z_j dt = dW_j(t), \tag{16}$$

its unique stationary solution is given by

$$z_j(\theta_t\omega_j) = -\delta\int_{-\infty}^0 e^{\delta s}(\theta_t\omega_j)(s)ds,\ s\in\mathbb{R},\ t\in\mathbb{R},\ \omega_j\in\Omega. \tag{17}$$

From [11,28], it is known that the random variable $|z_j(\omega_j)|$ is tempered and there is an invariant set $\bar{\Omega}\subseteq\Omega$ of full P measure such that $z_j(\theta_t\omega_j) = z_j(t,\omega_j)$ is continuous in t, for every $\omega\in\bar{\Omega}$. For convenience, it should be written $\bar{\Omega}$ as Ω. Follows from proposition 3.4 in [28], that for any $\epsilon > 0$, there exists a tempered function $\Upsilon(\omega) > 0$ such that

$$\sum_{j=1}^{m}(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^{\gamma+2}) \leq \Upsilon(\omega), \tag{18}$$

whereas $\Upsilon(\omega)$ satisfies for, p-a.e. $\omega \in \Omega$,

$$\Upsilon(\theta_t\omega) \leq e^{\epsilon|t|}\, \Upsilon\,(\omega),\ t \in \mathbb{R}. \tag{19}$$

Then, it follows from the above inequality, for p-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^{\gamma+2}) \leq e^{\epsilon|t|}\, \Upsilon\,(\omega),\ t \in \mathbb{R}, \tag{20}$$

put $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$, which solves $dz + \delta z dt = \sum_{j=1}^{m} h_j \dot{W}_j(t)$.

To define a cocycle for the problem (14)-(15), let $v(t,\tau,x,\omega) = \xi(t,\tau,x,\omega) - \varepsilon z(\theta_t\omega)$, then (14)-(15) can be rewritten as the equivalent system with random coefficients but without white noise

$$\begin{cases} u_t + \delta u = v + \varepsilon z(\theta_t\omega), \\ v_t - \delta v + \delta^2 u - \alpha\triangle(v - \delta u + \varepsilon z(\theta_t\omega)) - \beta\triangle u - \int_0^\infty \mu(s)\triangle\eta(s)ds + f(u) = g(x,t) + 2\delta\varepsilon z(\theta_t\omega), \\ \eta_t + \eta_s = -\delta u + v + \varepsilon z(\theta_t\omega), \\ u(x,\tau) = u_0(x), u_t(\tau,x) = u_1(\tau,x), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, \\ v(x,\tau) = v_0(x) = u_1(x) + \delta u_0(x) - \varepsilon z(\theta_t\omega), \\ \eta_0 = \eta(x,\tau,s) = u_0(x,\tau) - u_0(x,\tau - s), x \in \mathbb{R}^n,\ \tau \in\ \mathbb{R}, s \in\ \mathbb{R}^+. \end{cases} \tag{21}$$

Let $E = E(\mathbb{R}^n) = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_\mu$, and endow with the usual inner products and norms $E(\mathbb{R}^n)$, respectively,

$$\|y\|_{E(\mathbb{R}^n)} = (\|v\|^2 + \delta^2\|u\|^2 + \|\nabla u\|^2 + \|\eta\|^2_{\mu,1})^{\frac{1}{2}} \tag{22}$$

Then, the following equation is equivalent to the system (21)

$$\begin{cases} \varphi' + L\varphi = Q(\varphi,t,\omega) \\ \varphi(\tau,\omega) = (u_0(x), u_1(x) + \delta u_0(x) - \varepsilon z(\theta_t\omega), \eta_0)^\top,\ \varphi = (u,v,\eta)^\top, \end{cases} \tag{23}$$

whereas

$$\varphi = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix},$$

$$L\varphi = \begin{pmatrix} \delta u - v \\ \delta(\delta - \alpha A)u + \beta A u - (\delta - \alpha A)v + \int_0^\infty \mu(s)A\eta(s)ds \\ \delta u - v + \eta_s \end{pmatrix}$$

and

$$Q(\varphi,\omega,t) = \begin{pmatrix} \varepsilon z(\theta_t\omega) \\ -f(u) + g(x,t) + 2\delta\varepsilon z(\theta_t\omega) - \alpha\varepsilon A z(\theta_t\omega) \\ \varepsilon z(\theta_t\omega) \end{pmatrix}.$$

It is known from [11,29] that L is the infinitesimal generators of $\mathbf{C}^0$ - semigroup e^{Lt} on $E(\mathbb{R}^n)$. It is not difficult to check that the function $Q(\varphi,\omega,t) : E \to E$ is locally Lipschitz continuous with respect to φ and bounded for every $\omega \in \Omega$. By the classical semigroup theory of existence and uniqueness of solutions of evolution differential equations [11,29], the random partial differential equation (23 has a unique solution in the mild sense.

Theorem 1. *Put $\varphi(t,\tau,\omega,\varphi_0) = (u(t,\tau,\omega,u_0), v(t,\tau,\omega,v_0), \eta(t,\tau,s,\omega,\eta_0))^\top$ and let (6)-(12) hold. Then, for every $\omega \in \Omega$ and $\varphi_0 \in E(\mathbb{R}^n)$, the problem (23) has a unique solution $\varphi(t,\tau,\omega,\varphi_0)$ which is continuous with respect to $(u_0,v_0,\eta_0)^\top$ in $E(\mathbb{R}^n)$ such that $\varphi(\tau,\tau,\omega,\varphi_0) = \varphi_0$ and $\varphi(t)$ satisfies the integral equation.*

$$\varphi(t,\tau,\omega,\varphi_0) = e^{-L(t-\tau)}\varphi_0(\omega) + \int_\tau^t e^{-L(t-r)} Q(\varphi,r,\omega)dr. \tag{24}$$

For any $\varphi(t,\tau,\omega,\varphi_0) \in E(\mathbb{R}^n)$, it could be proved that for P-a.s. each $\omega \in \Omega$, the solutions satisfy the following properties for all $\mathrm{T} > 0$:

(1)- if $\varphi_0(\omega) \in E$, then $\varphi(\mathrm{T},\tau,\omega,\varphi_0) = \varphi(\mathrm{T},\omega,\varphi_0) \in C([\tau,\tau+\mathrm{T}); E)$,

(2)- $\varphi(t,\omega,\varphi_0)$ is jointly continuous into t and measurable in $\varphi_0(\omega)$,

(3)- the solution mapping of (24) holds the properties of continuous cocycle.

From the Theorem1, it can define a continuous random dynamical system over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, that is,

$$\begin{aligned} &\bar{\Phi}(t,\omega) : \mathbb{R}\times\Omega\times E \mapsto E, t \geq \tau, \\ &\varphi(\tau,\omega) = (u_0,v_0,\eta_0)^\top \mapsto (u(t,\omega), v(t,\omega), \eta(t,\omega))^\top = \varphi(t,\omega), \end{aligned} \tag{25}$$

it generates a random dynamical system; moreover,

$$\Phi(t,\omega) : \varphi(\tau,\omega) + (0, \varepsilon z(\theta_\tau\omega), 0)^\top \mapsto \varphi(t,\omega) + (0,\varepsilon z(\theta_t\omega),0)^\top. \tag{26}$$

To show the conjugation of the solution for the stochastic partial differential equation (21) and the random partial differential equation (23), introducing the homeomorphism $P(\theta_t\omega)(u,v,\eta(s))^\top = (u, u_t + \delta u - \varepsilon z(\theta_t\omega), \eta(s))^\top$, $(u,v,\eta(s))^\top \in E(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_t\omega)(u,v,\eta(s))^\top = (u, u_t - \delta u + \varepsilon z(\theta_t\omega), \eta(s))^\top$. Then, the transformation

$$\bar{\Phi}(\tau,t,\omega) = P(\theta_t\omega)\Phi(t,\omega)P^{-1}(\theta_t\omega), E \mapsto E, t \geq \tau. \tag{27}$$

Consider the equivalent RDS $\Psi(\tau,t,\omega) = P_\delta(\theta_t\omega)\Phi(t,\omega)P_{-\delta}(\theta_t\omega))$, where $\Phi(\tau,t,\omega)$ is decided by

$$\begin{cases} \psi' + H\psi = \bar{Q}(\psi,t,\omega), \\ \psi(\tau,\omega) = (u_0(x), u_1(x) + \delta u_0(x), \eta_0)^\top, \end{cases} \tag{28}$$

where

$$\psi = (u,w,\eta)^\top = (u, u_t + \delta u, \eta)^\top,$$

$$H\psi = \begin{pmatrix} \delta u - v \\ \delta(\delta - \alpha A)u + \beta A u - (\delta - \alpha A)v + \int_0^\infty \mu(s) A\eta(s)ds \\ \delta u - v + \eta_s \end{pmatrix}$$

and

$$\bar{Q}(\psi,\omega,t) = \begin{pmatrix} 0 \\ -f(u) + g(x,t) + \varepsilon z(\theta_t\omega) \\ 0 \end{pmatrix}.$$

4. Random absorbing set

In this section, to drive uniform estimates on the solutions of the stochastic strongly damped wave equations (14)-(15) defined on $\mathbb{R}^n$, when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equations. In particular, it will be shown that the tails of the solutions, i.e., solutions evaluated at large values of $|x|$, are uniformly become less when the time is sufficiently large. it is always assumed that $\mathcal{D}$ is the collection of all tempered subsets of $E(\mathbb{R}^n)$ from now on. Let $\delta \in (0,1)$ be small enough such that

$$(\beta - \alpha\delta - \frac{2m_0\delta}{\sigma} - \frac{\delta}{4}) > 0\,, \; \frac{3\alpha\lambda_0}{2} - \delta > 0. \tag{29}$$

The next Lemma shows that Φ has a random absorbing set in $\mathcal{D}$.

Lemma 2. *Let (6)-(12) and $h_j \in H^2(\mathbb{R}^n)$ hold. There exists a random ball $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ centered at 0 with random radius $M(\omega) \geq 0$ such that $\{K(\omega)\}$ is a random absorbing set for Φ in $\mathcal{D}$, that is for any B=$\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exists $T = T(\tau,\omega,B) > 0$ and $\varphi_0(\omega) \in B(\omega)$ such that*

$$\Phi(t,\tau,\theta_{-t}\omega, B(\tau,\theta_{-t}\omega)) \subseteq K(\tau,\omega) \;\; for\ all\ t \geq T. \tag{30}$$

Beside, we have

$$\int_{-\infty}^{0} e^{\delta(s-\tau)}\|\nabla v\|^2 ds < R(\tau,\omega)$$

Proof. Taking the inner product of the second term of (21) with v in $L^2(\mathbb{R}^n)$, we fine that

$$\begin{aligned}&(v_t,v)-\delta\,(v,v)+\delta^2\,(u,v)+\alpha\|\nabla v\|^2-\alpha\delta(Au,v)+\beta\,(Au,v)+\int_0^\infty \mu(s)\,(A\eta(s),v)\,ds\\&=(g(x,t),v)-(f(u),v)-\alpha\varepsilon(Az(\theta_t\omega),v)+2\delta\varepsilon\,(z(\theta_t\omega),v)\,.\end{aligned} \tag{31}$$

Substituting the following Eq. (32) into the third, fourth and fifth term on the left hand side of (31), it obtained

$$v=\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega), \tag{32}$$

$$(u,v)=\left(u,\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)\leq\frac{1}{2}\frac{d}{dt}\|u\|^2+\frac{\delta}{2}\|u\|^2-\frac{1}{2\delta}|\varepsilon|^2|z(\theta_t\omega)|^2, \tag{33}$$

$$(Au,v)=(-\triangle u,v)=\left(\nabla u,\nabla(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)\leq\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\frac{\delta}{2}\|\nabla u\|^2-\frac{1}{2\delta}|\varepsilon|^2|\nabla z(\theta_t\omega)|^2. \tag{34}$$

Using second Eq. (4), similar to the estimates of [15,26,27], it get

$$\begin{aligned}&\int_0^\infty \mu(s)\,(A\eta(s),v)\,ds=\int_0^\infty \mu(s)\,(-\triangle\eta(s),v)\,ds=\int_0^\infty \mu(s)\left(\nabla\eta(s),\nabla(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)ds\\&=\textstyle\int_0^\infty \mu(s)(\nabla\eta(s),\nabla u_t)ds+\delta\int_0^\infty \mu(s)(\nabla\eta(s),\nabla u)ds-\varepsilon\int_0^\infty \mu(s)(\nabla\eta(s),\nabla z(\theta_t\omega))ds.\end{aligned} \tag{35}$$

Integrating by parts and using second Eq. (4) and Young inequality, to show

$$\int_0^\infty \mu(s)(\nabla\eta(s),\nabla u_t)ds\geq\frac{1}{2}\frac{d}{dt}\|\eta(s)\|^2_{\mu,1}+\frac{\sigma}{2}\|\eta(s)\|^2_{\mu,1}. \tag{36}$$

Also, integrating by parts and using (7) and (8), it follows that

$$\delta\int_0^\infty \mu(s)(\nabla\eta(s),\nabla u)ds\geq-\frac{\sigma}{8}\|\eta(s)\|^2_{\mu,1}-\frac{2m_0\delta^2}{\sigma}\|\nabla u\|^2. \tag{37}$$

According to (8) and Young inequality, yields

$$-\varepsilon\int_0^\infty \mu(s)(\nabla\eta(s),\nabla z(\theta_t\omega))ds\geq-\frac{\sigma}{8}\|\eta(s)\|^2_{\mu,1}-\frac{2m_0\varepsilon^2}{\sigma}\|\nabla z(\theta_t\omega)\|^2\,. \tag{38}$$

Combining with (36)-(38) and (35), it is arrived

$$\int_0^\infty \mu(s)(A\eta(s),v)ds\geq\frac{1}{2}\frac{d}{dt}\|\eta(s)\|^2_{\mu,1}+\frac{\sigma}{4}\|\eta(s)\|^2_{\mu,1}-\frac{2m_0\delta^2}{\sigma}\|\nabla u\|^2-\frac{2m_0\varepsilon^2}{\sigma}\|\nabla z(\theta_t\omega)\|^2. \tag{39}$$

In addition,

$$(-f(u),v)=\left(-f(u),\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)=-\frac{d}{dt}\int_{\mathbb{R}^n}F(u)dx-\delta\int_{\mathbb{R}^n}f(u)udx+\varepsilon\int_{\mathbb{R}^n}f(u)z(\theta_t\omega)dx. \tag{40}$$

Combining with (9)-(10) and Young inequality, it follows that

$$\begin{aligned}&\varepsilon\left(f(u),z(\theta_t\omega)\right)\leq\varepsilon C\int_{\mathbb{R}^n}\left(|u|+|u|^{\gamma+1}\right)z(\theta_t\omega)dx\leq C\left|\varepsilon\right|\left|z(\theta_t\omega)\right|\|u\|+C|\varepsilon|\left(\int_{\mathbb{R}^n}|u|^{\gamma+2}dx\right)^{\frac{\gamma+1}{\gamma+2}}|z(\theta_t\omega)|_{\gamma+2}\\&\leq C\left|\varepsilon\right|\left|z(\theta_t\omega)\right|\|u\|+C\left|\varepsilon\right|\left(\tfrac{1}{C_3}\left(\int_{\mathbb{R}^n}F(u)dx+C_4\left|u\right|^2\right)\right)^{\frac{\gamma+1}{\gamma+2}}|z(\theta_t\omega)|_{\gamma+2}\leq\tfrac{C}{2\delta}\left|\varepsilon\right|^2\left|z(\theta_t\omega)\right|^2+\tfrac{C\delta}{2}\|u\|^2\\&+\tfrac{CC_4C_3^{-1}\delta}{2}\|u\|^2+\tfrac{CC_3^{-1}\delta}{2}\int_{\mathbb{R}^n}F(u)dx+\tfrac{C}{2\delta}\left|\varepsilon\right|^{\gamma+2}|z(\theta_t\omega)|_{\gamma+2}^{\gamma+2},\end{aligned}\tag{41}$$

and

$$\delta\int_{\mathbb{R}^n}f(u)udx\geq\frac{\delta}{C_1}\int_{\mathbb{R}^n}F(u)dx-\frac{C_2\delta}{C_1}\int_{\mathbb{R}^n}|u|^2dx,\tag{42}$$

substituting (41)-(42) into (40), there for

$$\begin{aligned}(-f(u),v)\geq-\frac{d}{dt}\int_{\mathbb{R}^n}F(u)dx-\frac{\delta\left(2C_1^{-1}-CC_3^{-1}\right)}{2}\int_{\mathbb{R}^n}F(u)dx+\frac{\delta\left(2C_2C_1^{-1}+C+CC_4C_3^{-1}\right)}{2\lambda_0}\|\nabla u\|^2\\+\tfrac{C}{2\delta}\left|\varepsilon\right|^2\left|z(\theta_t\omega)\right|^2+\tfrac{C}{2\delta}\left|\varepsilon\right|^{\gamma+2}|z(\theta_t\omega)|_{\gamma+2}^{\gamma+2}.\end{aligned}\tag{43}$$

By means of the Cauchy-Schwartz inequality and the Young inequality, it result

$$(g(x,t),v)\leq\|g(x,t)\|\|v\|\leq\frac{\|g(x,t)\|^2}{2\delta}+\frac{\delta}{2}\|v\|^2,\tag{44}$$

$$(2\delta\varepsilon z(\theta_t\omega)+\alpha\varepsilon Az(\theta_t\omega),v)\leq\delta\left|\varepsilon\right|^2\left|z(\theta_t\omega)\right|^2+\delta\|v\|^2+\frac{\alpha}{2}\left|\varepsilon\right|^2\left|\nabla z(\theta_t\omega)\right|^2+\frac{\alpha}{2}\|\nabla v\|^2.\tag{45}$$

Thus, substituting (33)-(45) into (31) and due to $\tilde{F}(u)=\int_{\mathbb{R}^n}F(u)dx$, it obtained

$$\begin{aligned}&\frac{d}{dt}\left(\|v(t)\|^2+\delta^2\|u(t)\|^2+(\beta-\alpha\delta)\|\nabla u(t)\|^2+\|\eta(s)(t)\|_{\mu,1}^2+2\tilde{F}(u(t))\right)\\&+\delta\left(\|v(t)\|^2+\delta^2\|u(t)\|^2+(\beta-\alpha\delta-\frac{2m_0\delta}{\sigma}-\frac{(2C_2C_1^{-1}+C+CC_4C_3^{-1})\delta}{\lambda_0})\|\nabla u(t)\|^2\right)\\&+\frac{\sigma}{2}\|\eta(s)(t)\|_{\mu,1}^2+\delta(2C_1^{-1}-CC_3^{-1})\tilde{F}(u(t))\\&\leq C_\nu\left(|z(\theta_t\omega)|^2+|\nabla z(\theta_t\omega)|^2+|z(\theta_t\omega)|_{\gamma+2}^{\gamma+2}+\|g(x,t)\|^2\right)+\left(3\delta-\frac{3\alpha\lambda_0}{2}\right)\|v(t)\|^2.\end{aligned}\tag{46}$$

In condition to choose δ small enough and using $\|\nabla v\|^2\geq\lambda_0\|v\|^2$, λ_0 is the first eigenvalue of $-\triangle$, such that

$$\begin{aligned}&\delta\leq\sigma\,,\ \delta\leq\frac{\alpha}{2},\ \left(\beta-\alpha\delta-\frac{2m_0\delta}{\sigma}-\frac{\left(2C_2C_1^{-1}+C+CC_4C_3^{-1}\right)\delta}{\lambda_0}\right)\leq\beta-\alpha\delta,\\&C_\nu=\max\left[\beta-\alpha\delta-\tfrac{2m_0\varepsilon^2}{\sigma},\tfrac{C}{2\delta}\left|\varepsilon\right|^2,\tfrac{C}{2\delta}\left|\varepsilon\right|^{\gamma+2},\tfrac{1}{2\delta},\delta\left|\varepsilon\right|^2\right].\end{aligned}\tag{47}$$

Let

$$\sigma_1=\min\{\delta\,,\delta\left(\beta-\alpha\delta\right),\delta\left(2C_1^{-1}-CC_3^{-1}\right),\frac{\sigma}{2}\}.\tag{48}$$

By using (11)-(12) and (18), the equation could be written as follows

$$\Gamma\left(t,(\theta_t\omega)\right)=2C_\nu\left(\Upsilon(\theta_t\omega)+\|g(x,t)\|^2\right).\tag{49}$$

From (47)-(49)and (25), we can recall the norm $\|'\|_{E(\mathbb{R}^n)}^2$, which it equivalents to (46) such that

$$\frac{d}{dt}\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)\leq\sigma_1\left(\|\varphi\|_E^2+2\tilde{F}(u)\right)\leq\Gamma\left(t,\omega\right).\tag{50}$$

Applying Gronwall's lemma over $[\tau\,,t]$, to fine that, for all t$\geq$ 0,

$$\|\varphi(t,\tau,\omega,\varphi_0(\omega))\|_E^2+\tilde{F}(u(t,\tau,\omega,\varphi_0(\omega))\leq e^{-\sigma_1(\tau-t)}(\|\varphi_0\|_E^2+\tilde{F}(u_0))+\int_\tau^t\Gamma(r,\theta_r\omega)e^{-\sigma_1(t-r)}dr.\tag{51}$$

By replacing ω *by* $\theta_{-t}\omega$, to get from (51) such that for all $t \geq 0$

$$\begin{aligned}&\|\varphi(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2+\tilde{F}(u(t,\tau,\theta_{-t}\omega,u_0))\\&\leq e^{-\sigma_1(\tau-t)}\left(\|\varphi_0(\theta_{-t}\omega)\|_E^2+\tilde{F}(u_0)\right)+\int_{\tau-t}^0\Gamma(r-t,\theta_{r-t}\omega)e^{\sigma_1(r-t)}dr.\end{aligned} \tag{52}$$

Note that (9)-(10) implied that there is a constant $C > 0$ such that

$$\tilde{F}(u_0) \leq C\left(\|u_0\|^2+\|u_0\|_{\gamma+2}^{\gamma+2}\right). \tag{53}$$

For any set $B \in \mathcal{D}$, which is tempered with respect to the norm of $E(\mathbb{R}^n)$, since $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, there exists constant $C_0 > 0$ and a $T = T(\tau,\omega,B) > 0$ such that

$$e^{-\sigma t}\left(\|\varphi_0\|_E^2+\tilde{F}(u_0)\right) \leq 1. \tag{54}$$

for all $t \geq T$. Substitute (54) into the first term of right- hand side of (52), and note that (9) implies

$$2\tilde{F}(u(t,\tau,\theta_{-t}\omega,u_0(\omega))) \leq 2C\left(\|u(t,\tau,\theta_{-t}\omega,u_0(\omega))\|^2+\|u(t,\tau,\theta_{-t}\omega,u_0(\omega))\|_{\gamma+2}^{\gamma+2}\right). \tag{55}$$

Since $\Upsilon(\omega)$ is tempered random variable by (12) and (52) it then follows from (16)-(20), $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and that due to $\varphi_0 = (u_0,v_0,\eta_0)^\top \in B(\tau,\theta_{-t}\omega)$ and $B \in \mathcal{D}$, we get from (50)

$$\lim_{r\to-\infty} e^{-\sigma r}\left(\|\varphi_0\|_E^2+\tilde{F}(u_0)\right)=0, \int_{\tau-t}^0\Gamma(r-t,\theta_{r-t}\omega)e^{\sigma_1(r-t)}dr \leq \int_{-\infty}^0\Gamma(r-t,\theta_{r-t}\omega)e^{\sigma_1(r-t)}dr < +\infty, \tag{56}$$

then

$$\|\varphi(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2+\|u(t,\tau,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_{\gamma+2}^{\gamma+2} \leq e^{-\sigma_1(\tau-t)}(\|\varphi_0(\theta_{-t}\omega)\|_E^2+\tilde{F}(u_0))+M(\omega), \tag{57}$$

whereas $\int_{-\infty}^0\Gamma(r-t,\theta_{r-t}\omega)e^{\sigma_1(r-t)}dr = M(\omega)$.

Obviously, by Theorem1, (11)-(12) and (56)-(57), it is concluded that there is a closed measurable $\mathcal{D}$-pullback absorbing set for the continuous cocycle associated with problem (21) in $\mathcal{D}$, that is for, every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,B) > 0$, such that for all $t \geq T$,

$$\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)) \subseteq K(\tau,\omega).$$

The proof is completed. □

In order to verify the asymptotically compact property, it is chosen a smooth function ρ defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}$ and

$$\rho(s) = \begin{cases} 0, \ \forall\, 0 < |s| \leq 1, \\ 1, \ \forall\, |s| \geq 2. \end{cases} \tag{58}$$

Then there exist constants μ_1 and μ_2 such that $|\rho'(s)| \leq \mu_1, |\rho''(s)| \leq \mu_2$ for any $s \in \mathbb{R}$, given $k \geq 1$, denoted by $\mathbb{H}_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\{\mathbb{R}^n \backslash \mathbb{H}_k\}$ the complement of $\mathbb{H}_k$. First it will be proved by the following Lemma.

Lemma 3. *Assume that(6)-(12) and $h_j \in H^2(\mathbb{R}^n)$ hold. Let B=$\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\varphi_0(\omega) \in B(\omega)$. There exists $\tilde{T} = \tilde{T}(B,\omega) > 0$ and $\tilde{\mathbb{K}} = \tilde{\mathbb{K}}(\omega) > 1$ such that the solution $\varphi(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))$ of (21) satisfies for P-a.e $\omega \in \Omega$, $\forall\, t \geq \tilde{T}, k \geq \tilde{\mathbb{K}}$*

$$\|\varphi(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_{E(\mathbb{R}^n\backslash\mathbb{H}_k)}^2 \leq \zeta. \tag{59}$$

Proof. Multiplying the second term of (21) by $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$ and integrating over $\mathbb{R}^n$, to find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx-\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx+\delta^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uvdx\\ &+\alpha\int_{\mathbb{R}^n}A\left(v-\delta u+\varepsilon z(\theta_t\omega)\right)\rho\left(\frac{|x|^2}{k^2}\right)vdx+\beta\int_{\mathbb{R}^n}Au\rho\left(\frac{|x|^2}{k^2}\right)vdx\\ &+\int_{\mathbb{R}^n}\int_0^\infty\mu(s)(A\eta(s))\rho\left(\frac{|x|^2}{k^2}\right)vdsdx+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(u)vdx\\ &=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)vdx+2\delta\varepsilon\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)z(\theta_t\omega)vdx. \end{aligned} \tag{60}$$

Substituting v in first term of (21) into the third, fourth and fifth term on the left- hand side of (60), to obtain

$$\begin{aligned} &\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uvdx=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)u\left(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)dx\\ &\leq\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{2}\frac{d}{dt}|u|^2+\frac{\delta}{2}|u|^2-|\varepsilon|^2|z(\theta_t\omega)|^2\right)dx, \end{aligned} \tag{61}$$

$$\begin{aligned} &\int_{\mathbb{R}^n}(-\triangle u)\rho\left(\frac{|x|^2}{k^2}\right)vdx=\int_{\mathbb{R}^n}(\nabla u)\nabla\left(\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)\right)dx\\ &=\int_{\mathbb{R}^n}\nabla u\left(\frac{2x}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)v\right)dx\quad+\int_{\mathbb{R}^n}\nabla u\left(\rho\left(\frac{|x|^2}{k^2}\right)\nabla\left(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)\right)dx\\ &\leq\int_{k<|x|<2\sqrt{2}k}\frac{2x}{k^2}\mu_1\nabla uvdx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx\\ &\quad+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx-\varepsilon\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)||\nabla u|dx\\ &\leq\frac{\sqrt{2}}{k}\mu_1\left(\|\nabla u\|^2+\|v\|^2\right)+\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\frac{d}{dt}|\nabla u|^2dx\\ &\quad+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx-\frac{\varepsilon^2}{2\delta}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx. \end{aligned} \tag{62}$$

From second Eq. (4), similar to the estimate of (35), to get

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^\infty\mu(s)(-\triangle\eta(s))\rho\left(\frac{|x|^2}{k^2}\right)vdsdx\leq\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\nabla\eta(s)\left((\nabla)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)\right)dsdx\\ &=\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\nabla\eta(s)\left(\frac{2x}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)v\right)dsdx\quad+\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\nabla\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\left(\frac{du}{dt}+\delta u-\varepsilon z(\theta_t\omega)\right)dsdx\\ &\leq\int_{k<|x|<2\sqrt{2}k}\frac{2x}{k^2}\mu_1\int_0^\infty\mu(s)\nabla\eta(s)vdsdx\quad+\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla u_tdsdx\\ &\quad+\delta\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\left(\rho\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla udsdx\\ &\quad-\varepsilon\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla z(\theta_t\omega)dsdx. \end{aligned} \tag{63}$$

Integrating by parts, by using second equations (4), (7)-(8) and Young inequality, it follows that

$$\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla u_tdsdx\leq\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx+\frac{\sigma}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx, \tag{64}$$

and

$$\delta\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla udsdx\geq-\frac{\sigma}{8}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx\quad-\frac{2m_0\delta^2}{\sigma}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx. \tag{65}$$

Due to (8) and Young inequality, we confirm that

$$\begin{aligned} &-\varepsilon\int_{\mathbb{R}^n}\int_0^\infty\mu(s)\rho\left(\frac{|x|^2}{k^2}\right)\nabla\eta(s)\nabla z(\theta_t\omega)dsdx\geq\frac{\sigma}{8}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx\\ &-\frac{2m_0\varepsilon^2}{\sigma}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx. \end{aligned} \tag{66}$$

Thus, applying (64)-(66) and (63), to find

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^\infty\mu(s)(-\triangle\eta(s))\rho\left(\frac{|x|^2}{k^2}\right)vdsdx\leq\frac{\sqrt{2}}{k}\mu_1(\|\nabla\eta\|^2_\mu+\|v\|^2)+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx\\ &+\frac{\sigma}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx-\frac{2m_0\delta^2}{\sigma}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx-\frac{2m_0\varepsilon^2}{\sigma}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx, \end{aligned} \tag{67}$$

in addition

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(u)vdx=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(u)\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx\\&=\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(u)dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(u)udx\quad-\varepsilon\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(u)z(\theta_t\omega)dx.\end{aligned}\tag{68}$$

According to (10) and (42), we have

$$\delta\int_{\mathbb{R}^n}f(u)udx\geq\frac{\delta}{C_1}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(u)dx-\frac{C_2\delta}{C_1}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2dx.\tag{69}$$

From (9)-(10) and Young inequality, to concentrate

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(f(u),z(\theta_t\omega))\leq\varepsilon C_3\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(u)z(\theta_t\omega)dx\\&\leq C|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|dx\\&\quad+C|\varepsilon|\left(\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^{\gamma+2}dx\right)^{\frac{\gamma+1}{\gamma+2}}|z(\theta_t\omega)|_{\gamma+2}\\&\leq C|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|dx\\&\quad+C|\varepsilon|\left(\frac{1}{C_3}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(F(u)+C_4|u|^2)\right)^{\frac{\gamma+1}{\gamma+2}}|z(\theta_t\omega)|_{\gamma+2}dx\\&\leq\frac{C}{2\delta}|\varepsilon|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|^2dx+\frac{C\delta}{2}\left(1+C_4C_3^{-1}\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2dx\\&\quad+\frac{CC_3^{-1}\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(u)dx+\frac{C}{2\delta}|\varepsilon|^{\gamma+2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|_{\gamma+2}^{\gamma+2}dx,\end{aligned}\tag{70}$$

applying (69)-(70) and (68), we get

$$\begin{aligned}&-\rho\left(\frac{|x|^2}{k^2}\right)f(u)v\geq-\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(u)dx-\frac{\delta\left(2C_1^{-1}-CC_3^{-1}\right)}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(u)dx\\&+\frac{\delta\left(2C_2C_1^{-1}+C+CC_4C_3^{-1}\right)}{2\lambda_0}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx\\&+\frac{C}{2\delta}|\varepsilon|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|^2dx+\frac{C}{2\delta}|\varepsilon|^{\gamma+2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|_{\gamma+2}^{\gamma+2}dx.\end{aligned}\tag{71}$$

By the Cauchy-Schwartz inequality and the Young inequality, we deduce that

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)vdx\leq\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,t)||v|dx\leq\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\frac{|g(x,t)|^2}{2\delta}dx+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx,\tag{72}$$

$$2\varepsilon\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)z(\theta_t\omega)vdx\leq\delta|\varepsilon|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|z(\theta_t\omega)|^2dx\quad+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2dx,\tag{73}$$

$$\begin{aligned}\int_{\mathbb{R}^n}(-\triangle z(\theta_t\omega))\rho\left(\frac{|x|^2}{k^2}\right)vdx&=\int_{\mathbb{R}^n}\nabla z(\theta_t\omega)\nabla\rho\left(\frac{|x|^2}{k^2}\right)vdx\\&=\int_{\mathbb{R}^n}\frac{2|x|}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|vdx+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)||\nabla v|dx\\&\leq\int_{k<|x|<\sqrt{2}k}\frac{2x}{k^2}\mu_1|\nabla z(\theta_t\omega)v|dx+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla v||\nabla z(\theta_t\omega)|dx\\&\leq\frac{\sqrt{2}}{k}\mu_1(\|\nabla z(\theta_t\omega)\|^2+\|v\|^2)+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx\\&\quad+\frac{1}{2\delta}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx.\end{aligned}\tag{74}$$

$$\begin{aligned}&\alpha\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)A\left(v-\delta u+\varepsilon z(\theta_t\omega)\right)vdx\leq\alpha\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|Av||v|dx\quad-\alpha\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|Au||v|dx\\&+\alpha\varepsilon\int_{\mathbb{R}^n}(\rho\left(\frac{|x|^2}{k^2}\right)|v||Az(\theta_t\omega)|dx,\end{aligned}\tag{75}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}(-\triangle v)\rho\left(\frac{|x|^2}{k^2}\right)vdx=\int_{\mathbb{R}^n}\nabla v\left(\nabla\left(\rho\left(\frac{|x|^2}{k^2}\right)v\right)\right)dx\\&\leq\int_{\mathbb{R}^n}\nabla v\left(\frac{2x}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)v\right)dx+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx\leq\int_{k<|x|<\sqrt{2}k}\frac{2x}{k^2}\mu_1|\nabla v||v|dx+\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx\\&\leq\frac{\sqrt{2}}{k}\mu_1(\|\nabla v\|^2+\|v\|^2)+\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx.\end{aligned}\tag{76}$$

Collecting (61)-(76) and (60), to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\delta^2|u|^2+(\beta-\alpha\delta)|\nabla u|^2+|\eta(s)|^2_{\mu,1}+2\tilde{F}(u)\right)dx\\&+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\delta^2|u|^2+(\beta-\alpha\delta-\frac{\delta(2C_2C_1^{-1}+C+CC_4C_3^{-1})}{2\lambda_0})|\nabla u|^2\right)dx\\&+\frac{\delta}{2}(2C_1^{-1}-CC_3^{-1})\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\tilde{F}(u)dx+\sigma\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,1}dx\\&\leq\frac{\sqrt{2}}{k}\mu_1\left(\alpha(\|\nabla v\|^2+\|v\|^2)+(\beta-\alpha\delta)(\|\nabla u\|^2+\|v\|^2)+\alpha\|v\|^2\right)+\sigma\|\eta\|^2_{\mu,1}+\|v\|^2\\&+2C_\varepsilon\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(1+2|z(\theta_t\omega)|^2+|\nabla z(\theta_t\omega)|^2+2|z(\theta_t\omega)|^{\gamma+2}\right)dx\\&+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left((2(\delta-\frac{3\alpha\lambda_0}{2})+\delta)|v|^2+|g(x,t)|^2\right)dx.\end{aligned}\tag{77}$$

Setting $\sigma_2=\min\frac{\sqrt{2}}{k}\mu_1(\beta-\alpha\delta,\alpha,\sigma)$, C_ε is a positive constant depending on $\beta,\alpha,\frac{2m_0\varepsilon^2}{\sigma},\frac{C}{2\delta},\frac{C}{2\delta}|\varepsilon|^{\gamma+2},\frac{1}{2\delta},4\varepsilon^2\delta$. Moreover, using by $\|\nabla v\|^2\geq\lambda\|v\|^2$, (47)-(48) and (18), equations could be written the follows

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\delta^2|u|^2+(\beta-\alpha\delta)|\nabla u|^2+|\eta(s)|^2_{\mu,1}+2\tilde{F}(u)\right)dx\\&+\sigma_1\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\delta^2|u|^2+(\beta-\alpha\delta)|\nabla u|^2+\tilde{F}(u)+|\eta(s)|^2_{\mu,1}\right)dx\\&\leq\left(\sigma_2(\|\nabla v\|^2+\|\nabla u\|^2+\|\nabla z(\theta_t\omega)\|^2+\|\eta\|^2_{\mu,1}+\|v\|^2\right)+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\Gamma(t,\omega)dx.\end{aligned}\tag{78}$$

Whereas

$$\Gamma(t,\omega)=2C_\varepsilon\left[1+2|z(\theta_t\omega)|^2+|\nabla z(\theta_t\omega)|^2+2|z(\theta_t\omega)|^{\gamma+2}+\|g(x,t)\|^2\right],$$

and let

$$\begin{aligned}\mathbf{X}(t,\tau,\omega,\mathbf{X}_0(\omega))&=|v(t,\tau,\omega,v_0(\omega))|^2+\delta^2|u(t,\tau,\omega,u_0(\omega))|^2\\&+(\beta-\alpha\delta)|\nabla u(t,\tau,\omega,u_0(\omega))|^2+|\eta(t,\tau,\omega,\eta_0(\omega),s)|^2_{\mu,1}.\end{aligned}\tag{79}$$

Substituting (79) into (78), then applying Gronwall's lemma over $[\tau,t]$, we find that, for all t≥ 0,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\mathbf{X}(t,\tau,\omega,\mathbf{X}_0)|^2_E+2\tilde{F}(u(t,\tau,\omega,u_0)\right)dx\leq e^{2\sigma_1(t-\tau)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\mathbf{X}_0|^2_E+\tilde{F}(u_0))\right)dx\\&+\beta_2\int_\tau^t e^{2\sigma_1(r-t)}\left(\|\nabla v(r,\tau,\omega,v_0)\|^2+\|\nabla u(r,\tau,\omega,u_0)\|^2\right.\\&\left.+\|\eta(t,\tau,s,\omega,\eta_0)\|^2_{\mu,1}+\|v(r,\tau,\omega,v_0)\|^2\right)dr+\int_\tau^t e^{2\sigma_1(r-t)}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Gamma(r,\theta_r\omega)dxdr.\end{aligned}\tag{80}$$

By replacing ω *by* $\theta_{-t}\omega$, to get from (71), such that for all $t\geq 0$

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\mathbf{X}(t,\tau,\theta_{-t}\omega,\mathbf{X}_0(\theta_{-t}\omega))|^2_E+2\tilde{F}(u(t,\tau,\theta_{-t}\omega,u_0))dx\\&\leq e^{2\sigma_1(t-\tau)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\mathbf{X}_0(\theta_{-t}\omega)|^2_E+\tilde{F}(u_0))\,dx\\&+\int_{\tau-t}^0 e^{2\sigma_1(r-t)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\Gamma(r-t,\theta_{r-t}\omega)dxdr\\&+\beta_2\int_{\tau-t}^0 e^{2\sigma_1(r-t)}(\|\nabla v(r,\tau,\theta_{r-t}\omega,v_0)\|^2+\|\nabla u(r,\tau,\theta_{r-t}\omega,u_0)\|^2\\&+\|\eta(r,\tau,s,\theta_{r-t}\omega,\eta_0)\|^2_{\mu,1}+\|v(r,\tau,\theta_{r-t}\omega,v_0)\|^2)dr.\end{aligned}\tag{81}$$

Follows the procedures in the proof of Lemma 2, it is similar to (46) and (49)and from (30), it is estimated the term of right hand side of (81). By (9) for any initial data $\mathbf{X}_0 = (u_0, v_0, \eta_0)^\top \in B(\tau, \theta_{-t}\omega)$ and the the fact $\{B(\omega)\} \in \mathcal{D}$ is tempered, to have that,

$$\lim_{r\to-\infty} e^{-\sigma r} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) (|\mathbf{X}_0(\theta_{-t}\omega)|_E^2 + \tilde{F}(u_0)) \leq 2\zeta. \tag{82}$$

Then there exists $\tilde{T}_1 = \tilde{T}_1(\tau, B, \omega) > 0$ such that $t \geq \tilde{T}_1$.

For the second and last terms on the right hand side of (81), there exists $\tilde{\mathbb{K}}_1 = \tilde{\mathbb{K}}(\tau,\omega)_1 \geq 1$ such that for all $k \geq \tilde{K}_1$, by Lemma 2, (47)-(48), Theorem1 and (11)-(12), there are $\tilde{T}_2 = \tilde{T}_2(\tau, B, \omega) > 0$ and $\tilde{\mathbb{K}}_1 = \tilde{\mathbb{K}}_1(\tau,\omega) \geq 1$ such that for all $t \geq \tilde{T}_2$ and $k \geq \tilde{K}_1$

$$C_\varepsilon \int_{\tau-t}^{0} e^{2\sigma_1(r-t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Gamma(r-t, \theta_{r-t}\omega) dx dr \leq \zeta. \tag{83}$$

Next to estimate the third term on the right hand side of (81), by Lemma 2, there are $\tilde{T}_3 = \tilde{T}_3(\tau, B, \omega) > 0$ and $\tilde{\mathbb{K}}_2 = \tilde{\mathbb{K}}_2(\tau,\omega) \geq 1$ such that for all $t \geq \tilde{T}_3$ and $k \geq \tilde{\mathbb{K}}_2$

$$\begin{aligned} &\beta_2 \int_{\tau-t}^{0} e^{2\sigma_1(r-t)} \Big(\|\nabla v(r-t,\tau,\theta_{r-t}\omega, v_0)\|^2 + \|\nabla u(r-t,\tau,\theta_{r-t}\omega, u_0)\|^2 \\ &+ \|\eta(r-t,\tau,s,\theta_{r-t}\omega,\eta_0)\|_{\mu,1}^2 + \|v(r-t,\tau,\theta_{r-t}\omega, v_0)\|^2 \Big) dr \ \leq \zeta. \end{aligned} \tag{84}$$

Let

$$\begin{cases} \tilde{T} = \max\{\tilde{T}_1, \tilde{T}_2, \tilde{T}_3\} \\ \tilde{K} = \{\tilde{K}_1, \tilde{K}_2\}. \end{cases} \tag{85}$$

By applying (82)-(84) and (81), we have for all $t > \tilde{T}$ and $k > \tilde{k}$

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(|\mathbf{X}(t,\tau,\theta_{-t}\omega, \mathbf{X}_0(\theta_{-t}\omega)|_E^2 + 2\tilde{F}(u(t,\tau,\theta_{-t}\omega, u_0(\theta_{-t}\omega))) \right) dx \leq 4\zeta, \tag{86}$$

which implies

$$\|\mathbf{X}(t,\tau,\theta_{-t}\omega, \mathbf{X}_0(\theta_{-t}\omega))\|_{E(\mathbb{R}^n\setminus\mathbb{H}_k)}^2 \leq 4\zeta. \tag{87}$$

The proof is completed. □

5. Decomposition of Equations

In this section, the Eq. (21) is decomposed into two parts (91) and (92). Then to decompose the nonlinear growth term $f \in C^2$ in equation (21) into two parts $f = f_1 + f_2$, whereas f_1, f_2 satisfy the following (88)-(89) respectively.

$$\begin{cases} f_1(0) = 0, u f_1(u) \geq 0, \\ |f_1'(u)| \leq C_5(1+|u|^4), \forall\, u \in \mathbb{R}, \end{cases} \tag{88}$$

$$\begin{cases} f_2(0) = 0, \\ |f_2'(u)| \leq C_6(1+|u|^\gamma), \forall\, u \in \mathbb{R}, 0 \leq \gamma \leq 4, \end{cases} \tag{89}$$

and

$$k_0 u^{\gamma+2} - \alpha_0 u^2 \leq F_i(u) \leq k_1 u f_i(u) + \mu u^2, \forall\, u \in \mathbb{R}, 0 \leq \gamma \leq 4, \tag{90}$$

whereas $C_5, C_6, \mu, k_1, \alpha_0, k_0$ are positive constants. $F_2(u) = \int_0^u f_2(r)dr$ and $F_1(u) = \int_0^u f_1(r)dr$.

In order to prove the asymptotic compactness solution of stochastic PDE uniform estimate, it is decomposed the solution $\varphi(t,\omega) = (u(t,\omega), v(t,\omega), \eta^t(t,s,\omega))^\top$ into the sum

$$\begin{cases} \varphi(t,\omega) = \tilde{\varphi}(t,\omega) + \psi(t,\omega), \\ u = y + w, \\ \eta^t = \eta_1^t + \eta_2^t, \end{cases}$$

whereas $\tilde{\varphi}(t,\omega) = (y(t), y_t(t), \eta_1^t(t,s))^\top$ and $\psi(t,\omega) = \left(w(t,\omega), w_t(t,\omega), \eta_2^t(t,s,\omega)\right)^\top$, then the equation (4)-(5) could be transformed into the following system.

$$\begin{cases} y_{tt} - \beta\triangle y - \alpha\triangle y_t - \int_0^\infty \mu(s)\triangle\eta_1(s)ds + f_1(y) = 0, \\ \eta_{1t} = -\eta_{1s} + y_t, \\ y(\tau,x) = u_0(\tau,x),\ y_t(\tau,x) = u_1(\tau,x),\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, \\ \eta_{1\tau}(x,\tau,s) = u_0(x,\tau) - u_0(x,\tau-s),\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+, \end{cases} \tag{91}$$

$$\begin{cases} w_{tt} - \beta\triangle w - \alpha\triangle w_t - \int_0^\infty \mu(s)\triangle\eta_2(s)ds + f(u) - f_1(y) = g(x,t) + \varepsilon\sum_{j=1}^m h_j\dot{W}_j, \\ \eta_{2t} = -\eta_{2s} + w_t, \\ w(\tau,x) = 0,\ w_t(\tau,x) = 0,\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, \\ \eta_{2\tau}(x,\tau,s) = 0,\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+, \end{cases} \tag{92}$$

Let $y_1 = y$ and $y_{1t} = y_t$

$$y_2 = y_{1t} + \delta y_1, \tag{93}$$

then (91) could be rewritten as the following equivalent system.

$$\begin{cases} y_{2t} - \delta y_2 + \alpha A y_2 + \delta^2 y_1 + (\beta-\alpha\delta)Ay_1 + \int_0^\infty \mu(s)A\eta_1(s)ds + f_1(y_1) = 0, \\ \eta_{1t} = -\eta_{1s} + y_2 - \delta y_1, \\ y_1(x,\tau) = u_0(x),\ y_2(x,\tau) = v_0(x) = u_1(x) + \delta u_0(x), \\ \eta_{1\tau}(x,\tau,s) = u_0(x,\tau) - u_0(x,\tau-s),\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+, \end{cases} \tag{94}$$

Also supposed that $\psi_1 = w$ and $\psi_{1t} = w_t$

$$\psi_2 = \psi_{1t} + \delta\psi_1 - \varepsilon z_1(\theta_t\omega) \tag{95}$$

then (92) could be rewritten like (94).

$$\begin{cases} \psi_{2t} - \delta\psi_2 + \delta^2\psi_1 + \alpha A\psi_2 + (\beta-\alpha\delta)A\psi_1 + \int_0^\infty \mu(s)A\eta_1(s)ds + f(u) - f_1(\varphi_1), \\ \quad = 2\delta\varepsilon z_1(\theta_t\omega) + \varepsilon A z_1(\theta_t\omega) + g(x,t) \\ \eta_{2t} = -\eta_{2s} + \psi_2 - \delta\psi_1 + \varepsilon z_1(\theta_t\omega), \\ \psi_1(x,\tau) = 0,\ \psi_2(x,\tau) = 0,\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, \\ \eta_{2\tau}(x,\tau,s) = 0,\ x\in\mathbb{R}^n,\ \tau\in\ \mathbb{R}, s\in\ \mathbb{R}^+. \end{cases} \tag{96}$$

Lemma 4. *Under condition (6)-(12) and (88)-(90). Let $B_1(\tau,\omega)\subseteq B_0(\tau,\omega)$ and $B_1 = \{B_1(\tau,\omega)\}_{\omega\in\Omega}\in\mathcal{D}(E)$ and $\tilde{\varphi}_0(\omega)\in B_1(\tau,\omega)$.There exists $T_1 = T_1(B_1,\omega) > 0$ and $M_0(\omega)$, such that the solution $\tilde{\varphi}(T_1,\omega,\tilde{\varphi}_0(\omega))$ of (91) satisfies for P-a.e $\omega\in\Omega$, $\forall\ t\geq T_1$*

$$\|\tilde{\varphi}(t,\tau,\omega,\varphi_0(\omega))\|_E^2 \leq \|\tilde{\varphi}_0\|^2 e^{-2\sigma t} + \int_\tau^r r_1(\omega)dr \leq M_0(\omega) \tag{97}$$

Proof. Taking inner product of (94) with y_2 in $L^2(\mathbb{R}^n)$, to find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y_2\|^2 - \delta\|y_2\|^2 + \alpha(Ay_2,y_2) + \delta^2(y_1,y_2) + (\beta-\alpha\delta)(Ay_1,y_2) \\ &+\int_0^\infty \mu(s)(A\eta_1(s),y_2)ds + (f_1(y_1),y_2) = 0, \end{aligned} \tag{98}$$

using the Hölder inequality and the Young inequality, easily to get

$$(y_1,y_2) \leq \frac{1}{2}\frac{d}{dt}\|y_1\|^2 + \frac{\delta}{2}\|y_1\|^2, \tag{99}$$

$$(Ay_1, y_2) \leq \frac{1}{2}\frac{d}{dt}\|\nabla y_1\|^2 + \frac{\delta}{2}\|\nabla y_1\|^2, \tag{100}$$

$$\int_0^\infty \mu(s)(A\eta_1(s), y_2)ds \geq \frac{1}{2}\frac{d}{dt}\|\eta_1(s)\|^2_{\mu,1} + \frac{\sigma}{2}\|\eta_1(s)\|^2_{\mu,1} - \frac{2m_0\delta^2}{\sigma}\|\nabla y_1\|^2. \tag{101}$$

By using of (88) and (90), to conclude

$$\begin{aligned}(f_1(y_1), y_2) &= (f_1(y_1), y_{1t} + \delta y_1)\\ &\leq \frac{d}{dt}\int_{\mathbb{R}^n} F_1(y_1)\,dx + \frac{\delta}{k_1}\int_{\mathbb{R}^n} F_1(y_1)\,dx - \frac{\mu\delta}{k_1}\|\nabla y_1\|^2.\end{aligned} \tag{102}$$

Applying together (99)-(102)and (98), to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dr}\left(\|y_2(r)\|^2 + \delta^2\|y_1(r)\|^2 + (\beta - \alpha\delta)\|\nabla y_1(r)\|^2 + \|\eta_1\|^2_{\mu,1} + \tilde{F}_1(y_1(e))\right)\\ &+\tfrac{\delta}{2}\left(\|y_2(r)\|^2 + \delta^2\|y_1(r)\|^2 + (\beta - \tfrac{\mu\delta}{k_1} - \alpha\delta - \tfrac{2m_0\delta}{\sigma})\|\nabla y_1(r)\|^2\right)\\ &+\sigma\|\eta_1(r)\|^2_{\mu,1} + \tfrac{\delta}{k_1}\tilde{F}_1(y(r)) \leq 0,\end{aligned} \tag{103}$$

suppose that $\varphi(t,\tau,\omega,\varphi_0(\omega)) = \tilde{\varphi}(t,\tau,\omega,\varphi_0(\omega)) - (0, z(\theta_t\omega), 0) \in B_0(\tau,\omega)$, by definition of $B_0(\tau,\omega)$, it follows that $\|\tilde{\varphi}(t,\tau,\omega,\varphi_0(\omega))\|^2_E \leq M(\omega) + |z(\theta_t\omega)| = r_1(\omega)$.
Applying Gronwall's Lemma to $[t,\tau]$, to arrive to (97).

$$\|\tilde{\varphi}(t,\tau,\omega,\varphi_0(\omega))\|^2_E \leq M_0(\omega).$$

The proof is completed □

Lemma 5. *Assume that (88) hold, let* $\tilde{\varphi}_0 = (y_1(x,\tau), y_{1t}(x,\tau) + \delta y_1(x,\tau), \eta_1((x,\tau),s))^\top) \in B, \tau \in \mathbb{R}, \omega \in \Omega, B \in \mathcal{D}$ *and* $\tilde{\varphi}(t,\omega) = (y_1(t), y_2(t), \eta_1^t(t,s))^\top$ *satisfies the system of (91), where B is bounded non-random subset of E, there exists* $\sigma_\delta(\omega) > 0$ *and* $M_\delta(\omega) > 0$ *such that.*

$$\|\tilde{\varphi}(t,\tau,\theta_{-t}\omega,\tilde{\varphi}_0)\|^2_E \leq M^2_\delta(\omega)e^{-2\sigma_\delta(\omega)(t-\tau)},\ \forall t \geq \tau. \tag{104}$$

Proof. Let $\tilde{\varphi} = (y_1, y_2, \eta_1)^\top = (y_1, y_{1t} + \delta y_1, \eta_1)^\top$ be solution of (91). By (88) and Sobolev embedding relation $H^1(\mathbb{R}^n) \hookrightarrow L^6(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$, there exists $\bar{r}(\omega) > 0$ and $\bar{\sigma}(\omega)$, such that, we get that

$$\begin{aligned}&F_1(y_1) \geq 0, \quad f_1(y_1)y_1 \geq 0, \quad \forall\, y \in \mathbb{R},\ x \in \mathbb{R}^n,\\ &0 \leq \tilde{F}_1(y_1) = \int_{\mathbb{R}^n} F_1(y_1)\,dx\\ &\qquad\leq C_5(\|y_1\|^2 + \|y_1\|^6_6)\\ &\qquad\leq \bar{r}(\omega)\|y_1\|^2_{H^1},\\ &\frac{\bar{\sigma}}{\bar{r}(\omega)}\tilde{F}_1(y_1) \leq \bar{\sigma}\|y_1\|^2_{H^1}, \quad \forall\, t \geq 0.\end{aligned} \tag{105}$$

Togethers with $(88)_2$, (105) and (99)-(101), from (98) yields

$$\frac{d}{dt}(\|\tilde{\varphi}\|^2_E + 2\tilde{F}_1(y_1)) + 2\bar{\sigma}(\omega)\|\tilde{\varphi}\|^2_E + \frac{\bar{\sigma}(\omega)}{\bar{r}(\omega)}2\tilde{F}_1(y_1) \geq 0, \tag{106}$$

whereas $\sigma_\delta(\omega) = \min\left[\bar{\sigma}(\omega), \frac{\bar{\sigma}(\omega)}{2\bar{r}(\omega)}\right]$.
By using the Gronwall's Lemma, to arrive the following result

$$\|\tilde{\varphi}(t,\tau,\theta_{-t}\omega,\tilde{\varphi}_0)\|^2_E \leq M^2_\delta(\omega)e^{-2\sigma_\delta(\omega)(t-\tau)}, \forall t \geq \tau.$$

Then, the proof is completed □

Lemma 6. *Under conditions (6)-(12) and (88)-(90) hold. Let* $B_1(\tau,\omega) \subseteq B_0(\tau,\omega)$ *and* $B_1 = \{B_1(\tau,\omega)\}_{\omega\in\Omega} \in \mathcal{D}(E)$ *and* $\psi(\tau,\omega) \in B_1(\tau,\omega)$*. There exists* $T_2 = T_2(B_1,\tau,\omega,) > 0$ *and a random radius* $\tilde{r}(\tau,\omega)$*, such that the solution* $\psi(t,\tau,\omega,\psi_0(\omega))$ *of (95)-(96) satisfies for P-a.e* $\omega \in \Omega$*,* $\forall\, t \geq T_2$

$$\|\psi\,(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2 \leq \|\psi_0(\theta_{-t}\omega)\|^2 e^{-2\sigma(\tau-t)} + r_2(\omega) \leq \tilde{r}(\tau,\omega), t \geq \tau. \tag{107}$$

to denote that

$$\nu = \min\{\frac{1}{4}, \frac{5-\gamma}{2}\}\,,\ \forall\, 0 \leq \gamma \leq 4. \tag{108}$$

Proof. According to (57), (97), and $\psi = \varphi - \tilde{\varphi}$, there exists a random variable $\rho(\omega) > 0$, such as

$$\max\left\{\|\psi\,(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2, \|\varphi\,(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2\right\} \leq \rho^2(\omega).$$

Multiplying (96) by $A^{2\nu}\psi_2(r)$ and integrating over $\mathbb{R}^n$, to achieve that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^\nu\psi_2\|^2 - \delta\|A^\nu\psi_2\|^2 + \alpha\left\|A^{\frac{1+2\nu}{2}}\psi_2\right\|^2 + \delta^2\left(\psi_1, A^{2\nu}\psi_2\right)\\
&\quad+(\beta-\alpha\delta)\left(A\psi_1, A^{2\nu}\psi_2\right) + \textstyle\int_0^\infty \mu(s)\left(A\eta_2(s), A^{2\nu}\psi_2\right)ds\\
&\quad+\left(f(u)-f_1(y_1), A^{2\nu}\psi_2\right)\\
&= \left(2\delta\varepsilon z(\theta_t\omega) - \alpha\varepsilon(Az(\theta_t\omega), A^{2\nu}\psi_2\right) + \left(g(x,t), A^{2\nu}\psi_2\right),
\end{aligned} \tag{109}$$

to deal with each term in (109) one by one, to conclude the following inequalities.

$$\left(\psi_1, A^{2\nu}\psi_2\right) \leq \frac{1}{2}\frac{d}{dt}\|A^\nu\psi_1\|^2 + \frac{\delta}{2}\|A^\nu\psi_1\|^2 - \frac{1}{2}|\varepsilon|^2|A^\nu z(\theta_t\omega)|^2, \tag{110}$$

$$\left(A\psi_1, A^{2\nu}\psi_2\right) \leq \frac{1}{2}\frac{d}{dt}\|A^{\nu+\frac{1}{2}}\psi_1\|^2 + \frac{\delta}{2}\|A^{\nu+\frac{1}{2}}\psi_1\|^2 - \frac{1}{2}|\varepsilon|^2|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2, \tag{111}$$

by second Eq. (4), (7)-(8), to conclude that

$$\begin{aligned}
&\int_0^\infty \mu(s)\left(A\eta_2(s), A^{2\nu}\psi_2\right)ds \geq \frac{1}{2}\frac{d}{dt}\|A^\nu\eta_2(s)\|_{\mu,1}^2 + \frac{\sigma}{2}\|A^\nu\eta_2(s)\|_{\mu,1}^2\\
&-\tfrac{2m_0\delta^2}{\sigma}\|A^{\nu+\frac{1}{2}}\psi_1\|^2 - \tfrac{2m_0\varepsilon^2}{\sigma}|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2,
\end{aligned} \tag{112}$$

from (88)and the Cauchy-Schwartz inequality and the Young inequality, straightforward to show that

$$\begin{aligned}
&\left(2\delta\varepsilon z(\theta_t\omega) - \alpha\varepsilon Az(\theta_t\omega), A^{2\nu}\psi_2\right)\\
&\leq 2\delta|\varepsilon|^2\,|A^\nu z(\theta_t\omega)|^2 + \tfrac{\delta}{2}\,\|A^\nu\psi_2\|^2 - \tfrac{\alpha}{2}|\varepsilon|^2\left|A^{\nu+\frac{1}{2}}z(\theta_t\omega)\right|^2 - \tfrac{\alpha}{2}\left\|A^{\nu+\frac{1}{2}}\psi_2\right\|^2,
\end{aligned} \tag{113}$$

$$\left(g(x,t), A^{2\nu}\psi_2\right) \leq \frac{1}{2\delta}\,|A^\nu g(x,t)|^2 + \frac{\delta}{2}\,\|A^\nu\psi_2\|^2. \tag{114}$$

From (9), (89)-(90) and the Young inequality, it is to deduce that

$$\begin{aligned}
&\left(f(u)-f_1(y), A^{2\nu}\psi_2\right) = \left(f(u)-f_1(y), A^{2\nu}\left(\psi_{1t}+\delta\psi_1-\varepsilon z(\theta_t\omega)\right)\right)\\
&= \tfrac{d}{dt}\textstyle\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}\psi_1 dx + \delta\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}\psi_1 dx\\
&\quad-\textstyle\int_{\mathbb{R}^n}(f'(u)u_t - f_1'(y)y_t)A^{2\nu}\psi_1 dx - \varepsilon\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}z(\theta_t\omega)dx\\
&= \tfrac{d}{dt}\textstyle\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}\psi_1 dx + \delta\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}\psi_1 dx\\
&\quad-\textstyle\int_{\mathbb{R}^n}((f_1'(u)-f_1'(y))u_t + f_1'(y)\psi_{1t} + f_2'(u)u_t)A^{2\nu}\psi_1 dx\\
&\quad-\varepsilon\textstyle\int_{\mathbb{R}^n}(f(u)-f_1(y))A^{2\nu}z(\theta_t\omega)dx,
\end{aligned} \tag{115}$$

hence to get the following inequalities

$$\begin{aligned}\int_{\mathbb{R}^n}(f_1'(u)-f_1'(y))|u_t||A^{2\nu}\psi_1|\,dx &\le c_7\int_{\mathbb{R}^n} f_1''(u+\theta(u-y))|u-y||u_t||A^{2\nu}\psi_1|\,dx\\ &\le c_8\left(\int_{\mathbb{R}^n}(1+|u|^3+|y|^3)\,dx\right)\left(\int_{\mathbb{R}^n}|\psi_1|^{\frac{6}{1-4\nu}}\,dx\right)^{\frac{1-4\nu}{6}}\\ &\quad\times\left(\int_{\mathbb{R}^n}|A^{2\nu}\psi_1|^{\frac{6}{1+4\nu}}\,dx\right)^{\frac{1+4\nu}{6}}\left(\int_{\mathbb{R}^n}|u_t|^{\frac{1}{2}}\,dx\right)^2\\ &\le c_9\left(1+\|u\|_{L^6}^3+\|y\|_{L^6}^3\right)\|y\|_{L^{\frac{6}{1-4\nu}}}\|A^{2\nu}\psi_1\|_{L^{\frac{6}{1+4\nu}}}\|u_t\|^2\\ &\le c_{10}\left(1+\|\nabla u\|^3+\|\nabla y\|^3\right)\|\psi_1\|_{L^{\frac{6}{1-4\nu}}}\|A^{2\nu}\psi_1\|_{L^{\frac{6}{1+4\nu}}}\|u_t\|^2\\ &\le c_{11}M_1(t,\tau,\theta_t\omega)+\frac{\delta^3}{2}\|A^{\nu+\frac{1}{2}}\psi_1\|^2,\end{aligned}\tag{116}$$

where $\theta\in(0,1)$, after using the embedding theorem to get that $H_1\hookrightarrow L^6$ and $H_{1-2\nu}=D(A^{\frac{1-2\nu}{2}})\hookrightarrow L^{\frac{6}{1+4\nu}}$

$$\begin{aligned}&\int_{\mathbb{R}^n} f_2'(u)u_tA^{2\nu}\psi_1dx\le c_{12}\int_{\mathbb{R}^n}(1+|u|^\gamma)\,|u_t|\left|A^{2\nu}\psi_1\right|dx\le c_{13}\left(\int_{\mathbb{R}^n}(1+|u|^\gamma)^{\frac{6}{5-4\nu}}dx\right)^{\frac{5-4\nu}{6}}\\ &\times\left(\int_{\mathbb{R}^n}|A^{2\nu}\psi_1|^{\frac{6}{1+4\nu}}\,dx\right)^{\frac{1+4\nu}{6}}\left(\int_{\mathbb{R}^n}|u_t|^2dx\right)^{\frac{1}{2}}\\ &\le c_{14}\left(1+\|u\|^\gamma_{L^{\frac{6\gamma}{5-4\nu}}}\right)\|A^{2\nu}\psi_1\|_{L^{\frac{6}{1+4\nu}}}\le c_{15}(1+\|\nabla u\|^\gamma)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|\\ &\le c_{16}M_2^{2\gamma}(t,\tau,\theta_t\omega)+\frac{\delta^3}{8}\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2,\end{aligned}\tag{117}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n} f_1'(y)\psi_{1t}A^{2\nu}\psi_1dx\le c_{17}\int_{\mathbb{R}^n}(1+|y|^4)\,|\psi_{1t}|\left|A^{2\nu}\psi_1\right|dx\\ &\le c_{18}\left(\int_{\mathbb{R}^n}(1+|y|^4)^{\frac{6}{4}}dx\right)^{\frac{4}{6}}\left(\int_{\mathbb{R}^n}|A^{2\nu}\psi_1|^{\frac{6}{1+4\nu}}\,dx\right)^{\frac{1+4\nu}{6}}\left(\int_{\mathbb{R}^n}|\psi_{1t}|^{\frac{6}{1-4\nu}}\,dx\right)^{\frac{1-4\nu}{6}}\\ &\le c_{19}\left(1+\|y\|^4_{L^6}\right)\|A^{2\nu}\psi_{1t}\|_{L^{\frac{6}{1-4\nu}}}\|\psi_1\|_{L^{\frac{6}{1+4\nu}}}\le c_{20}(1+\|\nabla y\|^2)\|A^\nu\psi_2\|^2+\frac{\delta^3}{8}\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2,\end{aligned}\tag{118}$$

where to have employed the inequality $\frac{\gamma}{1+4\nu}\le 1$ and $\frac{6\gamma}{1+4\nu}\le 6$, and exploited the embedding $H_1\hookrightarrow L^6$ and $H_{1-2\nu}=D(A^{\frac{1-2\nu}{2}})\hookrightarrow L^{\frac{6}{1+4\nu}}$

$$\begin{aligned}&\left(f(u)-f_1(y),A^{2\nu}(\varepsilon z(\theta_t\omega))\right)\le c_{21}|\varepsilon|\left|A^{2\nu}z(\theta_t\omega)\right|\int_{\mathbb{R}^n}|f'(u+\theta(u-y))|\,|\psi_1|dx\\ &\le c_{22}|\varepsilon|\left|A^{2\nu}z(\theta_t\omega)\right|\left(\int_{\mathbb{R}^n}(1+|u|^4+|y|^4)^{\frac{6}{4}}dx\right)^{\frac{4}{6}}\times\left(\int_{\mathbb{R}^n}|\psi_1|^{\frac{6}{1+4\nu}}dx\right)^{\frac{1+4\nu}{6}}\\ &\le c_{23}|\varepsilon|\left|A^{2\nu}z(\theta_t\omega)\right|\left(1+\|\nabla u\|^4+\|\nabla y\|^4\right)\|\psi_1\|_{L^{\frac{6}{1+4\nu}}}\\ &\le c_{24}\left[M_3(t,\tau,\theta_t\omega)+|\varepsilon|^4|A^\nu z(\theta_t\omega)|^4\right]+\frac{\delta^3}{4}\|A^{1+2\nu}\psi_1\|^2.\end{aligned}\tag{119}$$

Thus, by putting (110)-(119) into (109), to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|A^\nu\psi_2\|^2+\delta^2\|A^\nu\psi_1\|^2+(\beta-\alpha\delta)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2+\|A^\nu\eta_2\|^2_{\mu,1}+f(u)-f_1(y)\right)\\ &\quad+\frac{\delta}{2}\left(\|A^\nu\psi_2\|^2+\delta^2\|A^\nu\psi_1\|^2+(\beta-\alpha\delta-\frac{2m_0\delta}{\sigma})\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2\right)\\ &\quad+\sigma\|A^\nu\eta_2\|^2_{\mu,1}+\delta c_m\left(f(u)-f_1(y)\right)+\frac{1}{2\delta}|A^\nu g(x,t)|^2+\frac{(4\delta-3\alpha\lambda_0)}{2}\|A^\nu\psi_2\|^2\\ &\le c_{25}[1+M_1(t,\tau,\theta_t\omega)+M_2^{2\gamma}(t,\tau,\theta_t\omega)+M_3(t,\tau,\theta_t\omega)\\ &\quad+|\varepsilon|^4|A^\nu z(\theta_t\omega)|^4+|\varepsilon|^2|A^\nu z(\theta_t\omega)|^2+|\varepsilon|^2|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2,\end{aligned}\tag{120}$$

let

$$\boldsymbol{\varnothing}=\|A^\nu\psi_2\|^2+\delta^2\|A^\nu\psi_1\|^2+(\beta-\alpha\delta)\|A^{\frac{1+2\nu}{2}}\psi_1\|^2+\|A^\nu\eta_2\|^2_{\mu,1}+(f(u)-f_1(y)).$$

Then

$$\frac{1}{2}\frac{d}{dt}\boldsymbol{\varnothing}+\sigma_2\boldsymbol{\varnothing}\le c_{25}[1+M_1(t,\tau,\theta_t\omega)+M_2^{2\gamma}(t,\tau,\theta_t\omega)+M_3(t,\tau,\theta_t\omega) \\ +|\varepsilon|^4|A^\nu z(\theta_t\omega)|^4+|\varepsilon|^2|A^\nu z(\theta_t\omega)|^2+|\varepsilon|^2|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2], \tag{121}$$

where c_{25} depends on $c_{11}, c_{16}, c_{20}, c_{24}, \frac{1}{2\delta}, c_{22}, (\beta-\alpha\delta-\frac{2m_0\delta}{\sigma}), \frac{\alpha}{2}$.
Applying Gronwall's Lemma over $[\tau\ , t]$ and replacing ω to $\theta_{-t}\omega$, to find that, for all $t\ge 0$,

$$\|\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2\le\|\boldsymbol{\varnothing}(t,\tau,\theta_{-t}\omega,\boldsymbol{\varnothing}_0(\theta_{-t}\omega))\|_E^2 \\ \le e^{\sigma_2(t-\tau)}\|\boldsymbol{\varnothing}_0(\theta_{-t}\omega)\|_E^2+\int_{\tau-t}^0\Gamma_2(r-t,\theta_{r-t}\omega)e^{\sigma_2(r-t)}dr, \tag{122}$$

where σ_2 depends on $[\frac{\delta}{4},\sigma,\delta c_m,(2\delta-\frac{3\alpha\lambda_0}{2})]$ and

$$\Gamma_2(r-t,\theta_{r-t}\omega)=c_{25}[1+M_1(r-t,\tau,\theta_{r-t}\omega)+M_2^{2\gamma}(r-t,\tau,\theta_{t-t}\omega) \\ +M_3(r-t,\tau,\theta_{r-t}\omega)+|\varepsilon|^4|A^\nu z(\theta_{r-t}\omega)|^4+|\varepsilon|^2|A^\nu z(\theta_{r-t}\omega)|^2+|\varepsilon|^2|A^{\nu+\frac{1}{2}}z(\theta_{r-t}\omega)|^2. \tag{123}$$

Similar to (119), it is easy to show

$$(f(u)-f_1(y),A^{2\nu}\psi_1)\le c_{26}\int_{\mathbb{R}^n}(|f'(u+\theta(u-y))|)|u-y||A^{2\nu}\psi_1|dx \\ \le c_{27}\left(\int_{\mathbb{R}^n}(1+|u|^4+|y|^4)^{\frac{6}{4}}dx\right)^{\frac{4}{6}}\times\left(\int_{\mathbb{R}^n}(|\psi_1|^{\frac{6}{1-4\nu}}dx\right)^{\frac{1-4\nu}{6}}\left(\int_{\mathbb{R}^n}|A^{2\nu}\psi_1|^{\frac{6}{1+4\nu}}dx\right)^{\frac{1+4\nu}{6}} \\ \le M_4(t,\tau,\theta_t\omega)+\frac{\delta}{16}\|A^{\nu+\frac{1}{2}}\psi_1\|^2, \tag{124}$$

similar to (56)-(57), we can attain

$$\|\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2\le\|\boldsymbol{\varnothing}(t,\tau,\theta_{-t}\omega,\boldsymbol{\varnothing}_0(\theta_{-t}\omega))\|_E^2 \\ \le e^{\sigma_2(t-\tau)}\|\boldsymbol{\varnothing}_0(\theta_{-t}\omega)\|_E^2+\int_{\tau-t}^0\Gamma_2(r-t,\theta_{r-t}\omega)e^{-\sigma_2(t-r)}dr\le\tilde{r}(\tau,\omega).$$

the proof is completed. □

Lemma 7. *(see [27] Lemma 4.3) Under condition (6)-(12),(88)-(90). Let ψ be solution of equation (29)-(30) with initial data ψ_τ, then for any $\beta>0$, there exists a random variable $C_\beta(\omega)$ and $M_\beta(\tau,\omega)$ such that*

$$u=y_1+w_1\ .$$

where y_1, w_1 satisfies as following

$$\|A^{\frac{1+\sigma}{2}}w_1\|^2\le M_\beta(\tau,\omega)\ ,\ \forall\tau\le r\le t, \tag{125}$$

$$\int_{r_1}^{r_2}\|\nabla y_1(r)\|^2dr\le\beta(r_1-r_2)+C_\beta(\omega),\ \forall\tau\le r_1\le r_2\le t, \tag{126}$$

whereas a random variable $M_\beta(\tau,\omega)$ (independent of t) and $C_\beta(\omega)$ (independent of t and τ) now to establish asymptotic regularity by using the technique in [2,4,27].

Lemma 8. *Under conditions (9)-(12) and (88)-(90) hold. Let $B_\nu(\tau,\omega)\subseteq B_1(\tau,\omega)$ and $\{B_\nu=\{B_\nu(\tau,\omega)\}_{\omega\in\Omega}\in\mathcal{D}(E^\nu)$ and $\psi(\tau,\omega)\in B_\nu(\tau,\omega)$. There exists a random variable time $T_\nu=T_\nu(B_1,\tau,\omega)>0$ and a tempered random variable $M_\nu(\tau,\omega)$ such that for every $t>0$, the solution $\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))$ of (95)-(96) satisfies for P-a.e $\omega\in\Omega$, $\forall\ t\ge T_\nu, M\ge M_\nu$, $\psi_0(\theta_{-t}\omega)\in\{B_\nu(\tau,\theta_{-t}\omega)$,*

$$\|A^\nu\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_E^2\le\|\psi(\tau)\|^2e^{-2\sigma_\nu(t-\tau)}+r_3(\omega)\le\ M_\nu(\tau,\omega)\ ,t\ge\tau.$$

Proof. After simple computation and combining (110)-(119) with (109), to obtain the following equation

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|A^{\nu}\psi_2\|^2+\delta^2\|A^{\nu}\psi_1\|^2+(\beta-\alpha\delta)\|A^{\frac{1+2\nu}{2}}\psi_1\|^2+\|A^{\nu}\eta_2\|^2_{\mu,1}+(f(u)-f_1(y)))\\&\quad+\tfrac{\delta}{2}(\|A^{\nu}\psi_2\|^2+\delta^2\|A^{\nu}\psi_1\|^2+(\beta-\alpha\delta-\tfrac{2m_0\delta}{\sigma})\|A^{\frac{1+2\nu}{2}}\psi_1\|^2)+\sigma\|A^{\nu}\eta_2\|^2_{\mu,1}+(f(u)-f_1(y))\\&\leq\tfrac{1}{2\delta}\left[\|A^{\nu}g(x,t)\|^2+2|\varepsilon|^2\left(\tfrac{\alpha}{2}|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2+|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2+|A^{\nu}z(\theta_t\omega)|^2\right)\right]\\&\quad+\int_{\mathbb{R}^n}\left((f_1'(u)-f_1'(y))\,u_t+f_1'(y)\psi_{1t}+f_2'(u)u_t\right)A^{2\nu}\psi_1dx\\&\quad-\int_{\mathbb{R}^n}(f(u)-f_1(y))\,A^{2\nu}z(\theta_t\omega)dx+\tfrac{(4\delta-3\alpha\lambda_0)}{2}\|A^{\nu}\psi_2\|^2,\end{aligned}\tag{127}$$

applying Lemma 7 and embedding theorem $H_1(\mathbb{R}^n)\hookrightarrow L^6(\mathbb{R}^n)$ and $H_{1-2\nu}=D(A^{\frac{1-2\nu}{2}})\hookrightarrow L^{\frac{6}{1+4\nu}}$, to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^n}(f_1'(u)-f_1'(y))\,u_tA^{2\nu}\psi_1\leq c_{28}\int_{\mathbb{R}^n}(f_1''(u+\theta(u-y))\,|u-y||u_t|\left|A^{2\nu}\psi_1\right|dx\\&\leq c_{29}\int_{\mathbb{R}^n}(1+|u|^3+|y|^3)\,|\psi_1|\left|A^{2\nu}\psi_1\right|\,|u_t|\,dx\\&\leq c_{30}\int_{\mathbb{R}^n}(1+|y_1|^3+|w_1|^3+|y|^3)\,|\psi_1|\left|A^{2\nu}\psi_1\right|\,|u_t|dx\\&\leq c_{31}\left(1+\|y_1\|^3_{L^6}+\int_{\mathbb{R}^n}\left(|w_1|^{\frac{6}{1-4\nu}}dx\right)^{\frac{(1-4\nu)}{6}}+\|y\|^3_{L^6}\right)\|A^{\nu}\psi_1\|_{L^{\frac{6}{1-4\nu}}}\|u_t\|\left\|A^{\frac{1+\nu}{2}}\psi_1\right\|_{L^{\frac{6}{1+4\nu}}}\\&\leq\tfrac{\delta}{4}\|A^{\nu}\psi_1\|^2_{L^2}+c_{32}M_5(r,\tau,\theta_t\omega)\left(\|\nabla y_1\|^2+\|A^{\frac{1+2\nu}{2}}w_1\|^2\right)\|A^{\frac{1+\nu}{2}}\psi_1\|^2,\end{aligned}\tag{128}$$

$$\begin{aligned}&\int_{\mathbb{R}^n}f_2'(u)u_tA^{2\nu}\psi_1dx\leq c_{33}\int_{\mathbb{R}^n}(1+|u|^{\gamma})\,|u_t||A^{2\nu}\psi_1|dx\leq c_{34}\left(\int_{\mathbb{R}^n}(1+|u|^{\gamma})^6dx\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^n}\left|A^{2\nu}\psi_1\right|^{\frac{6}{1+4\nu}}dx\right)^{\frac{1+4\nu}{6}}\\&\times\left(\int_{\mathbb{R}^n}|u_t|^6dx\right)^{\frac{1}{6}}\leq c_{35}\left(1+\|\nabla u\|^{\gamma}\right)\|\nabla u_t\|\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|\leq c_{36}M_6^{\gamma}(r,\tau,\theta_t\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|,\end{aligned}\tag{129}$$

whereas $\theta\in(0,1)$ depends on t, and the following inequality is similar to (128), we have

$$\begin{aligned}&\int_{\mathbb{R}^n}f_1'(y)\psi_{1t}A^{2\nu}\psi_1dx\leq c_{37}\int_{\mathbb{R}^n}\left(1+|y|^4\right)\left|\psi_{1t}||A^{2\nu}\psi_1\right|dx\\&\leq c_{38}\left(\int_{\mathbb{R}^n}(1+|y|^4)^{\frac{6}{4}}dx\right)^{\frac{4}{6}}\left(\int_{\mathbb{R}^n}|A^{2\nu}\psi_1|^{\frac{6}{1+4\nu}}dx\right)^{\frac{1+4\nu}{6}}\\&\quad\times\left(\int_{\mathbb{R}^n}|\psi_{1t}|^{\frac{6}{1-4\nu}}dx\right)^{\frac{1-4\nu}{6}}\leq c_{39}\left(1+\|y\|^4_{L^6}\right)\|\psi_{1t}\|_{L^{\frac{6}{1-4\nu}}}\left\|A^{2\nu}\psi_1\right\|_{L^{\frac{6}{1+4\nu}}}\\&\leq c_{40}\left(1+\|\nabla y\|^2\right)(\|A^{\nu}\psi_2\|+\delta)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|\\&\leq c_{41}(\omega)\,\|\nabla y\|^2\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2+\tfrac{\alpha\delta}{4}\|A^{\nu}\psi_2\|^2+c_{42}(\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2,\end{aligned}\tag{130}$$

and

$$\begin{aligned}\left(f(u)-f_1(y),\varepsilon A^{2\nu}|z(\theta_t\omega)|\right)&\leq c_{43}\int_{\mathbb{R}^n}\left(1+|u|^4+|y|^4\right)|\psi_1||\varepsilon|\left|A^{2\sigma}z(\theta_t\omega)\right|dx\\&\leq c_{44}(\omega)|\varepsilon|^2\left|A^{\nu}z(\theta_t\omega)\right|^2+M_8^2(r,\tau,\theta_t\omega),\end{aligned}\tag{131}$$

similar to (128), using Cauchy inequality and the embedding theorem, to get

$$\begin{aligned}&\left(f(u)-f_1(y),A^{2\nu}\psi_1\right)\leq c_{45}\int_{\mathbb{R}^n}(|f'(u+\theta(u-y))|)\left|\psi_1||A^{2\sigma}\psi_1\right|dx\\&\leq c_{46}\int_{\mathbb{R}^n}(1+|u|^4+|y|^4)\,|\psi_1|\,\left|A^{2\nu}\psi_1\right|dx\\&\leq c_{47}\left(1+\|y_1\|^4_{L^6}+\left(\int_{\mathbb{R}^n}|w_1|^{\frac{6}{4(5-4\nu)}}dx\right)^{\frac{4(5-4\nu))}{6}}+\|v\|^4_{L^6}\right)\times\|A^{\nu}\psi_1\|_{L^{\frac{6}{4(5-4\nu)}}}\left\|A^{\frac{1+\nu}{2}}\psi_1\right\|_{L^{\frac{6}{1+4\nu}}}\\&\leq c_{48}(\omega)\|A^{\nu}\psi_1\|^2+c_{49}(\omega)M_7(r,\tau,\theta_t\omega)\times\left(\|\nabla y_1\|^2+\|A^{\frac{1+2\nu}{2}}w_1\|^2\right)\|A^{\frac{1+\nu}{2}}\psi_1\|^2,\end{aligned}\tag{132}$$

let $\boldsymbol{\varnothing} = \|A^{\nu}\psi_2\|^2 + \delta^2\|A^{\nu}\psi_1\|^2 + (\beta - \alpha\delta)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2 + \|A^{\nu}\eta_2\|^2_{\mu,1} + (f(u) - f_1(y))$. Then, it is concluded that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\boldsymbol{\varnothing} + \frac{\delta}{2}\left(\|A^{\nu}\psi_2\|^2 + \delta^2\|A^{\nu}\psi_1\|^2 \quad +(\beta - \alpha\delta - \frac{2m_0\delta}{\sigma})\|A^{\frac{1+2\nu}{2}}\psi_1\|^2\right) + \sigma\|A^{\nu}\eta_2\|^2_{\mu,1} + c_2\delta(f(u) - f_1(y))\\ &\le \tfrac{1}{2\delta}[2\varepsilon^2(|A^{\nu}z(\theta_t\omega)|^2 + \tfrac{\alpha}{2}|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2 + |A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2\\ &\quad +|A^{\nu}z(\theta_t\omega)|^2)] + c_{32}M_5(r,\tau,\theta_r\omega) \quad \times\left(\|\nabla y_1\|^2 + \|A^{\frac{1+2\nu}{2}}w_1\|^2\right)\|A^{\frac{1+\nu}{2}}\psi_1\|^2 + \tfrac{\delta}{4}\|A^{\nu}\psi_1\|^2_{L^2}\\ &\quad +c_{36}M_6^{\gamma}(r,\tau,\theta_r\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\| + c_{41}(\omega)\|\nabla y\|^2\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2\\ &\quad +\tfrac{\alpha\delta}{4}\|A^{\nu}\psi_2\|^2 + c_{42}(\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\| + M_8^2(r,\tau,\theta_r\omega) \quad + c_{44}|\varepsilon|^2|A^{\nu}z(\theta_r\omega)|^2 + (2(\delta - \tfrac{3\alpha\lambda_0}{2}) + \delta)\|A^{\nu}\varphi_2\|^2,\end{aligned} \tag{133}$$

it could be rewritten that

$$\begin{aligned}&\frac{d}{dt}\boldsymbol{\varnothing} + 2\rho\boldsymbol{\varnothing} \le c_{50}(\omega)\left[|\varepsilon|^2\left(|A^{\nu+\frac{1}{2}}z(\theta_t\omega)|^2 + |A^{\nu}z(\theta_t\omega)|^2\right) \quad +M_8^2(r,\tau,\theta_r\omega)\right] + c_{32}(\omega)M_5(r,\tau,\theta_r\omega)\\ &\times\left(\|\nabla v_1\|^2 + \|A^{\frac{1+2\nu}{2}}w_1\|^2\right)\|A^{\frac{1+\nu}{2}}\psi_1\|^2\\ &+c_{42}(\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2 + c_{36}(\omega)M_6^{\gamma}(r,\tau,\theta_r\omega)\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2 \quad + c_{41}(\omega)\|\nabla v\|^2\left\|A^{\frac{1+2\nu}{2}}\psi_1\right\|^2,\end{aligned} \tag{134}$$

whereas $\rho = \left[\frac{\delta}{4}, \sigma, c_m\delta, (3\delta - \frac{3\alpha\lambda_0}{4}), \frac{\alpha\delta}{4}\right]$.

Applying Gronwall's Lemma over $[\tau, t]$, to show that, for all $t \ge \tau$,

$$\begin{aligned}&\|\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|^2_E \le \|\boldsymbol{\varnothing}(t,\tau,\theta_{-t}\omega,\chi_0(\theta_{-t}\omega))\|^2_E \le e^{\bar{\sigma}_2(t-\tau)}\|\boldsymbol{\varnothing}_0(\theta_{-t}\omega)\|^2_E\\ &\quad + \textstyle\int_{\tau-t}^{0}\Gamma_3(r-t,\theta_{-r+t}\omega)e^{\bar{\sigma}_2(r-t)}dr,\end{aligned} \tag{135}$$

where

$$\Gamma_3(r-t,\omega) = c_{50}(\omega)\left[|\varepsilon|^2\left(|A^{\nu+\frac{1}{2}}z(\theta_{r-t}\omega)|^2 + |A^{\nu}z(\theta_{r-t}\omega)|^2\right) + M_9^2(r-t,\tau,\theta_{r-t}\omega)\right]. \tag{136}$$

and

$$\bar{\sigma}_2(r) = \rho - \|\nabla v_1(r,\tau,\theta_{-r}\omega,\psi_0(\theta_{-r}\omega))\|^2 - c_{51}(\omega)\|\nabla v(r,\tau,\theta_{-r}\omega,\psi_0(\theta_{-r}\omega))\|^2$$

For $r \ge r_2 \ge r_1 \ge \tau$, from (97) and (125)-(126), to get

$$\begin{aligned}\int_{r_1}^{r_2}\bar{\sigma}_2(r)dr &= \int_{r_1}^{r_2}\Big(\rho(r) - \|\nabla v_1(r,\tau,\theta_{-r}\omega,\psi_0(\theta_{-r}\omega))\|^2\\ &\quad -c_{51}(\omega)\|\nabla v(r,\tau,\theta_{-r}\omega,\psi_0(\theta_{-r}\omega))\|^2\Big)dr\\ &\ge \rho(r_2 - r_1) - c_{\rho}(\omega) - c_{52}(\omega)r_1(\omega)\textstyle\int_{r_1}^{r_2}e^{\sigma(r-t)dr}\\ &\ge \rho(r_2 - r_1) - M_9(\omega).\end{aligned} \tag{137}$$

Thus, from (115)-(116), (135) and (137), to reach

$$\begin{aligned}&\|\psi(t,\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|^2_E \le \|\boldsymbol{\varnothing}(t,\tau,\theta_{-t}\omega,\chi_0(\theta_{-t}\omega))\|^2_E\\ &\le e^{\bar{\sigma}_2(t-\tau)}\|\boldsymbol{\varnothing}_0(\theta_{-t}\omega)\|^2_E + \textstyle\int_{\tau-t}^{0}\Gamma_3(r-t,\theta_{r-t}\omega)e^{\bar{\sigma}_2(r-t)}dr\\ &\le e^{\bar{\sigma}_2(t-\tau)}\|\boldsymbol{\varnothing}_0(\theta_{-t}\omega)\|^2_E + \textstyle\int_{-\infty}^{0}\Gamma_3(r-t,\theta_{r-t}\omega)e^{\bar{\sigma}_2(r-t)}dr\\ &\le M_{\nu}(\tau,\omega).\end{aligned} \tag{138}$$

Since that $\int_{-\infty}^{0}\Gamma_3(r-t,\theta_{r-t}\omega)e^{\bar{\sigma}_2(r-t)}dr = r_3(\omega)$. The proof is completed. □

Lemma 9. *Under conditions f, g, f_1, (3)-(12) and (88)-(89) satisfy. For each $\nu \le \kappa \le 1$ and let $B_{\kappa}(\tau,\omega) \subseteq B_1(\tau,\omega)$ and $B_{\kappa} = \{B_{\kappa}(\tau,\omega)\}_{\omega\in\Omega} \in D(E_{\kappa})$. Then there exist $T_{\kappa}(\tau,\omega,B_{\kappa})$ and tempered variable $M_{\kappa}(\omega) > 0$ such that, the solution $\tilde{\psi}(t,\omega,\psi_{\tau}(\omega))$ of (91) satisfies for P-a.e $\omega \in \Omega$, $\forall\, t \ge 0$,*

$$\|\psi(t,\tau,\theta_{-t}\omega,B_{\kappa}(\theta_{-t}\omega))\|_{E_{\kappa}} \le M_{\kappa}(\omega), t \ge T_{\kappa}. \tag{139}$$

Lemma 10. *Under conditions f, g, f_1, (6)-(12) and (88)-(89)satisfy. For any $\kappa \in [\nu, 1-\nu]$, if initial condition set $B_\kappa(\tau,\omega) \subseteq B_1(\tau,\omega)$ and$\{B_\kappa = \{B_\kappa(\tau,\omega)\}_{\omega\in\Omega} \in \mathcal{D}(E_\kappa)$, such that for P-a.e $\omega \in \Omega$, $\forall\, t \geq 0$,*

$$\begin{aligned}&\|\psi(t,\tau,\theta_{-t}\omega, B_\kappa(\theta_{-t}\omega))\|_{E^\kappa} = \|\psi(t,\tau,\theta_{-t}\omega, B_\kappa(\theta_{-t}\omega))\|_{E^{\kappa+\nu}} + \|\psi_t(t,\tau,\theta_{-t}\omega, B_\kappa(\theta_{-t}\omega))\|_{E^{\kappa+\nu-1}}\\ &\leq J_\kappa(\tau,\omega)\end{aligned}$$

whereas the positive random variable depends only on the E_κ-bound of B_κ.

Proof. From, Lemma 7 and using (116)-(124), to conclude directly the above result. The proof is completed. □

6. Random Attractors

In this section, to prove the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with the stochastic wave equations (14)-(15) on $\mathbb{R}^n$. It follows from Lemma 2, that Φ has a closed random absorbing set in $\mathcal{D}$, then apply Lemmas in section4, it is proved the existence of a random attractor by using tail estimates and the decompose technique of solutions. which along with the $\mathcal{D}$-pullback asymptotic compactness (see[3,15,19,25,28]). Let that for every $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, then to get that

$$\eta_2(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega),s) = \begin{cases} w(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) - w(t-s,\tau,\theta_{-t+s}\omega,\varphi_0(\theta_{-t+s}\omega)),\ s \leq t,\\ w(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)),\ t \leq s;\end{cases} \tag{140}$$

$$\eta_{2,s}(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) = \begin{cases} w_t(t-s,\tau,\theta_{-t+s}\omega,\varphi_0(\theta_{-t+s}\omega)),\ 0 \leq s \leq t,\\ 0,\quad t \leq s.\end{cases} \tag{141}$$

Lemma 11. *Let $E_\nu = H_{2\nu+1} \times H_{2\nu} \times L^2_\mu(\mathbb{R}^+, H_{2\nu+1}) \hookrightarrow L^2_\mu(\mathbb{R}^+, H_{2\nu+1})$ is projection operator, setting $Y = \psi(t, B_\nu(\tau,\omega))$ is a random bounded absorbing set from Lemma 8, $\psi(t)$ is the solution operators of (95)-(96) and under the assumption of and Lemma 7, there is a positive random*

$$\begin{cases} 1\ \ Y \text{ is bounded in } L^2_\mu(\mathbb{R}^+, H_{1+2\nu}) \bigcap H^1_\mu(\mathbb{R}^+, H_{2\nu})\,,\\ 2\ \ \sup\limits_{\eta\in Y, s\in\mathbb{R}^+} \|\eta(s)\|^2_{\mu,1} \leq M_\nu(\omega).\end{cases} \tag{142}$$

Denote by B_ν the closed ball of $H_{1+2\nu} \times H_{2\nu}$ of random variable radius $M_\nu(\tau\omega)$, since to apply on a finite domain. B_ν is compact subset of $H_{1+2\nu} \times H_{2\nu}$. Then to since that a set $\tilde{B}_\nu(\tau,\omega)$

$$\tilde{B}_\nu(\tau,\omega) = \overline{\bigcup_{\psi_0(\theta_{-t}\omega)\in B_1(\theta_{-t}\omega)} \bigcup_{t\geq 0} \eta_2(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega),s)}\ s \in \mathbb{R}^+\ \tau \in \mathbb{R}\,, \omega \in \Omega, \tag{143}$$

whereas ν is as in (113). Thus, using (3) and (142), to get that

$$\|\eta(s)\|^2_{\mu,1} = \int_\tau^{+\infty} \mu(s)\|\nabla\eta(s)\|^2 ds \leq M_\nu(\tau,\omega)\int_\tau^{+\infty} e^{\delta s} ds \leq \frac{M_\nu(\tau,\omega)}{\delta}. \tag{144}$$

To obtain our main result about the existence of a random attractor for Random Dynamical System Φ as following Lemma

Lemma 12. *Let $\psi(t,\tau,\omega)$ be solution of system (95)-(96) and the assumption of Lemma 8 hold, then for any $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, There exists a random set $\tilde{B}_\nu(\tau,\omega) \in D(E_\nu)$ with*

$$\|\tilde{B}_\nu(\tau,\omega)\|_{E_\nu} = \sup_{\tilde{\psi}\in\tilde{B}_\nu(\tau,\omega)} \|\tilde{\psi}\|_E \leq M_\delta(\tau,\omega)$$

is relatively compact in E. Now to show the following attraction property of $\bar{A}(\tau,\omega)$, for every $B(\tau,\theta_{-t}\omega) \in \mathcal{D}(E)$, if there exist a Positive number σ and a tempered random variable $Q(\omega) \geq 0$, such that for any $\tau \in \mathbb{R}, \omega \in \Omega$, it holds that

$$d_H(\Phi(t,\tau,\theta_{-t}\omega, B_1(\tau,\theta_{-t}\omega)), \tilde{B}_\nu(\tau,\omega)) \leq Q(\omega)e^{-\sigma t} \to 0, \text{ as } t \to +\infty \tag{145}$$

Proof. consider $\varphi_0(\theta_{-t}\omega) \in B_1(\tau,\theta_{-t}\omega)$ and using (140)-(142) and Lemma 8 it shows that $\tilde{B}_\nu(\tau,\omega)$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H_1)$ and consequently in $L^2_\mu(\mathbb{R}^+, H_1)$, let $B_\nu(\tau,\omega) \subset E_\nu \subset E$ be closed ball of E_ν of radius

$M_\nu(\tau,\omega)$ defined by (67), where ν is as in (108). Finally making compact set $\bar{A}(r,\omega)=\tilde{B}_\nu\times B_\nu\subset E$, then by Lemma 2 and $\varphi_0(\theta_{-t}\omega)\in B_0(\tau,\theta_{-t}\omega)$, there exist a random set $M(\tau,\omega)\in B_0\subseteq B(\tau,\omega)\in\mathcal{D}(E)$, to conclude the following

$$d_H\left(\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)),B_0(\tau,\omega)\right)\le M(\tau,\omega)e^{-\sigma t}\ \to 0,\ as\ t\to+\infty. \tag{146}$$

in addition, it follows from Lemma 5 to $\varphi_0(\theta_{-t}\omega)\in B_1(\tau,\theta_{-t}\omega)$ there exists a random set $M_\delta(\tau,\omega)>0$ such that $M_\delta\in B_1(\tau,\omega)\in\mathcal{D}(E)$, to get that

$$d_H(\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)),B_1(\tau,\omega))\le M_\delta(\tau,\omega)e^{-\sigma_1 t}\ \to 0,\ as\ t\to+\infty, \tag{147}$$

Next, by using Lemma 8 and (108) to $\varphi_0(\theta_{-t}\omega)\in B_\nu(\tau,\theta_{-t}\omega)$, there exists $T_\nu(\tau,\omega,B_\nu)$ and a random set $M_\nu(\omega)$ such that $M_\nu(\omega)\in B_\nu(\tau,\omega)\subset B_1(\tau,\omega)\in D(E_\nu)$, so we get that

$$d_H(\Phi(t,\tau,\theta_{-t}\omega,B_1(\tau,\omega)),B_\nu(\tau,\theta_{-t}\omega))\le M_\nu(\omega)(\tau,\omega)e^{-\sigma_\nu t}\ \to 0,\ as\ t\to+\infty. \tag{148}$$

Since $\kappa\ge 0$ is fixed, by Lemma 9, there exists $T_\kappa(\tau,\omega,B_\kappa)$ and a random set $M_\kappa(\tau,\omega)$ such that $M_\kappa(\tau,\omega)\in B_\kappa(\tau,\omega)\in D(E_\nu)$ and

$$d_H(\Phi(t,\tau,\theta_{-t}\omega,B_1(\tau,\theta_{-t}\omega)),B_\kappa(\tau,\omega))\le M_\kappa(\tau,\omega)e^{-\sigma_\kappa t}\ \to 0,\ as\ t\to+\infty. \tag{149}$$

by Lemma 10 and Lemma 11 and (140)-(142), $M_\nu(\omega)$ such that $M_\nu(\omega)\in B_1(\tau,\omega)\in D(E)$, to get that

$$d_H(\Phi(t,\tau,\theta_{-t}\omega,B_\nu(\tau,\theta_{-t}\omega)),B_\kappa(\tau,\omega))\le M_\nu(\omega)e^{-\sigma_\kappa t}\ \to 0,\ as\ t\to+\infty. \tag{150}$$

Then by using (146)-(150) and Lemma 11, to assume that.

$$\bar{A}(\tau,\omega)=\tilde{B}_\nu(\tau,\omega)\times B_\kappa(\tau,\omega), \tag{151}$$

from Lemma 3 there exists $\bar{T}=\bar{T}(\tau,\omega,B)\ge 0$, then to dedicate the following attraction property

$$\varphi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega))\subseteq B_0(\tau,\omega),\forall t\ge\bar{T}.$$

Suppose $t\ge\bar{T}$ and $\tilde{T}=t-\bar{T}\ge T(\tau,\omega,B_0)\ge 0$ using cocycle property (iii) of Φ, to show that

$$\begin{aligned}
&\varphi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega))=\varphi(\tau,\tau-t,\theta_{-\tau}\omega,B(\tau-t,\theta_{-\tau}\omega))\\
&=\varphi(t,\tau-\tilde{T}-\bar{T},\theta_{-t}\omega,B(\tau-\tilde{t}-\bar{T},\theta_{-\tau}\omega))\\
&=\varphi(\tau,\tau-\tilde{T},(\theta_{-\tau}\omega),\varphi(\tau-\tilde{T},\tau-\bar{T}-\tilde{T},\theta_{-\tau}\omega,B(\tau-\tilde{T}-\bar{T},\theta_{-\tau}\omega))\\
&\subseteq\varphi(\tau,\tau-\tilde{T},\theta_{-\tau}\omega,B_0(\theta_{-\tilde{T}}\omega))\subseteq B_1(\tau,\theta_{-\tau}\omega).
\end{aligned} \tag{152}$$

Take any $\varphi(t,\tau,(\theta_{-t}\omega),\varphi_0(\theta_{-t}\omega))\in\varphi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega))$, for $t\ge\bar{T}+T(\tau,\omega,B_0)$, whereas $\varphi_0(\theta_{-t}\omega)\in B(\tau-t,\theta_{-t}\omega)$. Due to Lemmas2,5,8 and (152) it is decided that

$$\psi(t,\tau,(\theta_{-t}\omega),\varphi_0(\theta_{-t}\omega)=(\varphi(t,\tau,(\theta_{-t}\omega),\varphi_0(\theta_{-t}\omega))-\tilde{\varphi}(t,\tau,(\theta_{-t}\omega),\psi_0(\theta_{-t}\omega)))\in\bar{A}(\tau,\omega). \tag{153}$$

Then, by using Lemma 5, to conclude that

$$\begin{aligned}
&\inf_{\tilde{\psi}\in\bar{A}(\tau,\omega)}\|\varphi(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))-\tilde{\psi}(t,\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\\
&\le\|\tilde{\varphi}(t,\tau,\theta_{-t}\omega,\tilde{\varphi}_0(\theta_{-t}\omega))\|_E^2\le M_\delta^2(\omega)e^{-\sigma_\delta t}\ ,\ \forall t>\tilde{t}+T(\tau,\omega,B_0).
\end{aligned} \tag{154}$$

Thus there holds

$$d_H(\Phi(t,\tau,\theta_{-t}\omega,B(\tau,\theta_{-t}\omega)),\bar{A}(\tau,\omega))\le M_\delta(\omega)e^{-\sigma_\delta t}\ \to 0\ as\ t\to+\infty\,. \tag{155}$$

Then, the proof is completed. □

Theorem 2. *Let that(6)-(12)and $h_j\in H^2(\mathbb{R}^n)$ hold. There is a continuous cocycle Φ associated with problem (24), has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}(\tau,\omega)\subseteq\bar{A}(\tau,\omega)\bigcap B_0(\omega),\mathcal{A}(\tau,\omega)\in\mathcal{D}$ in $\mathbb{R}^n$.*

Proof. To follow from Lemma 1, Lemma 2, Lemma 3, (108), (137), Lemma 11 and Lemma 12, the random dynamical system Φ associated with (24) possesses a $\mathcal{D}$-pullback random attractor $\mathcal{A}(\tau,\omega) \subseteq \bar{\mathcal{A}}(\tau,\omega) \bigcap B_0(\omega)$. The proof is completed. □

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