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# Solving third order ordinary differential equation with constant coefficients by Adomian decomposition method

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**Abstract:** The aim of this work is to present an efficient modification of the Adomian Decomposition Method (ADM) for solving third-order ordinary differential equations with constant coefficients. The proposed approach is applicable to both linear and nonlinear problems. To demonstrate the effectiveness of the method, several examples are provided, showcasing its capability to handle both linear and nonlinear ordinary differential equations.

**Keywords:** Adomian decomposition method, third ordered ordinary differential equations.

**MSC:** 46B20, 46C15

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## 1. Introduction

The laws of physics are often expressed in the form of differential equations, which makes them fundamental to virtually all branches of science and engineering. Mathematics serves as the universal language of science, and within this language, differential equations play a central role in modeling and solving real-world problems [1]. Ordinary differential equations (ODEs) are a class of differential equations where the unknowns are functions of a single variable. These equations are widely used, particularly in the study of dynamical systems and electrical networks.

In many fluid mechanics problems, differential equations are nonlinear and complex, compounded by naturally varying flow geometries. Despite these challenges, solving such equations provides significant insight and opens pathways to technological innovation and modernization [2].

Third-order differential equations arise in various fields of applied mathematics and physics, such as modeling deflections in beams, three-layer structures, electromagnetic waves, and gravity-driven flows [3]. Recently, there has been growing interest in third-order boundary value problems (BVPs). For instance, [4,5] explored several cases of third-order BVPs, while [6–9] focused on three-point third-order problems. In 2020, J.O. Kuboye and collaborators developed hybrid numerical models to solve third-order ordinary differential equations directly [10]. Similarly, [11] investigated solutions to third-order ODEs using Riccati equations, and [12] proposed a novel Falkner-type method for solving third-order ODEs. This method utilizes configuration and interpolation techniques and is implemented in block mode while approximating values at grid points.

More recently, M.K. Duromla and others introduced a linear hybrid multistep block method for numerically integrating third-order ODEs, particularly initial value problems (IVPs) [13]. Nonlinear phenomena, significant in various branches of science and technology, present additional challenges for both theoretical and numerical approaches. The pursuit of efficient and accurate methods for solving such nonlinear models has gained considerable attention.

One prominent technique in this domain is the Adomian Decomposition Method (ADM), a semi-analytical approach used for both linear and nonlinear equations [14]. ADM has been applied to various problems, including boundary value problems, algebraic equations, and partial differential equations [15]. This method enables the accurate computation of series solutions with fast convergence and has become a valuable tool in applied sciences. ADM has also been employed for solving third-order differential equations, including singular initial value problems [16].

Y.Q. Hsan studied the application of ADM to second-order ODEs with constant coefficients [17]. Building on this foundation, the focus here will be on solving third-order ODEs with constant coefficients using the Adomian Decomposition Method.

## 2. Adomian Decompostion Method

We study the third-order ordinary differential equation of the form:

$$y''' + (3n + 2m + k)y'' + (3n^2 + m^2 + 4mn + 2nk + mk)y' + (nm^2 + 2mn^2 + n^3 + n^2k + mnk)y = f(x, y), \quad (1)$$

where $m, k \neq 0$, and the initial conditions are given as $y(0) = A$, $y'(0) = B$, and $y''(0) = C$. Here, $f(x, y)$ is a nonlinear function, $g(x)$ is a given function, and $A$, $B$, $C$, $n$, $k$, $m$ are constants.

The purpose of this study is to introduce a new differential operator to analyze Eq. (1). The operator is defined as:

$$L(.) = e^{-nx}\frac{d}{dx}e^{-mx}\frac{d}{dx}e^{-kx}\frac{d}{dx}e^{(n+m+k)x}(.), \quad (2)$$

allowing us to rewrite Eq. (1) as:

$$Ly = g(x) + f(x, y), \quad (3)$$

where the inverse operator is defined as:

$$L^{-1}(.) = e^{-(n+m+k)x}\int_0^x e^{kx}\int_0^x e^{mx}\int_0^x e^{nx}(.). \quad (4)$$

By applying the inverse operator to both sides of Eq. (3), we have:

$$L^{-1}Ly = L^{-1}g(x) - L^{-1}f(x, y). \quad (5)$$

This yields:

$$y(x) = \phi(x) + L^{-1}g(x) + L^{-1}f(x, y), \quad (6)$$

where:

$$\phi(x) = y(0) + xy'(0). \quad (7)$$

The constants $y(0)$ and $y'(0)$ are determined from the initial conditions.

The Adomian decomposition method (ADM) is employed to express the solution $y(x)$ and the nonlinear function $f(x, y)$ as infinite series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (8)$$

$$f(x, y) = \sum_{n=0}^{\infty} A_n(x), \quad (9)$$

where the components $y_n(x)$ are determined iteratively. See [18–20] for specific algorithms to compute Adomian polynomials. The algorithm is outlined as follows:

$$A_0 = F(y_0),$$
$$A_1 = y_1F'(y_0),$$
$$\vdots$$

Thus, we have:

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + L^{-1}\sum_{n=0}^{\infty} A_n(x), \quad (10)$$

and the components $y_n$ can be obtained as:

$$y_0 = \phi(x) + L^{-1}f(x, y),$$
$$y_{n+1} = L^{-1}A_n, \quad n \geq 0.$$

For example:

$$\begin{aligned}
y_0 &= \phi(x) + L^{-1} f(x,y), \\
y_1 &= L^{-1} A_0, \\
y_2 &= L^{-1} A_1, \\
y_3 &= L^{-1} A_2.
\end{aligned}$$

## 3. Numerical Illustrations

This section demonstrates the numerical application of ADM to solve the differential equation under study. The validity of the solutions is verified through comparison with exact solutions.

**Example 1.** Consider the differential equation:

$$y''' + 7y'' + 14y' + 8y = 30e^x - x + \ln(y), \tag{11}$$

with initial conditions $y(0) = y'(0) = y''(0) = 1$. Comparing Eq. (9) with Eq. (1), we identify $n = m = 1$ and $k = 2$. The differential operator is then:

$$L(.) = e^{-x} \frac{d}{dx} e^{-x} \frac{d}{dx} e^{-2x} \frac{d}{dx} e^{4x}(.). \tag{12}$$

The inverse operator is given as:

$$L^{-1}(.) = e^{-4x} \int_0^x e^{2x} \int_0^x e^{x} \int_0^x e^{x}(.). \tag{13}$$

The solution becomes:

$$y = \phi(x) + L^{-1}(30e^x - x) + L^{-1}(\ln(y)), \tag{14}$$

where:

$$\phi(x) = 5e^{-x} + e^{-4x} - 5e^{-2x}. \tag{15}$$

The first approximation is:

$$y_0(x) = \phi(x) + L^{-1}(30e^x - x), \tag{16}$$

yielding:

$$y_0 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{2} - \frac{43x^5}{30} + \frac{583x^6}{360} - \frac{317x^7}{420}. \tag{17}$$

The nonlinear terms are:

$$\begin{aligned}
y_1 &= \frac{x^4}{24} - \frac{7x^5}{120} + \frac{7x^6}{144} - \frac{2x^7}{105}, \\
y_2 &= \frac{x^7}{5040} - \frac{19x^8}{40320} + \frac{211x^9}{362880} - \frac{871x^{10}}{1814400}.
\end{aligned}$$

The solution is:

$$y = y_0 + y_1 + y_2, \tag{18}$$

resulting in:

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{61x^4}{24} - \frac{179x^5}{120} + \frac{1201x^6}{720} - \frac{557x^7}{720} - \frac{19x^8}{40320} + \frac{211x^9}{362880} - \frac{871x^{10}}{1814400}. \tag{19}$$

**Example 2.** Consider the equation

$$y''' + 6y'' + 11y' + 6y = 6x^2 + 22x + 12 - x^4 + y^2 \tag{20}$$

$$y(0) = y'(0) = 0, y''(0) = 2.$$

**Table 1.** The comparison between exact solution $y(x) = e^x$ and ADM

| x | Exact | ADM | Absolut error |
|---|---|---|---|
| 0.0 | 1 | 1 | 0 |
| 0.1 | 1.10517 | 1.10541 | 0.00036 |
| 0.2 | 1.2214 | 1.22502 | 0.00362 |
| 0.3 | 1.34986 | 1.36751 | 0.01765 |
| 0.4 | 1.49182 | 1.54602 | 0.0542 |
| 0.5 | 1.64872 | 1.77809 | 0.12937 |

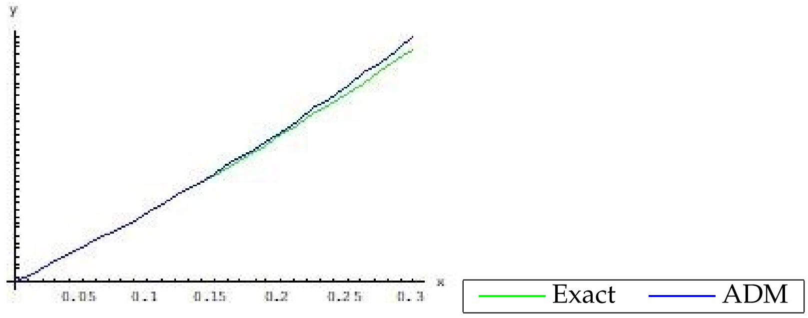


**Figure 1.** The exact solution $y = e^x$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

By comparing between Eq. (1) and Eq. (14), we get $n = m = k = 1$ and we get the operator as,

$$L(y) = 6x^2 + 22x + 12 - x^4 + y^2,$$

where the operator is as in Figure 1

$$L(.) = e^{-x}\frac{d}{dx}e^{-x}\frac{d}{dx}e^{-x}\frac{d}{dx}e^{3x}(.)$$

we use $L^{-1}$ for it and get,

$$y = \phi(x) + L^{-1}(6x^2 + 22x + 12 - x^4 + y^2),$$

where the inverse operator is as follows,

$$L^{-1}(.) = e^{-3x}\int_0^x e^x \int_0^x e^x \int_0^x e^x(.)$$

and

$$\phi(x) = e^{-x} + e^{-3x} - 2e^{-2x}.$$

Then value number one for $y$ is

$$y_0(x) = \phi(x) + L^{-1}(6x^2 + 22x + 12 - x^4),$$

$$y_0 = x^2 - \frac{x^7}{210} + \frac{x^8}{280} - \frac{5x^9}{3024} + \frac{x^{10}}{1680}, \tag{21}$$

and the nonlinear part is

$$y_{n+1} = L^{-1}(A_n), n \geq 0,$$

$$y_1 = \frac{x^7}{210} - \frac{x^8}{280} + \frac{5x^9}{3024} - \frac{x^{10}}{1680} \tag{22}$$

$$y_2 = \frac{x^{12}}{138600} - \frac{3x^{13}}{400400} + \frac{x^{14}}{360360} - \frac{37x^{15}}{50450400}. \tag{23}$$

then,

$$y = y_0 + y_1 + y_2$$

$$y = x^2 + \frac{x^{12}}{138600} - \frac{3x^{13}}{400400} + \frac{x^{14}}{360360} - \frac{37x^{15}}{50450400}. \tag{24}$$

**Table 2.** The comparison between exact solution $y(x) = x^2$ and ADM

| x | Exact | ADM | Absolute error |
|---|---|---|---|
| 0.0 | 0 | 0 | 0 |
| 0.1 | 1.01 | 1.01 | 0.00 |
| 0.2 | 0.04 | 0.04 | 0.00 |
| 0.3 | 0.09 | 0.09 | 0.00 |
| 0.4 | 0.16 | 0.16 | 0.00 |
| 0.5 | 0.25 | 0.25 | 0.00 |
| 0.6 | 0.36 | 0.36 | 0.00 |
| 0.7 | 0.49 | 0.49 | 0.00 |
| 0.8 | 0.64 | 0.64 | 0.00 |
| 0.9 | 0.81 | 0.81 | 0.00 |
| 1 | 1 | 1.000001764 | 0.000001764 |
| 1.5 | 2.25 | 2.24997 | 0.00003 |
| 2 | 4 | 3.9896078 | 0.0103922 |
| 2.5 | 6.25 | 5.991432 | 0.33568 |

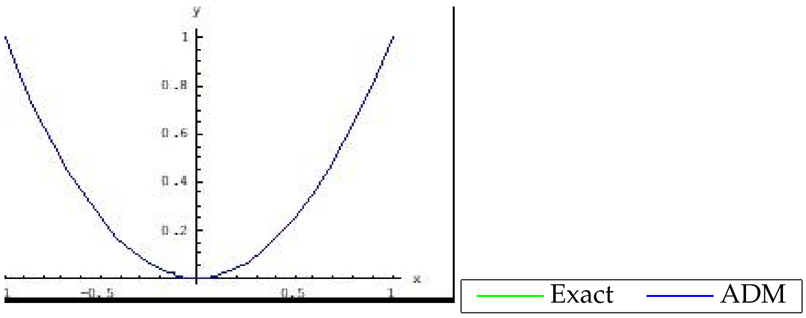


**Figure 2.** The exact solution $y = x^2$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$

**Example 3.** Consider the following nonlinear differential equation:

$$y''' + 12y'' + 44y' + 48y = 48(x+1) + 44 - (x+1)^3 + y^3, \tag{25}$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 0.$$

Using Eqs. (1) and (19), we find $n = m = k = 2$. Substituting these values, Eq. (19) is re-written as:

$$L(y) = 48(x+1) + 44 - (x+1)^3 + y^3,$$

where the operator $L(.)$ is defined as:

$$L(.) = e^{-2x}\frac{d}{dx}\left(e^{-2x}\frac{d}{dx}\left(e^{-2x}\frac{d}{dx}e^{6x}(.)\right)\right).$$

Using the inverse operator $L^{-1}$, we express the solution as:

$$y = \phi(x) + L^{-1}\left(48(x+1) + 44 - (x+1)^3 + y^3\right),$$

where the inverse operator is given by:

$$L^{-1}(.) = e^{-6x}\int_0^x e^{2x}\int_0^x e^{2x}\int_0^x e^{2x}(.)\,dx\,dx\,dx,$$

and

$$\phi(x) = \frac{17}{4}e^{-2x} - 5e^{-4x} + \frac{7}{4}e^{-6x}.$$

The initial approximation for $y$ is:

$$y_0(x) = \phi(x) + L^{-1}\left(48(x+1) + 44 - (x+1)^3\right),$$

which simplifies to:

$$y_0 = 1 + x - \frac{x^3}{6} + \frac{3x^4}{8} - \frac{7x^5}{12} + \frac{27x^6}{40} - \frac{199x^7}{315}. \quad (26)$$

The nonlinear part is recursively computed as:

$$y_{n+1} = L^{-1}(A_n), \quad n \geq 0.$$

For instance:

$$y_1 = \frac{x^3}{6} - \frac{3x^4}{8} + \frac{7x^5}{12} + \frac{163x^6}{240} + \frac{3223x^7}{5040}, \quad (27)$$

$$y_2 = \frac{x^6}{240} - \frac{13x^7}{1680} + \frac{x^8}{120} - \frac{779x^9}{120960} + \frac{763x^{10}}{172800}. \quad (28)$$

The solution is obtained as:

$$y = y_0 + y_1 + y_2,$$

which simplifies to:

$$y = 1 + x + \frac{x^8}{120} - \frac{779x^9}{120960} + \frac{763x^{10}}{172800}. \quad (29)$$

**Table 3.** Comparison between the exact solution $y(x) = 1 + x$ and the ADM solution, along with the absolute errors

| $x$ | Exact Solution | ADM Solution | Absolute Error |
|---|---|---|---|
| 0.0 | 1.00000 | 1.00000 | 0.00000 |
| 0.1 | 1.10000 | 1.10000 | 0.00000 |
| 0.2 | 1.20000 | 1.20000 | 0.00000 |
| 0.3 | 1.30000 | 1.30000 | 0.00000 |
| 0.4 | 1.40000 | 1.40000 | 0.00000 |
| 0.5 | 1.50000 | 1.50002 | 0.00002 |
| 0.6 | 1.60000 | 1.60010 | 0.00010 |
| 0.7 | 1.70000 | 1.70035 | 0.00035 |
| 0.8 | 1.80000 | 1.80101 | 0.00101 |
| 0.9 | 1.90000 | 1.90263 | 0.00263 |
| 1.0 | 2.00000 | 2.01413 | 0.01413 |
| 1.2 | 2.20000 | 2.22994 | 0.02994 |
| 1.3 | 2.30000 | 2.36055 | 0.06055 |
| 1.4 | 2.40000 | 2.51764 | 0.11764 |
| 1.5 | 2.50000 | 2.72061 | 0.22061 |

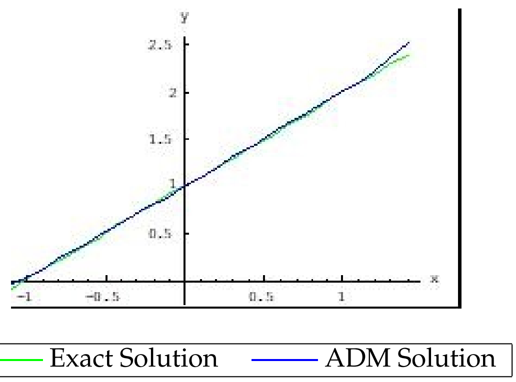


**Figure 3.** The exact solution $y = 1 + x$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$

## 4. Conclusion

The Adomian Decomposition Method (ADM) is a powerful tool for solving various functional equations, including ordinary differential equations, partial differential equations, and integral equations. This study demonstrates the effectiveness of ADM in solving third-order ordinary differential equations with constant coefficients. The results indicate that ADM provides accurate approximations for nonlinear differential equations.

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**Conflicts of Interest:** The authors declare no conflict of interest.

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