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Some geometric properties of a class of multivalent functions defined by an extended salagean differential operator

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Abstract: In the present investigation, the authors introduce a new class of multivalent analytic functions defined by an extended Salagean differential operator. Coefficient estimates, growth and distortion theorems for this class of functions are established. For this class, we also drive radius of starlikeness. Furthermore, the integral transforms of the class are obtained.

Keywords: analytic functions, multivalent functions, subordination, salagean differential operator.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let T_p be the subclass of $\mathcal{A}$ consisting of analytic p-valent functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad p \in \mathbb{N}. \tag{1}$$

The class T_p is referred to as the class of p-valent functions. A function $f(z)$ is called p-valent if it does not assume any value more than p times within U.

Definition 1. Let $p \in \mathbb{N}$, $\delta \in \mathbb{N} \setminus \{0\}$, and consider the function

$$f(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad p \in \mathbb{N}.$$

We denote by $D_*^\delta f(z)$ the differential operator defined by

$$D_*^\delta f(z) = D\left(D^{\delta-1} f(z)\right) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^\delta b_k z^k, \quad p \in \mathbb{N},\ \delta \in \mathbb{N} \setminus \{0\}. \tag{2}$$

Definition 2. For two functions s_1 and s_2, analytic in U, we say that the function $s_1(z)$ is *subordinate* to $s_2(z)$ in U, denoted by $s_1(z) \prec s_2(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that

$$s_1(z) = s_2(w(z)), \quad z \in U.$$

Furthermore, if s_2 is univalent in U, then the subordination $s_1(z) \prec s_2(z)$ is equivalent to

$$s_1(0) = s_2(0) \quad \text{and} \quad s_1(U) \subseteq s_2(U).$$

Definition 3. Ajab et al. [1] introduced the class $S_p(A,B,b,\lambda)$, utilizing the differential operator $D^{\lambda+p}f(z)$. This class consists of functions $f(z)\in S_p$ satisfying

$$1+\frac{1}{b}\left(\frac{z\left(D^{\lambda+p}f(z)\right)'}{D^{\lambda+p}f(z)}-p\right)\prec\frac{1+Az}{1+Bz}, \tag{3}$$

where $\prec$ denotes subordination, b is a nonzero complex number, and the parameters satisfy $-1\le B<A\le 1$.

We note that:

1. $S_1(1,-1,b,0)=C(b,1)$, which was studied by Wiatrowski [2].
2. $S_1(A,B,b,0)=C(A,B,b)$, which was studied by Ravichandran [3].

Motivated by the work of Ajab et al. [1], we define the following class $T_p(A,B,b,n,\alpha,\delta)$.

Definition 4. Let $T_p(A,B,b,n,\alpha,\delta)$ denote the class of functions $f(z)$ that satisfy the condition

$$1+\frac{1}{n}\left(\frac{z\left(D_*^{\delta}f(z)\right)'}{D_*^{\delta}f(z)}-p\right)\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha, \tag{4}$$

where $\prec$ denotes subordination, n is a positive real number, A and B are fixed real numbers with $-1\le B<A\le 1$, $0\le\alpha<1$, and $z\in U$.

The condition in Eq. (3) is equivalent to

$$\left|\frac{\frac{z\left(D_*^{\delta}f(z)\right)'}{D_*^{\delta}f(z)}-p}{n(A-B)(1-\alpha)-B\left(\frac{z\left(D_*^{\delta}f(z)\right)'}{D_*^{\delta}f(z)}-p\right)}\right|<1. \tag{5}$$

Remark 1. When $\delta=\lambda$, $n=b$, and $\alpha=0$, the class $T_p(A,B,b,n,\alpha,\delta)$ reduces to $S_p(A,B,b,\lambda)$ as studied by Ajab et al. [1]. Additionally, by varying the parameters A, B, n, δ, and α, we obtain the subclasses $C(b,1)$ and $C(A,B,b)$, which were studied by Ravichandran [3] and Wiatrowski [2], respectively. In this work, we discuss coefficient estimates, distortion and growth properties, the radius of starlikeness of the class $T_p(A,B,b,n,\alpha,\delta)$, and derive the integral transforms of the class.

2. Coefficient Estimates

Theorem 1. *A function defined by Eq. (1) belongs to the class $T_p(A,B,n,\delta,\alpha)$ if and only if*

$$\frac{\sum\limits_{k=p+1}^{\infty}\left[(k-p)(1-B)+n(A-B)(1-\alpha)\right]\left(\frac{k}{p}\right)^{\delta}b_k}{n(A-B)(1-\alpha)}\le 1. \tag{6}$$

Proof. Assume that $f(z)\in T_p(A,B,n,\delta,\alpha)$. By the definition of subordination, Eq. (4) can be expressed as

$$1+\frac{1}{n}\left(\frac{z(D_*^{\delta}f(z))'}{D_*^{\delta}f(z)}-p\right)=(1-\alpha)\frac{1+Aw(z)}{1+Bw(z)}+\alpha, \tag{7}$$

where $|w(z)|<1$ and $w(0)=0$.

From Eq. (7), we derive

$$\frac{z(D_*^{\delta}f(z))'}{D_*^{\delta}f(z)}-p=\frac{n(A-B)(1-\alpha)w(z)}{1+Bw(z)}. \tag{8}$$

Expanding and rearranging Eq. (8), we obtain

$$\frac{z(D_*^{\delta}f(z))'}{D_*^{\delta}f(z)}-p=\left[n(A-B)(1-\alpha)-B\left(\frac{z(D_*^{\delta}f(z))'}{D_*^{\delta}f(z)}-p\right)\right]w(z). \tag{9}$$

Substituting the differential operator $D^{\delta}f(z)$ defined by Eq. (2) into Eq. (9), we have

$$\frac{pz^p + \sum_{k=p+1}^{\infty} k\left(\frac{k}{p}\right)^{\delta} b_k z^k}{z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^{\delta} b_k z^k} - p = \left[n(A-B)(1-\alpha) - B\left(\frac{pz^p + \sum_{k=p+1}^{\infty} k\left(\frac{k}{p}\right)^{\delta} b_k z^k}{z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^{\delta} b_k z^k} - p\right)\right] w(z).$$

This simplifies to

$$\sum_{k=p+1}^{\infty} (k-p)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p} = \left[n(A-B)(1-\alpha) - \sum_{k=p+1}^{\infty} \left(B(k-p) - n(A-B)(1-\alpha)\right)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}\right] w(z). \tag{10}$$

Given that $|w(z)| < 1$, from Eq. (10) it follows that

$$|w(z)| = \left|\frac{\sum_{k=p+1}^{\infty} (k-p)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}}{n(A-B)(1-\alpha) - \sum_{k=p+1}^{\infty} \left(B(k-p) - n(A-B)(1-\alpha)\right)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}}\right| < 1.$$

Since $\Re(z) \leq |z|$ for all z, it follows that

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty} (k-p)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}}{n(A-B)(1-\alpha) - \sum_{k=p+1}^{\infty} \left(B(k-p) - n(A-B)(1-\alpha)\right)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}}\right\} \leq 1.$$

Taking the limit as $z \to 1^-$ along the real axis and clearing the denominator, we obtain

$$\sum_{k=p+1}^{\infty} [(k-p)(1-B) + n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k \leq n(A-B)(1-\alpha).$$

Therefore, the inequality (6) holds:

$$\frac{\sum_{k=p+1}^{\infty} [(k-p)(1-B) + n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k}{n(A-B)(1-\alpha)} \leq 1.$$

Conversely, assume that inequality (6) is satisfied. Then, from Eq. (4), we have

$$\left|z\left(D_*^{\delta}f(z)\right)' - pD_*^{\delta}f(z)\right| - \left|D_*^{\delta}f(z)\,[n(A-B)(1-\alpha)] - B\left[z\left(D_*^{\delta}f(z)\right)' - pD_*^{\delta}f(z)\right]\right| < 0, \tag{11}$$

provided that

$$\left|\sum_{k=p+1}^{\infty} (k-p)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}\right| - \left|n(A-B)(1-\alpha) - \sum_{k=p+1}^{\infty} \left(B(k-p) - n(A-B)(1-\alpha)\right)\left(\frac{k}{p}\right)^{\delta} b_k z^{k-p}\right| < 0. \tag{12}$$

For $|z| = r < 1$, the left-hand side of inequality (12) is bounded above by

$$\sum_{k=p+1}^{\infty} (k-p)\left(\frac{k}{p}\right)^{\delta} b_k r^{k-p} - n(A-B)(1-\alpha) - \sum_{k=p+1}^{\infty} [B(k-p) - n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k r^{k-p}.$$

This expression is less than

$$\sum_{k=p+1}^{\infty} [(k-p)(1-B) + n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k - n(A-B)(1-\alpha) \leq 0.$$

Hence, $f(z) \in T_p(A, B, n, \delta, \alpha)$. This completes the proof of Theorem 1. □

3. Growth and Distortion Theorems

We now establish the growth theorem for functions belonging to the class $T_p(A, B, b, n, \delta, \alpha)$.

Theorem 2. *Let $f(z)$ be defined by (1) and belong to the class $T_p(A, B, b, n, \delta, \alpha)$. Then, for $|z| = r$, the following inequality holds:*

$$r^p - r^{p+1}\left(\frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}\right) \le |f(z)| \le r^p + r^{p+1}\left(\frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}\right). \tag{13}$$

Furthermore, the bounds in (13) are attained for the functions $f(z)$ given by

$$f(z) = z^p + \left(\frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}\right) z^{p+1}, \quad |z| = r.$$

Proof. Let $f(z) \in T_p(A, B, b, n, \delta, \alpha)$. From Theorem 1, we have

$$\sum_{k=p+1}^{\infty} [(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k \le n(A-B)(1-\alpha).$$

Since

$$\sum_{k=p+1}^{\infty} [(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{k}{p}\right)^{\delta} b_k$$

is an increasing function of k, it follows that

$$[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta} \sum_{k=p+1}^{\infty} b_k \le n(A-B)(1-\alpha).$$

Hence,

$$\sum_{k=p+1}^{\infty} b_k \le \frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}. \tag{14}$$

Moreover, from Eqs. (1) and (14), and for $|z| = r$, it follows that

$$\begin{aligned} |f(z)| &\le r^p + \sum_{k=p+1}^{\infty} r^k b_k \\ &= r^p + r^{p+1} \sum_{k=p+1}^{\infty} b_k \\ &\le r^p + r^{p+1}\left(\frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} |f(z)| &\ge r^p - r^{p+1} \sum_{k=p+1}^{\infty} b_k \\ &\ge r^p - r^{p+1}\left(\frac{n(A-B)(1-\alpha)}{[(k-p)(1-B)+n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}}\right). \end{aligned}$$

Thus, the proof of Theorem 2 is complete. □

By setting $A = p = n = 1$ and $B = -1$ in Theorem 2, we obtain the following corollary:

Corollary 3. *If $f(z) \in T(\alpha, \delta)$, then for $|z| = r$, the inequality*

$$r - \frac{1-\alpha}{(2-\alpha)2^{\delta}} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(2-\alpha)2^{\delta}} r^2$$

holds.

This result was also obtained by Kadioglu [4].
By setting $A = p = n = 1$, $B = -1$, and $\delta = 0$ in Theorem 2, we obtain:

Corollary 4. *If $f(z) \in T(\alpha)$, then for $|z| = r$, the inequality*

$$r - \frac{1-\alpha}{2-\alpha} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2-\alpha} r^2$$

holds.

This result was also obtained by Silverman [5].

Theorem 5. *Let $f(z)$ be defined by (1) and belong to the class $A_p(A, B, n, \delta, \alpha)$. Then, for $|z| = r$, the following inequality holds:*

$$\begin{aligned} pr^{p-1} - r^p \left(\frac{(p+1)n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)] \left(\frac{p+1}{p}\right)^{\delta}} \right) &\leq |f'(z)| \\ \leq pr^{p-1} + r^p \left(\frac{(p+1)n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)] \left(\frac{p+1}{p}\right)^{\delta}} \right). \end{aligned} \tag{15}$$

Proof. From (14), we have

$$\sum_{k=p+1}^{\infty} k b_k \leq \frac{(p+1)n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)] \left(\frac{p+1}{p}\right)^{\delta}}. \tag{16}$$

Combining Eqs. (1) and (16), and for $|z| = r$, we obtain

$$\begin{aligned} |f'(z)| &\leq pr^{p-1} + \sum_{k=p+1}^{\infty} k b_k r^{k-1} \\ &= pr^{p-1} + r^p \sum_{k=p+1}^{\infty} k b_k \\ &\leq pr^{p-1} + r^p \left(\frac{(p+1)n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)] \left(\frac{p+1}{p}\right)^{\delta}} \right). \end{aligned}$$

Similarly,

$$\begin{aligned} |f'(z)| &\geq pr^{p-1} - \sum_{k=p+1}^{\infty} k b_k r^{k-1} \\ &= pr^{p-1} - r^p \sum_{k=p+1}^{\infty} k b_k \\ &\geq pr^{p-1} - r^p \left(\frac{(p+1)n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)] \left(\frac{p+1}{p}\right)^{\delta}} \right). \end{aligned}$$

Thus, the proof of Theorem 5 is complete. □

By setting $A = p = n = 1$ and $B = -1$ in Theorem 5, we obtain:

Corollary 6. *If $f(z) \in T(\delta, \alpha)$, then for $|z| = r$, the inequality*

$$1 - \frac{2(1-\alpha)}{(2-\alpha)2^{\delta}} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{(2-\alpha)2^{\delta}} r$$

holds.

This result was also obtained by Kadioglu [4].
By setting $A = p = n = 1$, $B = -1$, and $\delta = 0$ in Theorem 5, we obtain:

Corollary 7. *If $f(z) \in T(\alpha)$, then for $|z| = r$, the inequality*

$$1 - \frac{2(1-\alpha)}{2-\alpha} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{2-\alpha} r$$

holds.

This result was also obtained by Silverman [5].

4. Radius of Starlikeness of the class $T_p(A, B, n, \delta, \alpha, \beta)$

Theorem 8. *Let the function defined by Eq. (1) belong to the class $T_p(A, B, n, \delta, \alpha, \beta)$. Then $f(z)$ is p-valent and starlike of order β (where $0 \leq \beta < 1$) within the disk $|z| < r_2$, where*

$$r_2 = \inf_k \left[\frac{(p-\beta)\left[(k-p)(1-B) + n(A-B)(1-\alpha)\right]\left(\frac{k}{p}\right)^{\delta}}{(k-\beta)n(A-B)(1-\alpha)} \right]^{\frac{1}{k-p}}. \tag{17}$$

Proof. We aim to demonstrate that

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \beta \quad \text{for} \quad |z| < r_2.$$

From Eq. (1), we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq \frac{\sum\limits_{k=p+1}^{\infty} (k-p)|b_k||z|^{k-p}}{1 - \sum\limits_{k=p+1}^{\infty} |b_k||z|^{k-p}}.$$

Therefore, the inequality

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \beta$$

holds provided that

$$\frac{\sum\limits_{k=p+1}^{\infty} (k-\beta)|b_k||z|^{k-p}}{p-\beta} < 1. \tag{18}$$

Since

$$\frac{\sum\limits_{k=p+1}^{\infty} \left[(k-p)(1-B) + n(A-B)(1-\alpha)\right]\left(\frac{k}{p}\right)^{\delta} |b_k|}{n(A-B)(1-\alpha)} \leq 1, \tag{19}$$

the inequality (18) is satisfied if

$$\frac{\sum\limits_{k=p+1}^{\infty} (k-\beta)|b_k||z|^{k-p}}{p-\beta} < \frac{\sum\limits_{k=p+1}^{\infty} \left[(k-p)(1-B) + n(A-B)(1-\alpha)\right]\left(\frac{k}{p}\right)^{\delta} |b_k|}{n(A-B)(1-\alpha)}. \tag{20}$$

Solving for $|z|$ in inequality (20), we obtain

$$|z| < \left[\frac{(p-\beta)\left[(k-p)(1-B)+n(A-B)(1-\alpha)\right]\left(\frac{k}{p}\right)^{\delta}}{(k-\beta)n(A-B)(1-\alpha)} \right]^{\frac{1}{k-p}}.$$

Taking the infimum over all admissible values of k, we establish the radius r_2 as defined in equation (17). This completes the proof of Theorem 8. □

5. Integral Operators

In this section, we discuss the integral transforms of functions belonging to the class $T_p(A, B, n, \delta, \alpha, \beta)$.

Theorem 9. *Let the function $f(z)$ defined by Eq. (1) belong to the class $T_p(A, B, n, \delta, \alpha, \beta)$. Then the integral transform*

$$F(z) = \frac{c+p}{z^p} \int_0^z t^{c-1} f(t)\, dt, \quad c > -p,$$

belongs to the class $F_p^*(\gamma)$ for $0 \leq \gamma < p$, where

$$\gamma = \gamma(A, B, p, n, \alpha, \beta) = \frac{(c+k)(1-B)+n(A-B)(1-\alpha)-(c+p)(1-B)(1-\alpha)}{(c+k)(1-B)+n(A-B)(1-\alpha)}.$$

Moreover, this result is sharp for the function

$$f(z) = z^p + \frac{n(A-B)(1-\alpha)}{\left[(k-p)(1-B)+n(A-B)(1-\alpha)\right]\left(\frac{p+1}{p}\right)^{\delta}} z^{p+1}. \tag{21}$$

Proof. Let

$$f(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \in T_p(A, B, n, \delta, \alpha, \beta).$$

Then, the integral transform is given by

$$F(z) = \frac{c+p}{z^p} \int_0^z t^{c-1} f(t)\, dt, \quad c > -p,$$

which simplifies to

$$F(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{c+p}{c+k}\right) b_k z^k.$$

In view of Theorem 1, we seek to determine the largest γ for which

$$\frac{\sum_{k=p+1}^{\infty} \left[(k-p)(1-B)+n(A-B)(1-\gamma)\right]\left(\frac{k}{p}\right)^{\delta}|b_k|}{n(A-B)(1-\gamma)} \left(\frac{c+p}{c+k}\right) \leq 1.$$

It suffices to find the range of values for γ for each $k \in \mathbb{N}$ such that

$$\frac{(k-p)(1-B)+n(A-B)(1-\gamma)}{n(A-B)(1-\gamma)} \left(\frac{c+p}{c+k}\right) \leq \frac{(k-p)(1-B)+n(A-B)(1-\alpha)}{n(A-B)(1-\alpha)}. \tag{22}$$

Solving inequality (22) for γ, we obtain

$$\gamma \leq \frac{(c+k)(1-B)+n(A-B)(1-\alpha)-(c+p)(1-B)(1-\alpha)}{(c+k)(1-B)+n(A-B)(1-\alpha)}. \tag{23}$$

Since the right-hand side of Eq. (23) is an increasing function of k, substituting $k = p+1$ yields

$$\gamma \leq \frac{(c+p+1)(1-B)+n(A-B)(1-\alpha)-(c+p)(1-B)(1-\alpha)}{(c+p+1)(1-B)+n(A-B)(1-\alpha)}.$$

This establishes the desired bound for γ. Therefore, the integral transform $F(z)$ belongs to the class $F_p^*(\gamma)$ with the specified γ, and the result is sharp for the function given in Eq. (21). This completes the proof of Theorem 9. □

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