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Some new integral bounds for Godunova-Levin functions via fractional integral operators

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Abstract: This paper derives some new Hermite-Hadamard inequality and its different product versions, along with interesting non-trivial examples and remarks. Furthermore, we apply some of our results to special means as an application.

Keywords: Hermite-Hadamard, Fejér, Hölder's, Atangana-Baleanu, variable exponents.

MSC: 26D15,26A51, 46E35,26A33, 34A08

1. Introduction

Fractional calculus provides significant flexibility in modeling systems with memory effects, where the current state depends not only on the immediate past but also on a more extended history. This feature is particularly advantageous for systems exhibiting long-range interactions or substantial spatial correlations, as fractional-order derivatives and integrals offer a robust theoretical framework for explaining these non-local processes. Applications of fractional calculus are diverse, including signal processing, where fractional methods are effective in analyzing signals with long-range correlations or fractal-like characteristics, and electrochemistry, where it is employed to interpret impedance spectroscopy results and understand electrochemical processes such as charge transport in batteries and supercapacitors. Fractional differentiation and integration operators enhance signal processing and analysis by facilitating the extraction of valuable information, while broader uses span across various fields as noted in [1–5]. The study of fractional integral inequalities is a dynamic and evolving field, bridging classical analysis with modern mathematical techniques and uncovering new inequalities and applications that deepen our understanding of complex systems. Generalized convex mappings further extend the classical notion of convexity to broader contexts, enabling a wider range of functions to be classified as convex by relaxing certain conditions, incorporating more general forms of convex combinations, or introducing new parameters, as discussed in [6–10].

Assume $\mathscr{G} : \Omega \subseteq \mathtt{R} \to \mathtt{R}$ be a convex function defined over the interval Ω. Then, the following double inequality holds true:

$$\mathscr{G}\left(\frac{\mathbf{e_1}+\mathbf{f_1}}{2}\right) \leq \frac{1}{\mathbf{f_1}-\mathbf{e_1}}\int_{\mathbf{e_1}}^{\mathbf{f_1}} \mathscr{G}(\theta)\, d\theta \leq \frac{\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})}{2}. \tag{1}$$

This inequality has made significant contributions to mathematics in the areas of number theory and complex analysis, among many others. It is developed using various integral operators and generalized convex mappings in applications involving special means, probability density functions, numerical integration rules, etc. As a matter of fact, there are several notions for fractional integral operators (see refs. [11–14]). In addition to adding new parameters, one can generalize these fractional integrals and find inequalities such as Ostrowski, Grüss, Minkowski, Hermite–Hadamard, etc (see refs. [15–17]). Tariq et al. [18] provide a complete analysis of (H-H) type inequality involving several types of fractional integrals and demonstrate some applications using numerical calculations. Mohammed, Sarikaya and Baleanu [19] established the concept of incomplete gamma functions and studied the double inequality by using tempered fractional approach for convex mappings. Wang et al. [20] used two different forms of generalized convex mappings and generated numerous new bounds. Chu et al. [21] developed (H-H) inequality by using the notion of s-convex mappings via Hilfer integrals. Qi et al. [22] employed several types of exponential convex mappings to construct

Hermite-Hadamard's inequality with applications. Yildiz et al. [23] investigate the (H-H) inequalities for $\mathfrak{k}$-fractional integrals with the help of Green function. In [24], authors established various new inequalities for exponential s-convex mappings, utilizing Caputo fractional integrals, with several interesting applications.

In [25], the authors established the double inequality for s-convex functions in the second sense using the fractional integral operator. In [26], authors built a weighted double inequality whose second derivative was harmonic convex by using a Atangana-Baleanu integral operators. Sahoo et al. [27] developed a midpoint type fractional integral identity by combining various others well-known inequalities whose first-order derivative is convex. Afzal et al. [28,29] used h-Godunova-Levin functions with two different types of integral operators to establish various new integral bounds for (H-H) and its product forms with applications to special means. We refer to the following for some other recent developed outcomes related to these results (see refs. [30–34]).

Our study has the following characteristics that distinguish it from previous findings in the literature:

- Using different types of classical and fractional operators Afzal et al. [35–38] studied Godunova-Levin mappings and their associated inequalities in different perspectives of order relations with applications while this is the first time that we have utilized Atangana-Baleanu fractional integral operators with this class of generalized convex mappings to achieve desired results.
- Furthermore, for h-Godunova-Levin functions, it is innovative to create several new bounds of the (H-H) inequality by using other well-known inequalities such as Minkowski, Holder and Young.
- Given that some of our results are based on classical Holder's inequality, where exponent functions behave as constants, we left with an open problem: Would specific results hold if exponents are replaced from constants to variable?

This article is designed as follows. In Section 2, we will cover crucial ideas related to fractional calculus, including fundamental definitions and results. In Section 3, we develop our major findings. In Section 4, we discuss some of applications related to our main discoveries. In Section 5, we explore and conclude our key findings, provide some future recommendations, and leave an open question about variable exponent spaces.

2. Preliminaries

We provide some well-known definitions and facts in this part that can be utilized to bolster the paper's main conclusions.

Definition 1 (see [29]). Let $\mathscr{G} : \mathfrak{M} \subset \mathtt{R} \to \mathtt{R}$ defined on convex set $\mathfrak{M}$; then, $\mathscr{G}$ is said to be convex if

$$\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1 - \mathscr{D}\mathbf{f_1}) \leq \mathscr{D}\mathscr{G}(\mathbf{e_1}) + (1 - \mathscr{D})\mathscr{G}(\mathbf{f_1}),$$

holds for all $\mathbf{e_1}, \mathbf{f_1} \in \mathfrak{M} \subset \mathtt{R}$ and $\mathscr{D} \in [0, 1]$.

Definition 2 (see [29]). Consider h and $\mathscr{G}$ be non-negative mappings such that $\mathsf{h} : (0, 1) \to \mathtt{R}$ and $\mathscr{G} : \mathfrak{M} \subset \mathtt{R} \to \mathtt{R}$; then, $\mathscr{G}$ is said to be h-convex if

$$\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1 - \mathscr{D}\mathbf{f_1}) \leq \mathsf{h}(\mathscr{D})\mathscr{G}(\mathbf{e_1}) + \mathsf{h}(1 - \mathscr{D})\mathscr{G}(\mathbf{f_1}),$$

holds for all $\mathbf{e_1}, \mathbf{f_1} \in \mathfrak{M} \subset \mathtt{R}$ and $\mathscr{D} \in (0, 1)$.

Definition 3 (see [29]). Consider h and $\mathscr{G}$ be non-negative mappings such that $\mathsf{h} : (0, 1) \to \mathtt{R}$ and $\mathscr{G} : \mathfrak{M} \subset \mathtt{R} \to \mathtt{R}$; then, $\mathscr{G}$ is said to be h-Godunova-Levin if

$$\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1 - \mathscr{D}\mathbf{f_1}) \leq \frac{\mathscr{G}(\mathbf{e_1})}{\mathsf{h}(\mathscr{D})} + \frac{\mathscr{G}(\mathbf{f_1})}{\mathsf{h}(1 - \mathscr{D})},$$

holds for all $\mathbf{e_1}, \mathbf{f_1} \in \mathfrak{M} \subset \mathtt{R}$ and $\mathscr{D} \in (0, 1)$.

Remark 1.
- If $h_1(\mathscr{D}) = \frac{1}{\mathscr{D}^s}$, then Definition 3 leads to s-convex functions in the second sense [39].
- If $h(\mathscr{D}) = 1$, then Definition 3 leads to p-functions [40].
- If $h_1(\mathscr{D}) = \mathscr{D}^s, \mathsf{h}_2(\mathscr{D}) = 1$, then Definition 3 leads to s-Godunova-Levin functions [41].

According to reference [42], the left and right sides of Atangana-Baleanu fractional order integral operators are as follows:

$$
\begin{aligned}
{}^{\mathtt{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathtt{t}}\{\mathscr{G}(\mathtt{t})\} &= \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathtt{t}) + \frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_{\mathbf{e_1}}^{\mathtt{t}} \mathscr{G}(\mathscr{D})(\mathtt{t}-\mathscr{D})^{\mathscr{V}-1}\,\mathrm{d}\mathscr{D},\\
{}^{\mathtt{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathtt{t})\} &= \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathtt{t}) + \frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_{\mathtt{t}}^{\mathbf{f_1}} \mathscr{G}(\mathscr{D})(\mathscr{D}-\mathtt{t})^{\mathscr{V}-1}\,\mathrm{d}\mathscr{D},
\end{aligned}
$$

where $\mathbf{e_1} < \mathbf{f_1}, \mathscr{V} \in (0,1], \Gamma(\mathscr{V}) = \int_0^\infty \mathtt{t}^{\mathscr{V}-1}\mathtt{e}^{-\mathtt{t}}\,\mathtt{dt}$ is the Gamma function, $\mathbb{Q}(\mathscr{V}) > 0$ such that $\mathbb{Q}(0) = \mathbb{Q}(1) = 1, \|\mathbb{Q}(\mathscr{V})\| = 1$, and $\beta_{\mathtt{a}} = \beta_{\mathtt{a}}(\mathtt{p},\mathtt{q}) = \int_0^{\mathtt{a}} \mathscr{V}^{\mathtt{p}-1}(1-\mathscr{V})^{\mathtt{q}-1}\,\mathrm{d}\mathscr{V}$ is the incomplete beta integral. Since our findings are also presented in terms of special functions, we recall the very well known Gamma and Beta functions, respectively:

$$
\begin{aligned}
\Gamma(\mathscr{V}) &= \int_0^\infty e^{-\mathtt{t}}\mathtt{t}^{\mathscr{V}-1}\mathtt{dt},\\
\beta(\mathscr{V}_1,\mathscr{V}_2) &= \int_0^1 \mathtt{t}^{\mathscr{V}_1-1}(1-\mathtt{t})^{\mathscr{V}_2-1}\mathtt{dt} = \frac{\Gamma(\mathscr{V}_1)\Gamma(\mathscr{V}_2)}{\Gamma(\mathscr{V}_1+\mathscr{V}_2)}, \quad \mathscr{V}_1,\mathscr{V}_2 > 0.
\end{aligned}
$$

Another enhanced variation of Hölder's inequality has applications in a variety of mathematical analytical fields.

Theorem 1 (see [43]). *Let* $1 \le \mathtt{q}$. *Assume* $\mathscr{G}$ *and* $\aleph$ *are two mappings defined over* $[\mathbf{e_1},\mathbf{f_1}]$ *and* $|\mathscr{G}|, |\mathscr{G}||\aleph|^{\mathtt{q}}$ *are integrable over* $[\mathbf{e_1},\mathbf{f_1}]$, *then*

$$
\int_{\mathbf{e_1}}^{\mathbf{f_1}} |\mathscr{G}(\mathscr{D})\aleph(\mathscr{D})|\mathrm{d}\mathscr{D} \le \left(\int_{\mathbf{e_1}}^{\mathbf{f_1}} |\mathscr{G}(\mathscr{D})|\mathrm{d}\mathscr{D}\right)^{1-\frac{1}{\mathtt{q}}} \left(\int_{\mathbf{e_1}}^{\mathbf{f_1}} |\mathscr{G}(\mathscr{D})||\aleph(\mathscr{D})|^{\mathtt{q}}\mathrm{d}\mathscr{D}\right)^{\frac{1}{\mathtt{q}}}.
$$

Theorem 2 (see [43]). *(**Young's inequality**). Consider* $\mathtt{p},\mathtt{q}$ *be positive real numbers satisfying* $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$. *Then if* $\mathtt{f},\mathtt{g}$ *are nonnegative real numbers,*

$$
\mathtt{fg} \le \frac{\mathtt{f}^{\mathtt{p}}}{\mathtt{p}} + \frac{\mathtt{g}^{\mathtt{q}}}{\mathtt{q}},
$$

and equality holds iff $\mathtt{f}^{\mathtt{p}} = \mathtt{g}^{\mathtt{q}}$.

In [44], authors established new generalized inequalities for convexity utilizing a new identity, which is shown in the Lemma below.

Lemma 1 (see [44]). *Let* $\mathscr{G} : \mathfrak{M}^\circ \subset \mathtt{R} \to \mathtt{R}$ *is a differentiable mapping over* $\mathfrak{M}^\circ$. *If* $\mathscr{G}' \in \mathtt{L}[\mathbf{e_1},\mathbf{f_1}]$, *then one has*

$$
\begin{aligned}
\mathfrak{M}_{\mathfrak{k}}(\mathscr{G},\mathbf{e_1},\mathbf{f_1}) &= \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{1}{2\mathfrak{k}}\left[\mathscr{G}\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right) + \mathscr{G}\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)\right]\\
&\quad - \frac{1}{\mathbf{f_1}-\mathbf{e_1}}\int_{\mathbf{e_1}}^{\mathbf{f_1}} \mathscr{G}(\mathscr{V})d\mathscr{V}\\
&= \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1}-\mathbf{e_1}}{2\mathfrak{k}^2}\left[\int_0^1 (1-2\mathscr{D})\mathscr{G}'\left(\mathscr{D}\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} + (1-\mathscr{D})\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)\mathrm{d}\mathscr{D}\right].
\end{aligned}
$$

holds.

In [45], the authors introduced new inequalities that target s-convexity by using Lemma 1.

Theorem 3. *Let* $\mathscr{G} : \mathfrak{M} \subset \mathtt{R} \to \mathtt{R}$ *is a differentiable mapping on* $\mathfrak{M}^\circ$, *where* $\mathbf{e_1},\mathbf{f_1} \in \mathfrak{M}^\circ$, *with* $\mathbf{e_1} < \mathbf{f_1}$. *If* $|\mathscr{G}'|^{\mathtt{q}}$ *is* s*-convex on* $[\mathbf{e_1},\mathbf{f_1}]$ *for some* $\mathtt{q} > 1$, *then one has*

$$
\begin{aligned}
|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G},\mathbf{e_1},\mathbf{f_1})| &\le \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1}-\mathbf{e_1}}{2\mathfrak{k}^2}\left(\frac{1}{\mathtt{p}+1}\right)^{\frac{1}{\mathtt{p}}}\left(\frac{1}{\mathtt{s}+1}\right)^{\frac{1}{\mathtt{q}}}\\
&\quad \times \left[\left|\mathscr{G}'\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)\right|^{\mathtt{q}} + \left|\mathscr{G}'\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)\right|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}
\end{aligned}
$$

holds, where $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$.

The following lemma will be utilized throughout the key conclusions of the proof:

Lemma 2 (see [46]). *Let $\mathbf{e_1} < \mathbf{f_1}, \mathbf{e_1}, \mathbf{f_1} \in \mathrm{R}^+, \mathscr{G} : \mathrm{R}^+ \to \mathrm{R}^+$ is a differentiable. If $\mathscr{G}'' \in \mathrm{L}[\mathbf{e_1}, \mathbf{f_1}]$, for each $\mathscr{V} \in (0,1]$, then one has*

$$\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{AB}\mathrm{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{AB}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] - \frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)$$
$$= \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_0^1 \mathrm{w}^{\mathscr{V}}(\mathscr{D})\left[\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1}) + \mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\right]\mathrm{d}\mathscr{D},$$

where $\mathrm{w}^{\mathscr{V}}(\mathscr{D}) = \begin{cases} \mathscr{D}^{\mathscr{V}+1}, \mathscr{D} \in \left[0, \frac{1}{2}\right), \\ (1-\mathscr{D})^{\mathscr{V}+1}, \mathscr{D} \in \left[\frac{1}{2}, 1\right]. \end{cases}$

3. Main Results

This section aims to develop our major findings including the Hermite-Hadamard inequality in its weighted and product versions, using the h-Godunova-Levin function.

Theorem 4. *Let $\mathsf{h} : (0,1) \to \mathrm{R}^+$ and $\mathsf{h}\left(\frac{1}{2}\right) \neq 0$, and $\mathscr{G} : \mathrm{R}^+ \to \mathrm{R}^+$ is h-Godunova-Levin mapping, $\mathbf{e_1}, \mathbf{f_1} \in \mathrm{R}^+, \mathbf{e_1} < \mathbf{f_1}$ and $\aleph : [\mathbf{e_1}, \mathbf{f_1}] \to \mathrm{R}^+$ is symmetric about $\frac{\mathbf{e_1}+\mathbf{f_1}}{2}$. Then*

$$\begin{aligned} &\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\left[{}^{\mathcal{AB}}_{\mathbf{e_1}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\aleph(\mathbf{f_1})\} + {}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\aleph(\mathbf{e_1})\}\right] - \frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\aleph(\mathbf{e_1})+\aleph(\mathbf{f_1})] \\ &\quad + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1}) + \mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})] \le {}^{\mathcal{AB}}_{\mathbf{e_1}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{f_1})\} + {}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{e_1})\} \\ &\le \frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] \times \int_0^1 \mathscr{D}^{\mathscr{V}-1}\left[\frac{1}{\mathsf{h}(\mathscr{D})} + \frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D} \\ &\quad + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1}) + \mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})], \end{aligned} \tag{2}$$

where $\mathscr{V} \in (0,1]$.

Proof. As $\mathscr{G} \in \mathtt{SGX}(\mathsf{h}, [\mathbf{e_1}, \mathbf{f_1}], \mathrm{R}^+)$, we have

$$\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) \le \frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\left[\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1}) + \mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\right]. \tag{3}$$

Multiplying above inequality with $\mathsf{h}\left(\frac{1}{2}\right)\mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})$, and integrating the desired inequality on $(0,1)$, we have

$$\begin{aligned} &\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_0^1 \mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D} \\ &\quad \le \int_0^1 \mathscr{D}^{\mathscr{V}-1}\left[\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1}) + \mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}. \end{aligned}$$

Let $\mathrm{u} = \mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1}$, then the above inequality becomes

$$\begin{aligned} &\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\aleph(\mathrm{u})\mathrm{du} \\ &\quad \le \frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathbf{f_1}+\mathbf{e_1}-\mathrm{u})\aleph(\mathrm{u})\mathrm{du} + \int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})\mathrm{du}\right]. \end{aligned}$$

Let $\mathrm{v} = \mathbf{f_1} + \mathbf{e_1} - \mathrm{u}$, then we have $\aleph(\mathbf{f_1} + \mathbf{e_1} - \mathrm{v}) = \aleph(\mathrm{v})$, it follows that

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\aleph(\mathrm{u})d\mathrm{u}\\
&\quad\leq \frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{v})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})d\mathrm{v}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}\right].
\end{aligned}$$

Multiplying above inequality with $\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$ and add $\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]$ on both sides, we have

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\aleph(\mathrm{u})d\mathrm{u}+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})]\\
&\leq\frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{v})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})d\mathrm{v}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}\right]\\
&\quad+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})].
\end{aligned}$$

From this, it can be follows as

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right){}^{\mathcal{AB}}\mathrm{I}_{\mathbf{f_1}}^{\mathscr{V}}\{\aleph(\mathbf{e_1})\}\\
&\quad-\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\aleph(\mathbf{e_1})+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})]\\
&\leq{}_{\mathbf{e_1}}^{\mathcal{AB}}\mathrm{I}_{\mathbf{f_1}}^{\mathscr{V}}\{(\mathscr{G}\aleph(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathrm{I}_{\mathbf{f_1}}^{\mathscr{V}}\{(\mathscr{G}\aleph(\mathbf{e_1})\}.
\end{aligned}\tag{4}$$

Similarly, multiplying $\mathsf{h}\left(\frac{1}{2}\right)\mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})$ on both sides of (3) and integrate, we get

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_0^1\mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})d\mathscr{D}\\
&\quad\leq\int_0^1\mathscr{D}^{\mathscr{V}-1}[\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})+\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})]\aleph(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})d\mathscr{D}.
\end{aligned}$$

Let $\mathrm{u}=\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1}$, then the above inequality becomes

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\aleph(\mathrm{u})d\mathrm{u}\\
&\leq\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\mathscr{G}(\mathbf{f_1}+\mathbf{e_1}-\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}\right].
\end{aligned}$$

Let $\mathrm{v} = \mathbf{f_1} + \mathbf{e_1} - \mathrm{u}$, then we have $\aleph(\mathbf{f_1} + \mathbf{e_1} - \mathrm{v}) = \aleph(\mathrm{v})$, it follows that

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\aleph(\mathrm{u})d\mathrm{u}\\
&\quad\leq \frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{v}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})d\mathrm{v}\right].
\end{aligned}$$

Multiplying above inequality with $\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$ and add $\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]$ on both sides, we get

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\aleph(\mathrm{u})d\mathrm{u}+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})]\\
&\leq\frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{u})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})d\mathrm{u}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{v}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})d\mathrm{v}\right]\\
&\quad+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})].
\end{aligned}$$

From this, it can be follows that

$$\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right){}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\aleph(\mathbf{f_1})\}-\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\aleph(\mathbf{f_1})+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})]$$
$$\leq {}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{e_1})\}. \tag{5}$$

Adding (4) and (5), we can get the first inequality in (2). Now again taking into account Definition 3, we have

$$\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})\leq\frac{\mathscr{G}(\mathbf{e_1})}{\mathsf{h}(\mathscr{D})}+\frac{\mathscr{G}(\mathbf{f_1})}{\mathsf{h}(1-\mathscr{D})},$$
$$\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\leq\frac{\mathscr{G}(\mathbf{f_1})}{\mathsf{h}(\mathscr{D})}+\frac{\mathscr{G}(\mathbf{e_1})}{\mathsf{h}(1-\mathscr{D})},$$

adding the above two inequalities yields that

$$\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})+\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\leq\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right][\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})].$$

Multiplying aforementioned inequality with $\mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})$ and integrating, we have

$$\int_0^1\mathscr{D}^{\mathscr{V}-1}[\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})+\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}$$
$$\leq[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]\int_0^1\mathscr{D}^{\mathscr{V}-1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}.$$

Let $\mathrm{v}=\mathbf{f_1}+\mathbf{e_1}-\mathrm{u}$, then we have $\aleph(\mathbf{f_1}+\mathbf{e_1}-\mathrm{v})=\aleph(\mathrm{v})$, it follows

$$\frac{1}{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{v})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})\mathrm{dv}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})\mathrm{du}\right]$$
$$\leq[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]\int_0^1\mathscr{D}^{\mathscr{V}-1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}.$$

Multiplying above inequality with $\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$ and add $\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]$, we have

$$\frac{\mathscr{V}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathbf{f_1}-\mathrm{v})^{\mathscr{V}-1}\mathscr{G}(\mathrm{v})\aleph(\mathrm{v})\mathrm{dv}+\int_{\mathbf{e_1}}^{\mathbf{f_1}}(\mathrm{u}-\mathbf{e_1})^{\mathscr{V}-1}\mathscr{G}(\mathrm{u})\aleph(\mathrm{u})\mathrm{du}\right]$$
$$+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})]$$
$$\leq\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]$$
$$\times\int_0^1\mathscr{D}^{\mathscr{V}-1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}$$
$$+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})],$$

that is

$${}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{e_1})\}\leq\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})]$$
$$\times\int_0^1\mathscr{D}^{\mathscr{V}-1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}$$
$$+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})].$$

This completes the proof. □

Remark 2. If $\mathsf{h}(\mathscr{D})=1$, then Theorem 4 leads towards p-convex mappings:

$$\begin{aligned}
&\frac{1}{2}f\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\left[{}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\aleph(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\aleph(\mathbf{e_1})\}\right]-\frac{1}{2}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\aleph(\mathbf{e_1})+\aleph(\mathbf{f_1})\right]\\
&\quad+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})\right]\\
&\leq{}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{(\mathscr{G}\aleph(\mathbf{e_1})\}\\
&\leq\frac{2\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]\times\int_0^1\mathscr{D}^{\mathscr{V}-1}\aleph(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})\aleph(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\aleph(\mathbf{f_1})\right].
\end{aligned}$$

Theorem 5. *Let* $\mathsf{h}:(0,1)\to\mathsf{R}^+$ *where* $\mathsf{h}\left(\frac{1}{2}\right)\neq 0$, *and* $\mathscr{G}:\mathsf{R}^+\to\mathsf{R}^+$ *is* h*-Godunova-Levin mapping,* $\mathbf{e_1},\mathbf{f_1}\in\mathsf{R}^+$, $\mathbf{e_1}<\mathbf{f_1}$. *If* $\mathscr{G}\in\mathtt{L}[\mathbf{e_1},\mathbf{f_1}]$, *then the following double inequalities are obtained as follows:*

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]\\
&\quad\leq{}^{\mathcal{AB}}_{\mathbf{e_1}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}+{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{e_1})\}\\
&\quad\leq\left[\frac{\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})}{\mathbb{Q}(\mathscr{V})}\right]\left[1-\mathscr{V}+\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\Gamma(\mathscr{V})}\times\int_0^1\mathscr{D}^{\mathscr{V}-1}\left(\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right)\mathrm{d}\mathscr{D}\right],
\end{aligned}\tag{6}$$

where $\mathscr{V}\in(0,1)$.

Proof. As $\mathscr{G}\in\mathtt{SGX}(\mathsf{h},[\mathbf{e_1},\mathbf{f_1}],\mathsf{R}^+)$, we have

$$\mathscr{G}\left(\frac{\nu_1+\nu_2}{2}\right)\leq\frac{1}{\left[\mathsf{h}\left(\frac{1}{2}\right)\right]}\left[\mathscr{G}(\nu_1)+\mathscr{G}(\nu_2)\right],$$

let $\nu_1=\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1}$, $\nu_2=\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1}$, the above inequality becomes

$$\mathsf{h}\left(\frac{1}{2}\right)\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\leq\left[\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})+\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\right].\tag{7}$$

Multiplying with $\mathscr{D}^{\mathscr{V}-1}$ in (7) and integrate over interval (0,1), we have

$$\begin{aligned}
\frac{1}{\mathscr{V}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\leq&\frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\left[\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})\mathrm{d}\mathscr{D}\right.\\
&\left.+\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}\right],
\end{aligned}$$

that is

$$\frac{\mathsf{h}\left(\frac{1}{2}\right)}{\mathscr{V}}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\leq\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})\mathrm{d}\mathscr{D}+\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}.$$

Multiplying the above inequality with $\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$ and add $\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]$, we get

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]\\
&\quad\leq\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})\mathrm{d}\mathscr{D}\\
&\qquad+\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_0^1\mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D}+\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right].
\end{aligned}$$

Let $\mathsf{a} = \mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1}$ and $\mathbb{Q} = \mathscr{D}\mathbf{f_1} + (1-\mathscr{D})\mathbf{e_1}$ respectively, we have

$$\mathsf{h}\left(\frac{1}{2}\right)\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] \leq {}^{\mathcal{AB}}_{\mathbf{e_1}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\} + {}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{e_1})\},$$

so the initial inequality of (6) holds. For second part consider Definition 3, we have

$$\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1}) \leq \frac{\mathscr{G}(\mathbf{e_1})}{\mathsf{h}(\mathscr{D})} + \frac{\mathscr{G}(\mathbf{f_1})}{\mathsf{h}(1-\mathscr{D})}.$$

Multiplying above inequality with $\mathscr{D}^{\mathscr{V}-1}$, and integrate over (0,1), we have

$$\int_0^1 \mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1})\mathrm{d}\mathscr{D} \leq \mathscr{G}(\mathbf{e_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(\mathscr{D})} + \mathscr{G}(\mathbf{f_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}. \tag{8}$$

Multiplying both sides of (8) by $\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$ and add $\frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{f_1})$ on both sides, we get

$$\begin{aligned}\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_0^1 \mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1})\mathrm{d}\mathscr{D} + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{f_1}) \leq \frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\Bigg[\mathscr{G}(\mathbf{e_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(\mathscr{D})} \\ + \mathscr{G}(\mathbf{f_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\Bigg] + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{f_1}).\end{aligned} \tag{9}$$

Let $\mathsf{a} = \mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1}$, then the above inequality becomes

$${}^{\mathcal{AB}}_{\mathbf{e_1}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\} \leq \frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(\mathscr{D})} + \mathscr{G}(\mathbf{f_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right] + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{f_1}). \tag{10}$$

Again by Definition 3, we have

$$\mathscr{G}(\mathscr{D}\mathbf{e_1} + (1-\mathscr{D})\mathbf{f_1}) \leq \frac{\mathscr{G}(\mathbf{e_1})}{\mathsf{h}(\mathscr{D})} + \frac{\mathscr{G}(\mathbf{f_1})}{\mathsf{h}(1-\mathscr{D})}.$$

Multiplying aforementioned inequality with $\mathscr{D}^{\mathscr{V}-1}$, and integrating, we have

$$\begin{aligned}&\frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\int_0^1 \mathscr{D}^{\mathscr{V}-1}\mathscr{G}(\mathscr{D}\mathbf{f_1} + (1-\mathscr{D})\mathbf{e_1})\mathrm{d}\mathscr{D} + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{e_1}) \\ &\leq \frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\mathscr{G}(\mathbf{f_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(\mathscr{D})} + \mathscr{G}(\mathbf{e_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right] + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{e_1}).\end{aligned}$$

Let $\mathsf{v} = \mathscr{D}\mathbf{f_1} + (1-\mathscr{D})\mathbf{e_1}$ as a dummy variable, then the above inequality becomes

$${}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{e_1})\} \leq \frac{\mathscr{V}(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\mathscr{G}(\mathbf{f_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(\mathscr{D})} + \mathscr{G}(\mathbf{e_1})\int_0^1 \frac{\mathscr{D}^{\mathscr{V}-1}\mathrm{d}\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right] + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}\mathscr{G}(\mathbf{e_1}). \tag{11}$$

Adding (10) and (11), we can get that the second inequality of (6). This finishes the proof. □

Remark 3. If $\mathsf{h}(\mathscr{D}) = 1$, then Theorem 5 eads towards p-convex mappings:

$$\begin{aligned}&\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) + \frac{1-\mathscr{V}}{\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] \leq {}^{\mathcal{AB}}_{\mathbf{e_1}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\} + {}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{e_1})\} \\ &\leq \left[\frac{\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})}{\mathbb{Q}(\mathscr{V})}\right]\left[1-\mathscr{V}+\frac{2(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}}}{\Gamma(\mathscr{V})}\right].\end{aligned} \tag{12}$$

Theorem 6. *Let* $\mathsf{h} : (0,1) \to \mathsf{R}^+$ *such that* $\mathsf{h}\left(\frac{1}{2}\right) \neq 0$, *and* $\mathscr{G} : \mathsf{R}^+ \to \mathsf{R}^+$ *be a* h*-Godunova-Levin,* $\mathbf{e_1}, \mathbf{f_1} \in \mathsf{R}^+, \mathbf{e_1} < \mathbf{f_1}$. *If* $\mathscr{G}'' \in \mathsf{L}[\mathbf{e_1}, \mathbf{f_1}]$. *Then*

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|\right]\times\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D},
\end{aligned}
$$

where $\mathscr{V}\in(0,1]$.

Proof. Firstly, from Lemma 2, one has

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\times\int_0^1|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|\left[|\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|+|\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|\right]\mathrm{d}\mathscr{D}. \qquad (13)
\end{aligned}
$$

As $|\mathscr{G}''|$ is h-Godunova-Levin function, we have

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\times\left\{\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\right.\\
&+\int_{\frac{1}{2}}^1(1-\mathscr{D})^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\\
&+\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\\
&\left.+\int_{\frac{1}{2}}^1(1-\mathscr{D})^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\right\}\\
=&\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\\
&\times\left\{\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\right.\\
&\left.+\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{|\mathscr{G}''(\mathbf{f_1})|}{\mathsf{h}(\mathscr{D})}+\frac{|\mathscr{G}''(\mathbf{e_1})|}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}\right\}\\
=&\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|\right]\int_0^{\frac{1}{2}}\mathscr{D}^{\mathscr{V}+1}\left[\frac{1}{\mathsf{h}(\mathscr{D})}+\frac{1}{\mathsf{h}(1-\mathscr{D})}\right]\mathrm{d}\mathscr{D}.
\end{aligned}
$$

□

Remark 4. (i) If $\mathsf{h}(\mathscr{D})=1$, then Theorem 6 leads towards p-convex mappings:

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{\left(\frac{1}{2}\right)^{\mathscr{V}+1}}{(\mathscr{V}+1)(\mathscr{V}+2)}\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|\right].
\end{aligned}
$$

Theorem 7. *Let* $\mathsf{h} : (0,1) \to \mathsf{R}^+$ *such that* $\mathsf{h}\left(\frac{1}{2}\right) \neq 0$, *and* $\mathscr{G} : \mathsf{R}^+ \to \mathsf{R}^+$ *be a* h*-Godunova-Levin,* $\mathbf{e_1}, \mathbf{f_1} \in \mathsf{R}^+, \mathbf{e_1} < \mathbf{f_1}$. *If* $\mathscr{G}'' \in \mathsf{L}[\mathbf{e_1}, \mathbf{f_1}]$. *Then*

$$\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] \\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) \\
&\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathrm{p}+2\mathrm{p}}}{\mathscr{V}\mathrm{p}+\mathrm{p}+1}\right)^{\frac{1}{\mathrm{p}}}[|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|] \\
&\times\left[\left(\int_0^1 \frac{d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right)^{\frac{1}{\mathrm{q}}} + \left(\int_0^1 \frac{d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)^{\frac{1}{\mathrm{q}}}\right],
\end{aligned} \tag{14}$$

where $\mathscr{V} \in (0,1]$, $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$.

Proof. Considering Lemma 2 and Hölder's inequality for result (13), we have

$$\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] \\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) \\
&\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\int_0^1 |\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathrm{p}} d\mathscr{D}\right)^{\frac{1}{\mathrm{p}}}\left[\left(\int_0^1 |\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|^{\mathrm{q}} d\mathscr{D}\right)^{\frac{1}{\mathrm{q}}}\right. \\
&\left.+\left(\int_0^1 |\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|^{\mathrm{q}} d\mathscr{D}\right)^{\frac{1}{\mathrm{q}}}\right].
\end{aligned} \tag{15}$$

As $|\mathscr{G}''|^{\mathrm{q}}$ be h-Godunova and Levin function, then

$$\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] \\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) \\
&\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathrm{p}+2\mathrm{p}}}{\mathscr{V}\mathrm{p}+\mathrm{p}+1}\right)^{\frac{1}{\mathrm{p}}} \\
&\times\left\{\left[\int_0^1\left(\frac{|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}}{\mathsf{h}(\mathscr{D})} + \frac{|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}}{\mathsf{h}(1-\mathscr{D})}\right)d\mathscr{D}\right]^{\frac{1}{\mathrm{q}}}\right. \\
&\left.+\left[\int_0^1\left(\frac{|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}}{\mathsf{h}(\mathscr{D})} + \frac{|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}}{\mathsf{h}(1-\mathscr{D})}\right)d\mathscr{D}\right]^{\frac{1}{\mathrm{q}}}\right\}.
\end{aligned}$$

Then, we apply the fact that

$$\sum_{\mathfrak{k}=1}^{\mathbf{f_1}}(\mathrm{u}_{\mathfrak{k}}+\mathrm{v}_{\mathfrak{k}})^{\mathbf{e_1}} \leq \sum_{\mathfrak{k}=1}^{\mathbf{f_1}}\mathrm{u}_{\mathfrak{k}}{}^{\mathbf{e_1}} + \sum_{\mathfrak{k}=1}^{\mathbf{f_1}}\mathrm{v}_{\mathfrak{k}}{}^{\mathbf{e_1}},$$

for $0 < \mathbf{e_1} < 1, \mathrm{u}_1, \mathrm{u}_2, \cdots, \mathrm{u}_{\mathbf{f_1}} \geq 0, \mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_{\mathbf{f_1}} \geq 0$. So,one has the following inequality:

$$\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] \\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}p+2p}}{\mathscr{V}p+p+1}\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1\frac{|\mathscr{G}''(\mathbf{e_1})|^q}{h(\mathscr{D})}d\mathscr{D}\right)^{\frac{1}{q}}+\left(\int_0^1\frac{|\mathscr{G}''(\mathbf{f_1})|^q}{h(1-\mathscr{D})}d\mathscr{D}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\frac{|\mathscr{G}''(\mathbf{f_1})|^q}{h(\mathscr{D})}d\mathscr{D}\right)^{\frac{1}{q}}+\left(\int_0^1\frac{|\mathscr{G}''(\mathbf{e_1})|^q}{h(1-\mathscr{D})}d\mathscr{D}\right)^{\frac{1}{q}}\right]\\
&=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}p+2p}}{\mathscr{V}p+p+1}\right)^{\frac{1}{p}}\left[|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|\right]\\
&\times\left[\left(\int_0^1\frac{d\mathscr{D}}{h(\mathscr{D})}\right)^{\frac{1}{q}}+\left(\int_0^1\frac{d\mathscr{D}}{h(1-\mathscr{D})}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The finishes the proof. □

Remark 5. If $h(\mathscr{D})=1$, then Theorem 7 leads towards p-convex mappings:

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}I^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}I^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\frac{\left(\frac{1}{2}\right)^{\mathscr{V}+1}}{\mathscr{V}+2}\left[|\mathscr{G}''(\mathbf{e_1})|^q+|\mathscr{G}''(\mathbf{f_1})|^q\right]^{\frac{1}{q}}.
\end{aligned}\tag{16}
$$

Theorem 8. *Let* $h:(0,1)\to R^+$ *such that* $h\left(\frac{1}{2}\right)\neq 0$*, and* $\mathscr{G}:R^+\to R^+$ *be a* h*-Godunova-Levin,* $\mathbf{e_1},\mathbf{f_1}\in R^+,\mathbf{e_1}<\mathbf{f_1}$*. If* $\mathscr{G}''\in L[\mathbf{e_1},\mathbf{f_1}]$*. Then*

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}I^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}I^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{(\mathscr{V}+1)\left(\frac{q-p}{q-1}\right)}(q-1)}{(\mathscr{V}+1)(q-p)+q-1}\right)^{1-\frac{1}{q}}\\
&\times\left[|\mathscr{G}''(\mathbf{e_1})|^q+|\mathscr{G}''(\mathbf{f_1})|^q\right]^{\frac{1}{q}}\left[\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}p+p}d\mathscr{D}}{h(\mathscr{D})}+\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}p+p}d\mathscr{D}}{h(1-\mathscr{D})}\right]^{\frac{1}{q}},
\end{aligned}\tag{17}
$$

where $\mathscr{V}\in(0,1]$.

Proof. Considering the Holder's inequality and inequality (13), we have

$$
\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}I^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}I^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]\\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\\
&\times\int_0^1|w^{\mathscr{V}}(\mathscr{D})|^{\frac{q-p}{q}}\cdot|w^{\mathscr{V}}(\mathscr{D})|^{\frac{p}{q}}\left[|\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|+|\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|\right]d\mathscr{D}\\
&\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\int_0^1|w^{\mathscr{V}}(\mathscr{D})|^{\frac{q-p}{q-1}}d\mathscr{D}\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left[\left(\int_0^1 |\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|^{\mathtt{q}}d\mathscr{D}\right)^{\frac{1}{\mathtt{q}}}\right.$$
$$\left.+\left(\int_0^1 |\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|^{\mathtt{q}}d\mathscr{D}\right)^{\frac{1}{\mathtt{q}}}\right].$$

Since $|\mathscr{G}''|^{\mathtt{q}}$ is h-Godunova and Levin function, then

$$\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]$$
$$-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}$$
$$\times\int_0^1 |\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}}}\cdot|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\frac{\mathtt{p}}{\mathtt{q}}}\left[|\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|+|\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|\right]d\mathscr{D}$$
$$\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\int_0^1|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}}d\mathscr{D}\right)^{1-\frac{1}{\mathtt{q}}}\times\left[\left(\int_0^1\frac{|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right.\right.$$
$$\left.\left.+\int_0^1\frac{|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)^{\frac{1}{\mathtt{q}}}+\left(\int_0^1\frac{|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}+\int_0^1\frac{|\mathtt{w}^{\mathscr{V}}(\mathscr{D})|^{\mathtt{p}}|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)^{\frac{1}{\mathtt{q}}}\right]$$
$$=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{(\mathscr{V}+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\mathscr{V}+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}\left\{\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right.\right.\right.$$
$$\left.+\int_{\frac{1}{2}}^1\frac{(1-\mathscr{D})^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right)+|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}\left.\left(\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}+\int_{\frac{1}{2}}^1\frac{(1-\mathscr{D})^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)\right]^{\frac{1}{\mathtt{q}}}$$
$$+\left[|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}+\int_{\frac{1}{2}}^1\frac{(1-\mathscr{D})^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right)+|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right.\right.$$
$$\left.\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-\mathscr{D})^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)\right]^{\frac{1}{\mathtt{q}}}\right\}=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{(\mathscr{V}+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\mathscr{V}+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}$$
$$\times\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}+|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}\left[\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(\mathscr{D})}+\int_0^{\frac{1}{2}}\frac{\mathscr{D}^{\mathscr{V}\mathtt{p}+\mathtt{p}}d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right]^{\frac{1}{\mathtt{q}}}.$$

□

Remark 6. If $\mathsf{h}(\mathscr{D})=1$, then Theorem 8 leads towards p-convex mappings:

$$\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]$$
$$-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)$$
$$\leq\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{(\mathscr{V}+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\mathscr{V}+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}\times\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathtt{p}+\mathtt{p}}}{\mathscr{V}\mathtt{p}+\mathtt{p}+1}\right)^{\frac{1}{\mathtt{q}}}\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathtt{q}}+|\mathscr{G}''(\mathbf{f_1})|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}.$$

Theorem 9. *Let* $\mathsf{h}:(0,1)\to\mathsf{R}^+$ *such that* $\mathsf{h}\left(\frac{1}{2}\right)\neq0$, *and* $\mathscr{G}:\mathsf{R}^+\to\mathsf{R}^+$ *be a* h*-Godunova-Levin,* $\mathbf{e_1},\mathbf{f_1}\in\mathsf{R}^+,\mathbf{e_1}<\mathbf{f_1}$. *If* $\mathscr{G}''\in\mathtt{L}[\mathbf{e_1},\mathbf{f_1}]$. *Then*

$$\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathtt{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathtt{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]$$

$$
-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)
$$

$$
\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left\{\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathrm{p}+\mathrm{p}-1}}{(\mathscr{V}\mathrm{p}+\mathrm{p}+1)p}\right.
$$

$$
\left.+\frac{1}{\mathrm{q}}\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}+|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}\right]\int_0^1\left(\frac{1}{\mathrm{h}(\mathscr{D})}+\frac{1}{\mathrm{h}(1-\mathscr{D})}\right)d\mathscr{D}\right\},
$$

where $\mathscr{V}\in(0,1]$.

Proof. Considering Holder's inequality and result (13), related to Young's inequality: $\mathcal{AB}\leq\frac{1}{\mathrm{p}}\mathrm{a}^{\mathrm{p}}+\frac{1}{\mathrm{q}}{}^{\mathrm{q}}$, we obtain

$$
\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]
$$

$$
-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)
$$

$$
\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left[\frac{2}{\mathrm{p}}\int_0^1|\mathrm{w}^{\mathscr{V}}(\mathscr{D})|^{\mathrm{p}}d\mathscr{D}\right.
$$

$$
\left.+\frac{1}{\mathrm{q}}\left(\int_0^1|\mathscr{G}''(\mathscr{D}\mathbf{e_1}+(1-\mathscr{D})\mathbf{f_1})|^{\mathrm{q}}d\mathscr{D}+\int_0^1|\mathscr{G}''(\mathscr{D}\mathbf{f_1}+(1-\mathscr{D})\mathbf{e_1})|^{\mathrm{q}}d\mathscr{D}\right)\right].
$$

As $|\mathscr{G}''|^{\mathrm{q}}$ is h-Godunova and Levin, then

$$
\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]
$$

$$
-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)
$$

$$
\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left\{\frac{2}{\mathrm{p}}\int_0^1|\mathrm{w}^{\mathscr{V}}(\mathscr{D})|^{\mathrm{p}}d\mathscr{D}+\frac{1}{\mathrm{q}}\left[\int_0^1\frac{|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}d\mathscr{D}}{\mathrm{h}(\mathscr{D})}+\int_0^1\frac{|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}d\mathscr{D}}{\mathrm{h}(1-\mathscr{D})}\right.\right.
$$

$$
\left.\left.+\int_0^1\frac{|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}d\mathscr{D}}{\mathrm{h}(\mathscr{D})}+\int_0^1\frac{|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}d\mathscr{D}}{\mathrm{h}(1-\mathscr{D})}\right]\right\}
$$

$$
=\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left\{\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathrm{p}+\mathrm{p}-1}}{(\mathscr{V}\mathrm{p}+\mathrm{p}+1)p}+\frac{1}{\mathrm{q}}\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}+|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}\right]\times\int_0^1\left(\frac{1}{\mathrm{h}(\mathscr{D})}+\frac{1}{\mathrm{h}(1-\mathscr{D})}\right)d\mathscr{D}\right\}.
$$

The proof is completed. □

Remark 7. If $\mathrm{h}(\mathscr{D})=1$, then Theorem 9 leads towards p-convex mappings:

$$
\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathrm{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\}+{}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathrm{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right]
$$

$$
-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}\left[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})\right]-\frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right)
$$

$$
\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left\{\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathrm{p}+\mathrm{p}}}{(\mathscr{V}\mathrm{p}+\mathrm{p}+1)p}+\frac{1}{\mathrm{q}}\left[|\mathscr{G}''(\mathbf{e_1})|^{\mathrm{q}}+|\mathscr{G}''(\mathbf{f_1})|^{\mathrm{q}}\right]\right\}.
$$

Theorem 10. *Let* $\mathsf{h} : (0,1) \subset \mathtt{R} \to \mathtt{R}$ *and* $\mathscr{G} : [\mathbf{e_1}, \mathbf{f_1}] \to \mathtt{R}$ *be positive function with* $0 \leq \mathbf{e_1} < \mathbf{f_1}$ *and* $[\mathsf{h}(\mathscr{D})]^{\mathtt{q}} \in \mathtt{L}_1[0,1], \mathscr{G} \in \mathtt{L}_1[\mathbf{e_1}, \mathbf{f_1}]$. *If* $|\mathscr{G}'|$ *be a* h*-Godunova-Levin mapping on* $[\mathbf{e_1}, \mathbf{f_1}]$, *then*

$$|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| = \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\frac{1}{2}\right)^{\frac{\mathtt{q}-1}{\mathtt{q}}} \left(\int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right.\right.$$
$$\left.\left. + \int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right)^{\frac{1}{\mathtt{q}}} \right]$$

holds, where $1 < \mathtt{p}$ *and* $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$.

Proof. As $\mathtt{q} \geq 1$ and consider Lemma 1, then we have

$$\begin{aligned}
&|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left(\int_0^1 \left| (1-2\mathscr{D})\mathscr{G}' \left(\mathscr{D}\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} + (1-\mathscr{D})\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right| d\mathscr{D} \right) \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left(\int_0^1 |1-2\mathscr{D}| d\mathscr{D} \right)^{1-\frac{1}{\mathtt{q}}} \\
&\times \left(\int_0^1 |1-2\mathscr{D}| \left| \mathscr{G}' \left(\mathscr{D}\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} + (1-\mathscr{D})\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right)^{\frac{1}{\mathtt{q}}}.
\end{aligned}$$

As $|\mathscr{G}'|^{\mathtt{q}}$ is h-Godunova and Levin function, we have

$$\begin{aligned}
|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| &\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\int_0^1 |1-2\mathscr{D}| d\mathscr{D} \right]^{1-\frac{1}{\mathtt{q}}} \left[\int_0^1 |1-2\mathscr{D}| \left(\frac{1}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} \right.\right. \\
&\left.\left. + \frac{1}{\mathsf{h}(1-\mathscr{D})} \cdot \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} \right) d\mathscr{D} \right]^{\frac{1}{\mathtt{q}}} \\
&= \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left(\int_0^1 |1-2\mathscr{D}| d\mathscr{D} \right)^{1-\frac{1}{\mathtt{q}}} \left(\int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right. \\
&\left. + \int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right)^{\frac{1}{\mathtt{q}}} \\
&= \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\frac{1}{2}\right)^{\frac{\mathtt{q}-1}{\mathtt{q}}} \left(\int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(\mathscr{D})} \cdot \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right.\right. \\
&\left.\left. + \int_0^1 \frac{|1-2\mathscr{D}|}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} d\mathscr{D} \right)^{\frac{1}{\mathtt{q}}} \right].
\end{aligned}$$

□

Corollary 11. *If we apply* $\mathsf{h}(\mathscr{D}) = \frac{1}{\mathscr{D}}$ *in Theorem 10, one has*

$$|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| = \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{\mathfrak{k}^2 (2)^{2+\frac{1}{\mathtt{q}}}} \left(\left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{z} \right) \right|^{\mathtt{q}} + \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^{\mathtt{q}} \right)^{\frac{1}{\mathtt{q}}}$$

which appeared in [44].

Corollary 12. *If we apply* $\mathsf{h}(\mathscr{D}) = \frac{1}{\mathscr{D}^s}$ *in Theorem 10, one has*

$$|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| \leq \sum_{\mathscr{A}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{\mathfrak{k}^2 2^{2-\frac{1}{q}}} \left(\frac{1}{2^s(s+1)(s+2)} + \frac{s}{(s+1)(s+2)} \right)^{\frac{1}{q}} \times \left[\left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathscr{A})\mathbf{e_1} + \mathscr{A} r}{\mathfrak{k}} \right) \right|^q + \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathscr{A}-1)\mathbf{e_1} + (\mathscr{A}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q \right]^{\frac{1}{q}},$$

which appeared in [45].

Theorem 13. *Let* $\mathsf{h} : (0,1) \subset \mathtt{R} \to \mathtt{R}$ *and* $\mathscr{G} : [\mathbf{e_1}, \mathbf{f_1}] \to \mathtt{R}$ *be positive function with* $0 \leq \mathbf{e_1} < \mathbf{f_1}$ *and* $[\mathsf{h}(\mathscr{D})]^q \in \mathtt{L}_1[0,1], \mathscr{G} \in \mathtt{L}_1[\mathbf{e_1}, \mathbf{f_1}]$. *If* $|\mathscr{G}'|$ *is an* h*-Godunova-Levin mapping on* $[\mathbf{e_1}, \mathbf{f_1}]$, *then the following inequality*

$$|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| \leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\frac{1}{1+p} \right)^{\frac{1}{p}} \times \left(\int_0^1 \left[\frac{1}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q + \frac{1}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q \right] d\mathscr{D} \right)^{\frac{1}{q}} \right]$$

holds, where $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Suppose $1 < p$. Considering Lemma 1 and the Hölder inequality, then we have

$$\begin{aligned}
&|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\int_0^1 \left| (1-2\mathscr{D})\mathscr{G}' \left(\mathscr{D} \frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} + (1-\mathscr{D}) \frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right| d\mathscr{D} \right) \right] \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\int_0^1 |1-2\mathscr{D}|^p d\mathscr{D} \right)^{\frac{1}{p}} \right. \\
&\left. \times \left(\int_0^1 \left| \mathscr{G}' \left(\mathscr{D} \frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N} b}{\mathfrak{k}} + (1-\mathscr{D}) \frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1) b}{\mathfrak{k}} \right) \right|^q d\mathscr{D} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Since $|\mathscr{G}'|^q$ is h-Godunova and Levin function, then

$$\begin{aligned}
&\int_0^1 \left| \mathscr{G}' \left(\mathscr{D} \frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} + (1-\mathscr{D}) \frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right| d\mathscr{D} \\
&\leq \int_0^1 \left(\frac{1}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q + \frac{1}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right| \right) d\mathscr{D}.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
&|\mathfrak{M}_{\mathfrak{k}}(\mathscr{G}, \mathbf{e_1}, \mathbf{f_1})| \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\int_0^1 |1-2\mathscr{D}|^p d\mathscr{D} \right)^{\frac{1}{p}} \right. \\
&\left. \times \left(\int_0^1 \left(\frac{1}{\mathsf{h}(\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1} + \mathfrak{N}\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q + \frac{1}{\mathsf{h}(1-\mathscr{D})} \left| \mathscr{G}' \left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1} + (\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}} \right) \right|^q \right) d\mathscr{D} \right)^{\frac{1}{q}} \right] \\
&\leq \sum_{\mathfrak{N}=0}^{\mathfrak{k}-1} \frac{\mathbf{f_1} - \mathbf{e_1}}{2\mathfrak{k}^2} \left[\left(\frac{1}{1+p} \right)^{\frac{1}{p}} \right.
\end{aligned}$$

$$\times\left(\int_0^1\left(\frac{1}{h(\mathscr{D})}\left|\mathscr{G}'\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)\right|^q+\frac{1}{h(1-\mathscr{D})}\left|\mathscr{G}'\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)\right|^q\right)d\mathscr{D}\right)^{\frac{1}{q}}\Bigg].$$

□

Remark 8. If we apply $h(\mathscr{D})=\mathscr{D}^s$ in Theorem 13, one has Theorem 6 in [45].

4. Applications to Special Means

In this section, we describe some special means that can be used to interpret our main results described above. Let $\mathbf{e_1},\mathbf{f_1}\in\mathbb{R}$,

1. The arithmetic mean:

$$A=A(\mathbf{e_1},\mathbf{f_1}):=\frac{\mathbf{e_1}+\mathbf{f_1}}{2},\mathbf{e_1},\mathbf{f_1}\geq 0.$$

2. The harmonic mean:

$$H=H(\mathbf{e_1},\mathbf{f_1}):=\frac{2\mathbf{e_1}\mathbf{f_1}}{\mathbf{e_1}+\mathbf{f_1}},\mathbf{e_1},\mathbf{f_1}>0.$$

3. The logarithmic mean:

$$L=L(\mathbf{e_1},\mathbf{f_1}):=\begin{cases}\mathbf{e_1}, & \text{if } \mathbf{e_1}=\mathbf{f_1}\\ \frac{\mathbf{f_1}-\mathbf{e_1}}{\ln\mathbf{f_1}-\ln\mathbf{e_1}}, & \text{if } \mathbf{e_1}\neq\mathbf{f_1},\end{cases}\quad \mathbf{e_1},\mathbf{f_1}>0.$$

4. The p-logarithmic mean:

$$L_p=L_p(\mathbf{e_1},\mathbf{f_1}):=\begin{cases}\mathbf{e_1}, & \text{if } \mathbf{e_1}=\mathbf{f_1}\\ \left[\frac{r^{p+1}-\mathbf{e_1}^{p+1}}{(p+1)(\mathbf{f_1}-\mathbf{e_1})}\right]^{\frac{1}{p}}, & \text{if } \mathbf{e_1}\neq\mathbf{f_1},\end{cases}\quad p\in\mathbb{R}\backslash\{-1,0\},\mathbf{e_1},\mathbf{f_1}>0.$$

Proposition 1. *Let* $\mathbf{e_1},\mathbf{f_1}\in\mathbb{R}, 0<\mathbf{e_1}<\mathbf{f_1}$, *and* $i\in\mathbb{N}, i\geq 2$. *Then*

$$\begin{aligned}&\left|\sum_{\mathfrak{N}=0}^{\mathfrak{k}-1}\frac{1}{\mathfrak{k}\mathfrak{N}}A\left(\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)^m,\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)^m\right)-L_m{}^m(\mathbf{e_1},\mathbf{f_1})\right|\\ &\leq\sum_{\mathfrak{N}=0}^{\mathfrak{k}-1}\frac{(\mathbf{f_1}-\mathbf{e_1})m}{2^{2-\frac{1}{q}}\mathfrak{k}^2}\left[\left(\int_0^1\frac{|1-2\mathscr{D}|}{h(\mathscr{D})}d\mathscr{D}\right)\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)^{(m-1)q}\right.\\ &\quad\left.+\left(\int_0^1\frac{|1-2\mathscr{D}|}{h(1-\mathscr{D})}d\mathscr{D}\right)\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)^{(m-1)q}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. The proof of this result is follows from Theorem 10 with $\mathscr{G}(\mathscr{D})=\mathscr{D}^m,\mathscr{D}\in[\mathbf{e_1},\mathbf{f_1}], i\in\mathbb{N}, i\geq 2$. □

Proposition 2. *Let* $\mathbf{e_1},\mathbf{f_1}\in\mathbb{R}, 0<\mathbf{e_1}<\mathbf{f_1}$, *and* $i\in\mathbb{N}, i\geq 2$. *Then, the following*

$$\begin{aligned}&\left|\sum_{\mathfrak{N}=0}^{\mathfrak{k}-1}\frac{1}{\mathfrak{k}}A\left(\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)^m,\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)^m\right)-L_m{}^m(\mathbf{e_1},\mathbf{f_1})\right|\\ &\leq\sum_{\mathfrak{N}=0}^{\mathfrak{k}-1}\frac{(\mathbf{f_1}-\mathbf{e_1})m}{2^{2-\frac{1}{q}}\mathfrak{k}^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left[\left(\int_0^1\frac{d\mathscr{D}}{h(\mathscr{D})}\right)\left(\frac{(\mathfrak{k}-\mathfrak{N})\mathbf{e_1}+\mathfrak{N}\mathbf{f_1}}{\mathfrak{k}}\right)^{(m-1)q}+\left(\int_0^1\frac{d\mathscr{D}}{h(1-\mathscr{D})}\right)\left(\frac{(\mathfrak{k}-\mathfrak{N}-1)\mathbf{e_1}+(\mathfrak{N}+1)\mathbf{f_1}}{\mathfrak{k}}\right)^{(m-1)q}\right]^{\frac{1}{q}}\end{aligned}$$

holds, for all $1\leq q$.

Proof. The proof of this result follows from Theorem 13 with $\mathscr{G}(\mathscr{D})=\mathscr{D}^m,\mathscr{D}\in[\mathbf{e_1},\mathbf{f_1}], i\in\mathbb{N}, i\geq 2$. □

5. Conclusion

There has been considerable research into integral inequalities associated with different types of fractional operators. As we demonstrate in this note, we use h-Godunova-Levin mappings to develop different products forms of famous double inequality. In applications, we relate our newly developed results with special functions and special means. These results are the first to be developed for this generalized class of Godunova-Levin functions in the context of Atangana-Baleanu fractional operators, and we believe that this innovative concept will be used in the future in various other aspects.

6. Future recommendation

Variable Lebesgue spaces are a generalization of classical Lebesgue spaces in which the constant exponent p is replaced with variable exponent function $\mathsf{p}(\cdot)$. In the last few decades, many well known mathematical inequalities and classical results have been extended to the setting of variable exponent function spaces. In addition to these inequalities, the Holders inequality has also been extended to different variable exponent spaces. In variable exponent Lebesgue spaces, Holder inequality can be defined as follows:

Theorem 14 (see [47])**.** *Given* $\mathfrak{M}$ *and* $\mathsf{p}(\cdot) \in \mathsf{P}(\mathfrak{M})$*, for all* $\mathscr{G} \in \mathsf{L}^{\mathsf{p}(\cdot)}(\mathfrak{M})$ *and* $\aleph \in \mathsf{L}^{\mathsf{p}'(\cdot)}(\mathfrak{M}), \mathscr{G}, \aleph \in \mathsf{L}^1(\mathfrak{M})$ *and*

$$\int_{\mathfrak{M}} |\mathscr{G}(\mathscr{D})\aleph(\mathscr{D})| d\mathscr{D} \leq \mathsf{K}\|\mathscr{G}\|_{\mathsf{p}(\cdot)}\|\aleph\|_{\mathsf{p}'(\cdot)},$$

where

$$\mathsf{K} = \left(\frac{1}{\mathsf{p}_-} - \frac{1}{\mathsf{p}_+} + 1\right) \|\chi_{\mathfrak{M}_*}\|_\infty + \|\chi_{\mathfrak{M}_\infty}\|_\infty + \|\chi_{\mathfrak{M}_1}\|_\infty .$$

Remark 9. The final three terms in the definition of K are all equal to either 0 or 1, and at least one of them needs to be 1. So, if $p(\cdot)$ is not constant, $1 < \mathsf{K} \leq 4$.

As some notions are not explicitly defined here, please see page 27 of the book in the reference [47]. Moreover, there is a very recent work by Afzal et al. [48], in which authors extend the standard Hermite-Hadamard inequality by using the classical integral operator in variable exponent spaces, so we recommend readers extend the result below to the setup of variable exponent spaces for fractional integral operators.

$$\begin{aligned}
&\frac{1}{\mathbf{f_1}-\mathbf{e_1}}\left[{}^{\mathcal{AB}}\mathsf{I}^{\mathscr{V}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\{\mathscr{G}(\mathbf{e_1})\} + {}^{\mathcal{AB}}_{\frac{\mathbf{f_1}+\mathbf{e_1}}{2}}\mathsf{I}^{\mathscr{V}}_{\mathbf{f_1}}\{\mathscr{G}(\mathbf{f_1})\}\right] \\
&-\frac{1}{(\mathbf{f_1}-\mathbf{e_1})\mathbb{Q}(\mathscr{V})}[\mathscr{G}(\mathbf{e_1})+\mathscr{G}(\mathbf{f_1})] - \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{2^{\mathscr{V}-1}\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\mathscr{G}\left(\frac{\mathbf{f_1}+\mathbf{e_1}}{2}\right) \\
&\leq \frac{(\mathbf{f_1}-\mathbf{e_1})^{\mathscr{V}-1}}{(\mathscr{V}+1)\mathbb{Q}(\mathscr{V})\Gamma(\mathscr{V})}\left(\frac{\left(\frac{1}{2}\right)^{\mathscr{V}\mathsf{p}(\cdot)+2\mathsf{p}(\cdot)}}{\mathscr{V}\mathsf{p}(\cdot)+\mathsf{p}(\cdot)+1}\right)^{\frac{1}{\mathsf{p}(\cdot)}} [|\mathscr{G}''(\mathbf{e_1})|+|\mathscr{G}''(\mathbf{f_1})|] \\
&\times\left[\left(\int_0^1 \frac{d\mathscr{D}}{\mathsf{h}(\mathscr{D})}\right)^{\frac{1}{\mathsf{q}(\cdot)}} + \left(\int_0^1 \frac{d\mathscr{D}}{\mathsf{h}(1-\mathscr{D})}\right)^{\frac{1}{\mathsf{q}(\cdot)}}\right],
\end{aligned}$$

where $\mathscr{V} \in (0,1]$, $\frac{1}{\mathsf{p}(\cdot)} + \frac{1}{\mathsf{q}(\cdot)} = 1$.

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