Open Journal of
Mathematical Sciences

PSR Press

*Article*

# Some notes on an identity of Frisch

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Communicated By: Waqas Nazeer
Received: 02 January 2024; Accepted: 10 November 2024; Published: 31 December 2024.

**Abstract:** In this note, we show how a combinatorial identity of Frisch can be applied to prove and generalize some well-known identities involving harmonic numbers. We also present some combinatorial identities involving odd harmonic numbers which can be inferred straightforwardly from our results.

**Keywords:** harmonic number, odd harmonic number, binomial coefficient, binomial transform.

**MSC:** 05A10; 11B65.

## 1. Introduction

The binomial coefficients are defined, for non-negative integers $n$ and $m$, by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & \text{if } n \geq m; \\ 0, & \text{if } n < m. \end{cases}$$

More generally, for complex numbers $r$ and $s$, they are defined by

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)},$$

where the Gamma function, $\Gamma(z)$, is defined for $\Re(z) > 0$ by the integral [1]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

The Gamma function can be extended to the entire complex plane through analytic continuation. It possesses simple poles at each of the points $z = \ldots, -3, -2, -1, 0$. The Gamma function extends the classical factorial function to the complex plane via the relation $(z-1)! = \Gamma(z)$, thereby facilitating the computation of binomial coefficients for non-integer and non-real values.

Closely related to the Gamma function is the psi or digamma function, defined by $\psi(z) = \Gamma'(z)/\Gamma(z)$. It has the infinite series representation [1, p. 14]

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{k+z} \right), \tag{1}$$

where $\gamma$ is the Euler-Mascheroni constant.

Harmonic numbers, denoted by $H_x$ for $x \in \mathbb{C} \setminus \mathbb{Z}^-$, are defined by the recurrence relation

$$H_x = H_{x-1} + \frac{1}{x}, \quad H_0 = 0. \tag{2}$$

They are related to the digamma function through the fundamental relation

$$H_x = \psi(x+1) + \gamma. \tag{3}$$

When $x$ is a positive integer, say $n$, the harmonic numbers form the sequence $(H_n)_{n\in\mathbb{Z}^+}$, and the recurrence relation (2) yields

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad H_0 = 0. \tag{4}$$

The following combinatorial identity is attributed to Frisch [2]:

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{b+k}{c}} = \frac{c}{n+c} \cdot \frac{1}{\binom{n+b}{b-c}}, \quad \text{for } n \in \mathbb{Z}^+ \text{ and } b,\, c,\, b-c \in \mathbb{C}\setminus\mathbb{Z}^-. \tag{5}$$

This identity is listed in Gould's compendium [3] as Identity 4.2 and was recently utilized by Gould and Quaintance [4] to establish a new binomial transform identity. Abel [5] provided a concise proof of Frisch's identity and investigated its infinite variant.

In this note, we demonstrate how Frisch's identity (5) can be applied to prove and generalize several well-known identities involving harmonic numbers $H_n$. Additionally, we present some combinatorial identities involving odd harmonic numbers $O_n$, which can be readily derived from our results. Our approach primarily involves leveraging the fact that derivatives of generalized binomial coefficients yield harmonic numbers. This method is well-established and has been employed in significant earlier research by other mathematicians [6–9].

## 2. Results

**Theorem 1.** *For $n \in \mathbb{Z}^+$ and $b,\, c,\, b-c \in \mathbb{C}\setminus\mathbb{Z}^-$, we have*

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{b+k}{c}} \left(H_{k+b} - H_{k+b-c}\right) = \frac{c}{n+c} \frac{H_{n+b} - H_{b-c}}{\binom{n+b}{n+c}} \tag{6}$$

*and*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{c}{k+c} \frac{H_{k+b} - H_{b-c}}{\binom{k+b}{k+c}} = \frac{H_{n+b} - H_{n+b-c}}{\binom{b+n}{c}}. \tag{7}$$

*In particular, for $n \in \mathbb{Z}^+$ and $b \in \mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\binom{b+k}{k}} \left(H_{k+b} - H_k\right) = \frac{b}{n+b} H_{n+b} \tag{8}$$

*and*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{H_{k+b}}{k+b} = \frac{H_{n+b} - H_n}{b\binom{b+n}{b}}. \tag{9}$$

**Proof.** In Frisch's identity (5) treat $b$ and $c$ as complex numbers and differentiate w.r.t. $b$, using

$$\frac{d}{db} \binom{b+k}{c}^{-1} = \binom{b+k}{c}^{-1} \left(\psi(b+k+1-c) - \psi(b+k+1)\right) \tag{10}$$

and

$$\frac{d}{db} \binom{n+b}{n+c}^{-1} = \binom{n+b}{n+c}^{-1} \left(\psi(b+1-c) - \psi(b+n+1)\right), \tag{11}$$

simplify, making use of the fundamental relation (3). This gives (6). Identity (7) is the binomial transform of (6). □

**Corollary 2.** *For $n \in \mathbb{Z}^+$ and $b \in \mathbb{C}\setminus\mathbb{Z}^-$, $b \neq 0$ we have*

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{(b+k)^2} = \frac{1}{n+1} \frac{H_{n+b} - H_{b-1}}{\binom{n+b}{n+1}}. \tag{12}$$

Identity (12) generalizes the well-known identity

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{(k+1)^2}=\frac{H_{n+1}}{n+1}, \tag{13}$$

which can be proved directly via

$$\begin{aligned}(n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+1)^2}&=\sum_{k=0}^{n}\frac{n+1}{k+1}\binom{n}{k}\frac{(-1)^k}{k+1}\\&=\sum_{k=0}^{n}\binom{n+1}{k+1}\frac{(-1)^k}{k+1}\\&=\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}\\&=H_{n+1}.\end{aligned}$$

**Theorem 3.** *For $n\in\mathbb{Z}^+$ and $b,\ c,\ b-c\in\mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=0}^{n}(-1)^{k+1}\frac{\binom{n}{k}}{\binom{b+k}{c}}\left(H_{k+b-c}-H_c\right)=\frac{1}{(n+c)\binom{n+b}{n+c}}\left(\frac{n}{n+c}+c(H_{n+c}-H_{b-c})\right). \tag{14}$$

*In particular, for $n\in\mathbb{Z}^+$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}H_{k+b}=\frac{1}{n\binom{n+b}{n}}. \tag{15}$$

**Proof.** Differentiate Frisch's identity (5) w.r.t. $c$ using

$$\frac{d}{dc}\binom{b+k}{c}^{-1}=\binom{b+k}{c}^{-1}\left(\psi(c+1)-\psi(b+k+1-c)\right),$$

and the proof is completed. □

Identity (15) generalizes the well-known identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=-\frac{1}{n}, \tag{16}$$

which is the binomial transform of the sequence $H_n$ (see [10, p. 34]).

**Corollary 4.** *For $n\in\mathbb{N}_0$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{\binom{n}{k}}{k\binom{k+b}{k}}=H_{n+b}-H_b. \tag{17}$$

*In particular,*

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{\binom{n}{k}}{k\binom{k+n}{k}}=H_{2n}-H_n. \tag{18}$$

**Corollary 5.** *For $n\in\mathbb{Z}^+$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{\binom{k+b}{k}}H_{k+b}=\frac{b}{n+b}H_b-\frac{n}{(n+b)^2}. \tag{19}$$

**Proof.** Set $c=b$ in (14) to get

$$\sum_{k=0}^{n}(-1)^{k+1}\frac{\binom{n}{k}}{\binom{b+k}{b}}\left(H_k-H_b\right)=\frac{1}{n+b}\left(\frac{n}{n+b}+bH_{n+b}\right).$$

Use the fact that

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{\binom{b+k}{k}}=\frac{b}{n+b} \tag{20}$$

and combine with (8). □

**Corollary 6.** *For $n\in\mathbb{N}_0$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$ we have*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{k}{(k+b)^2}=\frac{H_{n+b}-H_b}{\binom{n+b}{n}}. \tag{21}$$

*In particular,*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{k}{(k+n)^2}=\frac{H_{2n}-H_n}{\binom{2n}{n}}. \tag{22}$$

**Proof.** Using the fact that the right hand-side of (19) is the binomial transform of $H_{n+b}/\binom{n+b}{n}$ we get

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{b}{k+b}H_b-\frac{k}{(k+b)^2}\right)=\frac{H_{n+b}}{\binom{n+b}{n}}.$$

Combine this with

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+b}=\frac{1}{b\binom{n+b}{b}} \tag{23}$$

and the proof is completed. □

**Remark 1.** Comparing (21) with (12) we see that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{b}{(b+k)^2}=\frac{H_{n+b}-H_{b-1}}{\binom{n+b}{n}}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{k}{(b+k)^2}=\frac{-H_{n+b}+H_b}{\binom{n+b}{n}},$$

which upon addition again yield (23).

**Theorem 7.** *If $n\in\mathbb{Z}^+$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{\binom{b+k}{k}}H_k=\frac{b}{n+b}\left(H_{n+b}-H_b\right)+\frac{n}{(n+b)^2}. \tag{24}$$

**Proof.** Set $c=b$ in (14) and use (20). □

**Theorem 8.** *If $n\in\mathbb{N}_0$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$, $b\neq 0$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+b)^3}=\frac{1}{2n+2}\binom{n+b}{n+1}^{-1}\left(\left(H_{n+b}-H_{b-1}\right)^2-H_{b-1}^{(2)}+H_{n+b}^{(2)}\right). \tag{25}$$

*In particular,*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+1)^3}=\frac{1}{2n+2}\left(H_{n+1}^2+H_{n+1}^{(2)}\right). \tag{26}$$

**Proof.** Write (12) as

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{(b+k)^2}=\frac{1}{n+1}\frac{\psi(n+b+1)-\psi(b)}{\binom{n+b}{n+1}}$$

and differentiate with respect to $b$ to obtain

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-1)^k}{(k+b)^3}\binom{n}{k}\\ &= \frac{1}{2n+2}\binom{n+b}{n+1}^{-1}\left(\psi(n+b+1)-\psi(b)\right)^2\\ &\quad -\frac{1}{2n+2}\binom{n+b}{n+1}^{-1}\left(\psi_1(n+b+1)-\psi_1(b)\right),\end{aligned}$$

where $\psi_1(x)$ is the trigamma function defined by

$$\psi_1(x)=\frac{d}{dx}\psi(x)=\sum_{k=0}^{\infty}\frac{1}{(k+x)^2}$$

and $H_r^{(2)}$ is the $r^{th}$ second order harmonic number,

$$H_r^{(2)}=\sum_{j=1}^{r}\frac{1}{j^2}.$$

The result follows upon using the fact that

$$\psi\left(x+1\right)-\psi\left(y+1\right)=H_x-H_y \tag{27}$$

and

$$\psi_1\left(x+1\right)-\psi_1\left(y+1\right)=H_y^{(2)}-H_x^{(2)}, \tag{28}$$

for $x,y\in\mathbb{C}\setminus\mathbb{Z}^-$. □

**Theorem 9.** *If $n\in\mathbb{N}_0$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}H_{k+b}^{(2)}=\frac{H_{n+b}-H_b}{n\binom{n+b}{n}},\quad n\neq 0, \tag{29}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_{k+b}-H_b}{k\binom{k+b}{k}}=H_{n+b}^{(2)}-H_b^{(2)}, \tag{30}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_{k+b}}{k\binom{k+b}{k}}=\left(H_{n+b}-H_b\right)H_b+H_{n+b}^{(2)}-H_b^{(2)}. \tag{31}$$

**Proof.** Differentiate (15) with respect to $b$ to get (29). Identity (30) is the inverse of (29). Identity (31) is obtained by using (17) in (30). □

**Theorem 10.** *If $n\in\mathbb{Z}^+$ and $b\in\mathbb{C}\setminus\mathbb{Z}^-$, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{\binom{b+k}{b}}H_kH_{k+b} &= \frac{1}{n+b}\left(bH_b-\frac{n}{n+b}\right)\left(H_{n+b}-H_b\right)\\ &\quad+\frac{b}{n+b}\left(H_{n+b}^{(2)}-H_b^{(2)}\right)+\frac{n}{(n+b)^2}\left(\frac{2}{n+b}+H_b\right).\end{aligned} \tag{32}$$

**Proof.** Write (24) as

$$\sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{\binom{b+k}{k}}H_k=\frac{b}{n+b}\left(\psi(n+b+1)-\psi(b+1)\right)+\frac{n}{(n+b)^2}$$

and differentiate with respect to $b$, using (10). Use (24) again in simplifying the left hand side of the resulting expression. □

In particular, for all positive integers $n$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k^2=\frac{H_n}{n}-\frac{2}{n^2} \tag{33}$$

and

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}}{\binom{n+k}{k}}H_kH_{k+n}\\ &=\frac{1}{2}\left(H_{2n}^{(2)}-H_n^{(2)}\right)+\frac{1}{2}H_{2n}\left(H_n-\frac{1}{2n}\right)-\frac{1}{2}H_nH_{n-1}+\frac{1}{4n^2}.\end{aligned} \tag{34}$$

**Corollary 11.** *If $n$ is a positive integer, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_k}{k}=H_n^{(2)}. \tag{35}$$

**Proof.** Treat identity (33) as the binomial transform of the sequence $H_n^2$. The inverse relation yields

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{H_k}{k}+2\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2}=H_n^2.$$

But (see Boyadzhiev's book [10, p. 64])

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2}=\frac{1}{2}\left(H_n^2+H_n^{(2)}\right)$$

and the proof is completed. □

**Remark 2.** Identity (35) can also be obtained by taking the limit of (9) as $b$ approaches zero as well as simply setting $b=0$ in (30) or (31).

## 3. Some Identities Involving Odd Harmonic Numbers

This section contains some combinatorial identities involving odd harmonic numbers $O_n$, which are defined by

$$O_n=\sum_{k=1}^{n}\frac{1}{2k-1},\quad O_0=0.$$

Obvious relations between harmonic numbers $H_n$ and odd harmonic numbers $O_n$ are given by

$$H_{2n}=\frac{1}{2}H_n+O_n\qquad\text{and}\qquad H_{2n-1}=\frac{1}{2}H_{n-1}+O_n. \tag{36}$$

Additional relations are contained in the next lemma.

**Lemma 1.** *If $n$ is an integer, then*

$$H_{n-1/2}=2O_n-2\ln 2, \tag{37}$$
$$H_{n-1/2}-H_{-1/2}=2O_n, \tag{38}$$
$$H_{n-1/2}-H_{1/2}=2\left(O_n-1\right), \tag{39}$$
$$H_{n+1/2}-H_{-1/2}=2O_{n+1}, \tag{40}$$
$$H_{n+1/2}-H_{1/2}=2\left(O_{n+1}-1\right), \tag{41}$$
$$H_{n+1/2}-H_{n-1/2}=\frac{2}{2n+1}, \tag{42}$$
$$H_{n-1/2}-H_{-3/2}=2\left(O_n-1\right), \tag{43}$$
$$H_{n+1/2}-H_{-3/2}=2\left(O_{n+1}-1\right). \tag{44}$$

**Proof.** Use (3) as the definition of the harmonic numbers for all complex $n$ (excluding zero and the negative integers) and use the known result for the digamma function at half-integer arguments [1, Eq. (51)], namely,

$$\psi(n+1/2) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}.$$

□

**Lemma 2** (Gould [3, Identities Z.45 and Z.51])**.** *If $r$ and $s$ are integers such that $0 \le s \le r$, then*

$$\binom{r+1/2}{s} = \binom{2r+1}{2s}\binom{r}{s}^{-1} 2^{-2s}\binom{2s}{s} \tag{45}$$

*and*

$$\binom{r-1/2}{s} = \binom{r}{s}\binom{2r-2s}{r-s}^{-1} 2^{-2s}\binom{2r}{r}. \tag{46}$$

*We also have*

$$\binom{r}{1/2} = \frac{2^{2r+1}}{\pi\binom{2r}{r}}, \quad [3, \text{Identity Z.48}], \tag{47}$$

$$\binom{r}{-1/2} = \frac{2^{2r+1}}{\pi(2r+1)\binom{2r}{r}}, \tag{48}$$

*and*

$$\binom{r-1/2}{r+1} = -\frac{1}{r+1}\binom{2r}{r}\frac{1}{2^{2r+1}}. \tag{49}$$

**Proof.** Identity (48) follows from the fact that

$$\binom{r}{-1/2} = \frac{r!}{(-1/2)!(r+1/2)!},$$

while (49) is a consequence of

$$\binom{r-1/2}{r+1} = -\frac{1}{2}\frac{(r-1/2)!}{(r+1)!\sqrt{\pi}}. \tag{50}$$

□

**Theorem 12.** *If $n$ is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(2k+1)^2} = \frac{2^{2n+1}}{n+1}\frac{O_{n+1}}{\binom{2(n+1)}{n+1}}, \tag{51}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{2^{2k+1}}{k+1}\frac{O_{k+1}}{\binom{2(k+1)}{k+1}} = \frac{1}{(2n+1)^2}, \tag{52}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{(2k-1)^2} = 2^{2n}\frac{O_n - 1}{\binom{2n}{n}}, \tag{53}$$

*and*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}2^{2k}\frac{O_k}{\binom{2k}{k}} = \frac{2n}{(2n-1)^2}. \tag{54}$$

**Proof.** The first identity is obtained by setting $b = 1/2$ in (12) and using (40) and (45); while the second is its inverse transform. Identity (53) is obtained by setting $b = -1/2$ in (12) and using (43) and (49). To prove identity (54) use the inverse binomial transform of (53) to get

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}2^{2k}\frac{O_k}{\binom{2k}{k}} - \sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}\frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{(2n-1)^2}.$$

But
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}\frac{2^{2k}}{\binom{2k}{k}}=\frac{1}{2n-1} \tag{55}$$
and the proof of (54) is completed. □

**Theorem 13.** *If n is a non-negative integer, then*
$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{2k-1}{2^{2(k-1)}}\binom{2(k-1)}{k-1}O_k=\frac{1}{2^{2(n-1)}}\binom{2(n-1)}{n-1}\left(\frac{2n}{2n-1}-O_n\right), \tag{56}$$
$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{2k+1}{2^{2k}}\binom{2k}{k}O_{k+1}=\frac{1}{2n-1}\frac{\binom{2n}{n}}{2^{2n}}\left(\frac{4n-1}{2n-1}-O_n\right), \tag{57}$$
$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}2^{-2k-2}\binom{2(k+1)}{k+1}(O_{k+1}-1)=\frac{2^{-2n-2}}{2n+1}\binom{2(n+1)}{n+1}\left(O_{n+1}-\frac{1}{2n+1}\right), \tag{58}$$
$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}2^{-2k}\binom{2k}{k}(O_k-1)=2^{-2n}\binom{2n}{n}\left(\frac{2n}{2n+1}+O_{n+1}\right). \tag{59}$$

**Proof.** These results follow from (14). To obtain (56), set $b=-1$, $c=-1/2$ in (14) and use (38) and (48). Identity (57) comes from setting $b=0$, $c=-1/2$ in (14) and using (40), (47) and (48). □

**Theorem 14.** *If n is a positive integer, then*
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}O_k=\frac{2^{2n-1}}{n\binom{2n}{n}}, \tag{60}$$
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{2^{2k-1}}{k\binom{2k}{k}}=O_n, \tag{61}$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}O_{k+1}=\frac{2^{2n-1}}{n(2n+1)\binom{2n}{n}}, \tag{62}$$
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{2^{2k-1}}{k(2k+1)\binom{2k}{k}}=O_{n+1}-1. \tag{63}$$

**Proof.** Write (15) as
$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}(H_{k+b}-H_c)=\frac{1}{n\binom{n+b}{n}}.$$
Evaluate at $(b,c)=(-1/2,-1/2)$ and at $(b,c)=(1/2,-1/2)$ to obtain (60) and (62). Identity (61) is the binomial transform of (60) while (63) is the transform of (62). □

**Theorem 15.** *If n is a non-negative integer, then*
$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}k}{(2k+1)^2}=\frac{2^{2n-1}(O_{n+1}-1)}{(2n+1)\binom{2n}{n}}, \tag{64}$$
$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}k}{(2k-1)^2}=\frac{2^{2n-1}O_n}{\binom{2n}{n}}, \tag{65}$$
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{2^{2k-1}(O_{k+1}-1)}{(2k+1)\binom{2k}{k}}=\frac{n}{(2n+1)^2}, \tag{66}$$
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{2^{2k-1}O_k}{\binom{2k}{k}}=\frac{n}{(2n-1)^2}. \tag{67}$$

**Proof.** The first two results follow from (21). The other two identities are the inverse binomial transforms of the former. □

**Remark 3.** Identity (67) is a rediscovery of identity (54). Also, combining the proof of (54) with (67) yields (55).

**Theorem 16.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}\frac{2^{2k}H_k}{(2k+1)\binom{2k}{k}}=\frac{2O_{n+1}}{2n+1}-\frac{2}{(2n+1)^2}, \tag{68}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{2^{2k}H_k}{\binom{2k}{k}}=\frac{2O_{n}}{2n-1}-\frac{4n}{(2n-1)^2}, \tag{69}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{O_{k+1}}{2k+1}=\frac{2^{2n+1}}{n+1}\frac{O_{n+1}}{\binom{2n+2}{n+1}}-\frac{2^{2n-1}}{2n+1}\frac{H_n}{\binom{2n}{n}}, \tag{70}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{O_{k}}{2k-1}=\frac{2^{2n-1}}{\binom{2n}{n}}\left(H_n-2O_n\right), \tag{71}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{2^{2k}}{\binom{2k}{k}}H_{2k}=\frac{4n}{(2n-1)^2}-\frac{O_n}{2n-1}. \tag{72}$$

**Proof.** The first two results follow from (24). Identity (70) is a consequence of the inverse binomial relation of (68) in conjunction with identity (51). Identity (71) is a consequence of the inverse binomial relation of (69) in conjunction with identity (65). Identity (72) is a consequence of (67) and (69) on account of the first identity in (36). □

**Theorem 17.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{(2k-1)^3}=\frac{2^{2n-1}}{\binom{2n}{n}}\left((O_n-1)^2+O_n^{(2)}+1\right). \tag{73}$$

**Proof.** Set $b=-1/2$ in (25) and use Lemma 1 and identity 46. Use also

$$H_{n-1/2}^{(2)}=-2\zeta(2)+4O_n^{(2)} \tag{74}$$

which follows from the known value $H_{-1/2}^{(2)}=-2\zeta(2)$ since

$$H_{n-1/2}^{(2)}-H_{-1/2}^{(2)}=4O_n^{(2)},$$

where $O_n^{(2)}$ is the $n^{th}$ second order odd harmonic number defined by

$$O_n^{(2)}=\sum_{k=1}^{n}\frac{1}{(2k-1)^2}.$$

□

**Theorem 18.** *If n is a positive integer, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}O_k^{(2)}=\frac{2^{2n-1}}{\binom{2n}{n}}\frac{O_n}{n}, \tag{75}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{2^{2k-1}}{\binom{2k}{k}}\frac{O_k}{k}=O_n^{(2)}. \tag{76}$$

**Proof.** Set $b=-1/2$ in (29) and use Lemma 1 and also (46) and (74) to obtain (75). Identity (76) is the inverse of (75). □

**Theorem 19.** *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}\frac{2^{2k}}{\binom{2k}{k}}H_kO_k=\frac{8n}{(2n-1)^3}-\frac{4n\,O_n}{(2n-1)^2}-\frac{2O_n^{(2)}}{(2n-1)}, \tag{77}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{2^{2k}}{\binom{2k}{k}}H_k=\frac{2O_n}{(2n-1)}-\frac{4n}{(2n-1)^2}. \tag{78}$$

**Proof.** Set $b=-1/2$ in (32). Use Lemma 1, (46) and (74). Equate rational coefficients from both sides. Identity (78) is a rediscovery of (69). □

## 4. Final Comments

In these notes, we have demonstrated how a combinatorial identity attributed to Frisch can be used to prove a series of harmonic number and odd harmonic number identities. Some of the results that we derived are known and here we provide final comments and give further references.
We begin by pointing out that identity (33) can also be found in a recent article by Batir [11, Identity 18]. Next, identity (23) can be restated as

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{b}{k+b}=\prod_{k=1}^{n}\frac{k}{b+k}.$$

This identity is known and two probabilistic proofs were given recently by Peterson [12] and Nakata [13]. Peterson [12] writes (12) in the form

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{b}{b+k}\right)^2=\prod_{k=1}^{n}\frac{k}{b+k}\left(1+\sum_{k=1}^{n}\frac{b}{b+k}\right),$$

and also states an expression for the generalization involving an additional parameter $m$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{b}{b+k}\right)^m,\qquad m\geq 1. \tag{79}$$

In 2019, Bai and Luo [14] derived a new expression for Peterson's identity (79) using a partial fraction decomposition and involving generalized harmonic numbers. As applications of their main result, they stated some harmonic number identities. One of their special cases is our identity (26).

It is remarkable that a simple expression for (79) is also hidden in Frisch's identity (5). Indeed, setting $c=1$ in (5) yields (23), i.e.,

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{b+k}=\frac{1}{b\binom{n+b}{b}}.$$

Differentiating both sides $m$ times w.r.t. $b$ gives

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(b+k)^m}=\frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{db^{m-1}}\frac{1}{b\binom{n+b}{b}}, \tag{80}$$

which shows such a simple expression.

**Conflicts of Interest:** The authors declare no conflict of interest.

**Acknowledgments:** The authors would like to thank the referee for his/her valuable comments that resulted in the present improved version of the article.

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