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A bisection-type method based on α -dense curves

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Abstract: Let (X, d) be a metric space, $D \subset (X, d)$ and $f : D \rightarrow \mathbb{R}$ a continuous function (with respect to the metric d and the Euclidean metric on $\mathbb{R}$). Under suitable conditions on the function f and the set D , we prove the existence of zeros of the function f on D by using the so-called α -dense curves. To be more precise, we prove that if $f(x_1)f(x_2) < 0$ for some $x_1, x_2 \in D$ and D is densifiable then f has some zero in D . Moreover, from this result we derive a numerical method to approximate, with arbitrarily small error, at least one of these zeros. In particular, as a compact interval $[a, b]$ is densifiable, our method generalizes the well known bisection method. The feasibility and reliability of the proposed method is illustrated by several numerical examples.

Keywords: zeros of continuous functions, approximation of zeros, numerical methods, bisection method, α -dense curves

MSC: 35B05, 65Y15, 26A30, 94A11

1. Introduction

The well known bisection method (see, for instance, [1,2]) is a robust numerical method used to approximate the zeros of a real-valued function. It is based on the Intermediate Value Theorem, which states that if a continuous function $f : [a, b] \rightarrow \mathbb{R}$ (we assume $\mathbb{R}$ is endowed with the Euclidean metric) has different signs at the endpoints of an interval $[a, b]$, then there exists at least one zero of f within that interval. That is to say, there exists $x^* \in [a, b]$ such that $f(x^*) = 0$. The bisection method iteratively refines the interval in which a zero exists until a desired level of accuracy is reached. Some generalizations and improvements of this method can be found in [3–7] and references therein. For completeness, we show in Algorithm 1 the pseudo-code of the bisection method.

Let us note that if we remove the condition $b - a \geq \delta$ in the *while* loop of Algorithm 1, theoretically we obtain $c \in [a, b]$ such that $f(c) \leq \varepsilon$. As proven in [1, p. 262], under this condition, the convergence of the bisection method is at least linear. Also, the number of iterations in the bisection method to attain the desired approximation is

$$n := \left\lceil \frac{\ln \left(\frac{b-a}{2\varepsilon} \right)}{\ln(2)} \right\rceil,$$

where $\lceil x \rceil$ denotes the *ceiling* of x , i.e., the smallest integer greater than or equal to x . However, for computational reasons that condition can not be removed, because the computer works with a finite number of decimal places. So, if we remove the condition $b - a \geq \delta$, the *loop* of Algorithm 1 may never end.

On the other hand, in the present paper we consider a continuous function $f : D \subset (X, d) \rightarrow \mathbb{R}$, where (X, d) is a metric space ($\mathbb{R}$ is assumed to be endowed with the Euclidean metric) and $D \neq \emptyset$, and we analyze, under suitable conditions, the problem of finding some zero of f in D . For this, we will use so-called α -dense curves, explained in detail in §2. To be more precise, if there are $x_1, x_2 \in D$ such that $f(x_1)f(x_2) < 0$ and D is densifiable (i.e., for any arbitrarily small $\alpha > 0$, there exists an α -dense curve in D), then we prove in Theorem 1 that f has some zero in D . Moreover, from the proof of Theorem 1 we propose a numerical method

to approximate some zeros of f in the densifiable set D . Noticing that a compact interval $[a, b]$ of the real line is, in particular, a densifiable set (see Proposition 1), our result is a generalization of the bisection method.

We provide in §4 some numerical examples to illustrate the proposed method and, to conclude our exposition, in §5 we discuss briefly its advantages and drawbacks.

Algorithm 1 Bisection method

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1: INPUT: A continuous function  $f : [a, b] \rightarrow \mathbb{R}$  with  $f(a)f(b) < 0$  and the stop criteria numbers  $\varepsilon, \delta > 0$ 
2:
3: THE LOOP.
4:
5: while  $b - a \geq \delta$  or  $(f(a) \geq \varepsilon$  and  $f(b) \geq \varepsilon)$  do
6:
7:    $c \leftarrow \frac{a+b}{2}$ .
8:
9:   if  $f(c) \leq \varepsilon$  then
10:
11:     break (exit the while loop).
12:
13:   end if
14:
15:   if  $f(a)f(c) < 0$  then
16:
17:      $b \leftarrow c$ .
18:
19:   else
20:
21:      $a \leftarrow c$ .
22:
23:   end if
24:
25: end while
26:
27: OUTPUT: Approximation of a zero of  $h(t)$  in  $[a, b]$  and the value where it is attained.
28:
29: Return  $\{c, f(c)\}$ .

```

2. α -dense curves and densifiable sets

We need to recall the concepts of α -dense curves and densifiable sets, introduced in [8]. In what follows, (X, d) will be a metric space and $\mathcal{B}(X)$ the class of the non-empty and bounded subsets of X . Also, in what follows, all subsets of $\mathbb{R}^d$, for a given integer $d \geq 1$, are assumed endowed with the Euclidean metric if otherwise is not specified.

Definition 1. Let $\alpha > 0$ and $D \in \mathcal{B}(X)$. A continuous function $\gamma : [0, 1] \rightarrow (X, d)$ is said to be an α -dense curve in D if the following conditions are satisfied:

- (i) $\gamma([0, 1]) \subset D$.
- (ii) For every $x \in D$ there exists $y \in \gamma([0, 1])$ such that $d(x, y) \leq \alpha$.

If for each $\alpha > 0$ there is an α -dense curve in D , then D is said to be densifiable.

Let us note that, for a given $D \in \mathcal{B}(X)$, fixed $x_0 \in D$, the mapping defined as $\gamma(t) := x_0$ for all $t \in [0, 1]$ is, trivially, an α -dense curve in D , for any $\alpha \geq \text{Diam}(D)$, the diameter of D . Therefore, the class of α -dense curves in D is non-empty for each $\alpha \geq \text{Diam}(D)$. Also, taking into account that fixed an integer $n \geq 1$ a continuous function from $[0, 1]$ onto $[0, 1]^n$, is called a space-filling curve (see [9]), it is clear that the α -dense curves are a generalization of the space-filling curves. Moreover, Mora [10] proved that the space-filling curves can be characterized in terms of the α -dense curves. A detailed exposition of the α -dense curves as well as their applications can be found in [8,10–19] and references therein.

In the present paper, and according with Schaeffer's book [20], we mean that a subset B of (X, d) is *precompact* (some authors call it *totally bounded*) if for each $\varepsilon > 0$ there is a finite set $\{x_1, \dots, x_n\} \subset B$ such that

$$B \subset \bigcup_{i=1}^n \bar{B}(x_i, \varepsilon),$$

where $\bar{B}(x, r)$ stands for the closed ball centered at x and radius $r > 0$. We have the following result to characterize the precompact sub-sets of a metric space (see [13,21]).

Proposition 1. Let $D \in \mathcal{B}(X)$ be arc-wise connected. Then, D is densifiable if and only if it is precompact.

We show in the below example that, in general, the arc-wise connectedness in the above result can not be removed.

Example 1. In the Euclidean plane consider the following compacts, connected but not arc-wise connected sets

$$D_1 := \left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) : x \in (0, 1] \right\} \cup \{(0, y) : y \in [-1, 1]\},$$

$$D_2 := \left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) : x \in [-1, 0) \cup (0, 1] \right\} \cup \{(0, y) : y \in [-1, 1]\}.$$

Then, it is easy to check that D_1 is densifiable but there is not an α -dense curve in D_2 for any $0 < \alpha < 1$. Consequently, D_2 is not densifiable.

We provide some examples of α -dense curves, in the specified sets, that will be used in §4.

Example 2. Let two integers $n, m > 1$, $B := \prod_{i=1}^n [a_i, b_i]$ for certain $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for $i = 1, \dots, n$, and $\alpha := M \frac{\sqrt{n-1}}{m}$ where $M := \max\{b_i - a_i : i = 1, \dots, n\}$. Define the function $\gamma_\alpha : [0, 1] \rightarrow \mathbb{R}^n$ as

$$\gamma_\alpha(t) := \left(a_1 + (b_1 - a_1)t, a_2 + \frac{b_2 - a_2}{2} (1 - \cos(m\pi t)), \dots, a_n + \frac{b_n - a_n}{2} (1 - \cos(m^{n-1}\pi t)) \right),$$

for each $t \in [0, 1]$. Then, γ_α is an α -dense curve in B (see [11, Corollary 9.5.5]). For $B := [0, 1]^n$, we show in Figure 1 the graphs of some of these curves, for the indicated values of m and n .

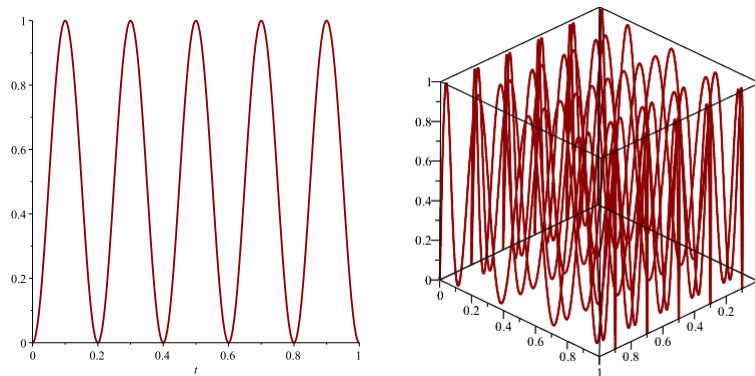


Figure 1. Graphs of γ_α for $m = 10$ and $n = 2, 3$

As we will see in the below examples, the above α -dense curve are useful to define other α -dense curves in other metric spaces.

Example 3. Consider $\mathbb{R}^{\mathbb{N}}$, the product of $\mathbb{N}$ -copies of $\mathbb{R}$, endowed with the metric

$$d_{\mathbb{N}}(\mathbf{x}, \mathbf{y}) := \sum_{n \geq 1} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \text{for all } \mathbf{x} := (x_n)_{n \geq 1}, \mathbf{y} := (y_n)_{n \geq 1} \in \mathbb{R}^{\mathbb{N}}.$$

Fixed an integer $n_0 > 1$, let $B := \prod_{n \geq 1} [a_n, b_n]$, with $a_n, b_n \in \mathbb{R}$ with $a_n < b_n$ for each $n \geq 1$, and $B_0 := \prod_{n=1}^{n_0} [a_n, b_n] \subset \mathbb{R}^{n_0}$. Let $\gamma_{\alpha_0} = (\gamma_{\alpha_0}^1, \dots, \gamma_{\alpha_0}^{n_0})$ be an α_0 -dense curve in B_0 , for some $\alpha_0 > 0$. Let $\alpha := \alpha_0 + \sum_{n \geq n_0+1} 2^{-n}$ and define $\gamma_\alpha := (\gamma_\alpha^n)_{n \geq 1} : [0, 1] \rightarrow (\mathbb{R}^{\mathbb{N}}, d_{\mathbb{N}})$ as

$$\gamma_\alpha^n(t) := \begin{cases} \gamma_{\alpha_0}^n(t), & \text{for } n \leq n_0 \\ \xi_n(t), & \text{for } n \geq n_0 + 1 \end{cases} \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, 1],$$

where, for each $n \geq n_0 + 1$, $\xi_n : [0, 1] \rightarrow [a_n, b_n]$ is an arbitrary continuous function. Then, γ_α is an α -dense curve in B (see [13,18]).

Example 4. We recall that for each integer $n \geq 0$, the n -th Chebyshev polynomial is defined as $T_n(x) := \cos(n \arccos(x))$, for each $x \in [-1, 1]$. Let us denote by $\mathcal{C}[-1, 1]$ the Banach space of the continuous functions defined on $[-1, 1]$, endowed its usual supremum norm. It is a known fact (see, for instance, [22, Chapter 4]) that given $h \in \mathcal{C}[-1, 1]$ it can be written as a series of Chebyshev polynomials. That is to say,

$$h(x) = \sum_{n \geq 0} a_n T_n(x) \quad \text{with } a_n := \int_{-1}^1 \frac{h(x) T_n(x)}{\sqrt{1-x^2}} dx, \quad \text{for all } n \geq 0.$$

Fixed a sequence $(r_n)_{n \geq 0}$ of positive numbers such that $\sum_{n \geq 0} r_n < \infty$, consider the set

$$D := \left\{ \sum_{n \geq 0} a_n T_n(x) : |a_n| \leq r_n \text{ for each } n \geq 0 \right\} \subset \mathcal{C}[-1, 1].$$

Let us note that as $\sum_{n \geq 0} r_n < \infty$, each series in D is uniformly convergent and therefore, $D \subset \mathcal{C}[-1, 1]$ as indicated above. Clearly, D is convex. Also, given any sequence $(h_k(x))_{k \geq 1} \subset K$, put $h_k(x) := \sum_{n \geq 0} a_n^{(k)} T_n(x)$, with $|a_n^{(k)}| \leq r_n$ for each $n \geq 0$ and $k \geq 1$, $(h_k(x))_{k \geq 1}$ has a converging sub-sequence. Indeed, let c_0 be the Banach space of the null sequences, endowed its usual supremum norm and consider the set

$$A := \{(a_n^{(k)})_{k \geq 1} : n \geq 1\} \subset c_0.$$

Then, by [23, Theorem II.4.1], A is precompact. Consequently, A has convergent sub-sequence, put $(a_n^{(k_j)})_{j \geq 1}$, and thus $(h_{k_j}(x))_{j \geq 1}$ converges (uniformly) to a function in $\bar{K}$, the closure of K . In other words, D is precompact. So, by Proposition 1, the set D is densifiable. In fact, we show in the following lines an example of α -dense curve in D for an arbitrary $\alpha > 0$.

Fixed an integer $n_0 \geq 1$, let $\gamma_{\alpha_0} := (\gamma_{\alpha_0}^0, \dots, \gamma_{\alpha_0}^{n_0})$ be an $\frac{\alpha_0}{n_0+1}$ -dense curve in $\prod_{n=0}^{n_0} [-r_n, r_n] \subset \mathbb{R}^{n_0+1}$ for some $\alpha_0 > 0$. Define $\alpha := \alpha_0 + 2 \sum_{n \geq n_0+1} r_n$ and $\gamma_\alpha := (\gamma_\alpha^n)_{n \geq 1} : [0, 1] \rightarrow (\mathbb{R}^{\mathbb{N}}, d_{\mathbb{N}})$, $d_{\mathbb{N}}$ being the metric given in Example 3, as

$$\gamma_\alpha^n(t) := \sum_{n=0}^{n_0} \gamma_\alpha^n(t) T_n(x) + \sum_{n \geq n_0+1} \xi_n(t) T_n(x) \quad \text{for all } t \in [0, 1],$$

where, for each $n \geq n_0 + 1$, $\xi_n : [0, 1] \rightarrow [-r_n, r_n]$ is an arbitrary continuous function. It is easy to check that γ_α is an α -dense curve in D . Also, it is clear that $\alpha \rightarrow 0$ as $n_0 \rightarrow \infty$, that is, α can be arbitrarily small.

3. The main result

As in the previous section (X, d) is a metric space. Assume $f : D \subseteq X \rightarrow \mathbb{R}$ is a given function. Denote by $\mathcal{Z}_D(f)$ the zeros of f in the set D , that is to say

$$\mathcal{Z}_D(f) := \{x \in D : f(x) = 0\}.$$

Assume $\gamma_\alpha : [0, 1] \rightarrow (X, d)$ is an α -dense curve in D , for some $\alpha > 0$. Assume $\mathcal{Z}_D(f) \neq \emptyset$. If $x^* \in \mathcal{Z}_D(f)$, there is $t \in [0, 1]$ such that $d(x^*, \gamma_\alpha(t)) \leq \alpha$. In particular, if D is densifiable, we can take α arbitrarily small. This means that, for a given $x^* \in \mathcal{Z}_D(f)$ and a prescribed $\alpha > 0$, there is $y^* \in \gamma_\alpha([0, 1])$ that is at a distance from x^* less than or equal to α . This observation can be formally proved in the following result.

Proposition 2. Let $f : D \subseteq X \rightarrow \mathbb{R}$ be a function such that $\mathcal{Z}_D(f) \neq \emptyset$. Assume that D is densifiable, and let $(\alpha_n)_{n \geq 1}$ be a sequence of positive numbers such that $\lim_n \alpha_n = 0$. Also, for each $n \geq 1$, let $\gamma_{\alpha_n} : [0, 1] \rightarrow (X, d)$ be an α_n -dense curve in D . Then, if $x^* \in \mathcal{Z}_D(f)$ there exists a sequence $(t_n)_{n \geq 1} \subset [0, 1]$ such that

$$\lim_n d(\gamma_n(t_n), x^*) = 0.$$

Proof. Let $x^* \in \mathcal{Z}_D(f)$. As, for each $n \geq 1$, γ_{α_n} is an α_n -dense curve in D there is $t_n \in [0, 1]$ such that $d(x^*, \gamma_{\alpha_n}(t_n)) \leq \alpha_n$. As $\lim_n \alpha_n = 0$, the result follows. $\square$

In the above result, we assume $\mathcal{Z}_D(f) \neq \emptyset$ and D is densifiable, and we prove that a given $x^* \in \mathcal{Z}_D(f)$ can be approximated by using α -dense curves. In the following result, we state sufficient conditions on f and D to ensure that $\mathcal{Z}_D(f) \neq \emptyset$. Moreover, under such conditions, we provide below (see Algorithm 2) a numerical method to, by using α -dense curves, approximate some $x^* \in \mathcal{Z}_D(f)$.

Now, we can prove our main result, which reads as follows.

Theorem 1. Let $f : D \subseteq (X, d) \rightarrow \mathbb{R}$ be a continuous function. Assume the following conditions:

(1) There are $x_1, x_2 \in D$ such that $f(x_1)f(x_2) < 0$.

(2) The set D is densifiable.

Then, $\mathcal{Z}_D(f) \neq \emptyset$. Moreover, there is a small enough $\alpha > 0$, an α -dense curve in D , put γ_α , such that $\gamma_\alpha(t^*) \in \mathcal{Z}_D(f)$ for some $t^* \in [0, 1]$.

Proof. According to (1), let $x_1, x_2 \in D$ such that $f(x_1)f(x_2) < 0$, without loss of generality assume $f(x_1) > 0$ and $f(x_2) < 0$. By the continuity of f , there are $r_1, r_2 > 0$ such that

$$f(x) > 0 \text{ for all } x \in B(x_1, r_1) \cap D, \quad f(x) < 0 \text{ for all } x \in B(x_2, r_2) \cap D \quad (1)$$

where $B(x, r)$ stands for the open ball centered at x and radius $r > 0$.

Let $0 < \alpha < \min\{r_1, r_2\}$ and γ_α an α -dense curve in D . The existence of such curve is guaranteed by condition (2). Let $t_1, t_2 \in (0, 1)$ such that

$$d(x_1, \gamma_\alpha(t_1)) \leq \alpha \quad \text{and} \quad d(x_2, \gamma_\alpha(t_2)) \leq \alpha. \quad (2)$$

Then, noticing (1) and (2), we have

$$f(\gamma_\alpha(t_1))f(\gamma_\alpha(t_2)) < 0. \quad (3)$$

Therefore, by the Intermediate Value Theorem there is $t^* \in (t_1, t_2)$ such that $f(\gamma_\alpha(t^*)) = 0$ and the proof is completed. $\square$

Let us note that in the above result, the densifiability of the set D , in general, can not be removed, as we show in the following simple example.

Example 5. Let $D := \{(-1, 1), (-1, -1), (1, -1), (1, 1)\} \subset \mathbb{R}^2$, and $f : D \rightarrow \mathbb{R}$ the function $f((x_i, x_j)) := x_i x_j$, for each $(x_i, x_j) \in D$. It is clear that f is continuous and $f((-1, -1))f((1, -1)) = -1 < 0$ but $\mathcal{Z}_D(f) = \emptyset$.

The set D is not densifiable. Indeed, if γ is an α -dense curve in D , since $\gamma([0, 1])$ is arc-wise connected, then $\gamma(t)$ must be equal to (x_i, x_j) for some $(x_i, x_j) \in D$ and all $t \in [0, 1]$. Therefore, $\alpha = 2\sqrt{2}$ and consequently D can not be densifiable.

On the other hand, it is important to stress that Theorem 1, implicitly, provides us a method to approximate some zero of the function f . Indeed, let $f : D \subseteq (X, d) \rightarrow \mathbb{R}$ be as in Theorem 1. Then, there are $\alpha > 0$ and $t^* \in (t_1, t_2)$, for certain $t_1, t_2 \in (0, 1)$, such that $f(\gamma_\alpha(t^*)) = 0$, γ_α being an α -dense curve in D . Also, noticing (3) and reasoning as in the bisection method, there exists $\eta > 0$ such that

$$f(\gamma_\alpha(t_1^*))f(\gamma_\alpha(t_2^*)) < 0, \quad \text{whenever } t_1^* \in (t^* - \eta, t^*), t_2^* \in (t^*, t^* + \eta). \quad (4)$$

So, if we find $t_1^*, t_2^* \in (0, 1)$ obeying the above inequality, by applying the bisection method to the function $f \circ \gamma_\alpha : [t_1^*, t_2^*] \rightarrow \mathbb{R}$ we can obtain an approximation of $\gamma_\alpha(t^*) \in \mathcal{Z}_D(f)$. A way to find some $\alpha > 0$ and $t_1^*, t_2^* \in (0, 1)$ satisfying (4) is explained in the below lines.

Let $(N_k)_{k \geq 1}$ a sequence of positive integers such that $\lim_n N_k = \infty$ and $\gamma_{\alpha_{N_k}}$ an α_{N_k} -dense curve in D . Assume $\lim_k \alpha_{N_k} = 0$. Construct a set of points

$$S_{N_k} := \left\{ t_{1,n}^{N_k}, t_{2,n}^{N_k} : n = 1, \dots, N_k \right\} \subset [0, 1],$$

such that S_{N_k} becomes into a dense sequence in $[0, 1]$ when $k \rightarrow \infty$. For instance, for each N_k we can take

$$t_{1,n}^{N_k} \sim \mathcal{U}\left[\frac{n-1}{N_k}, \frac{n}{N_k}\right] \text{ and } t_{2,n}^{N_k} \sim \mathcal{U}\left[\frac{n-1}{N_k}, \frac{n}{N_k}\right], \quad \text{for each } n = 1, \dots, N_k,$$

where $t \sim \mathcal{U}[a, b]$ means that t is a sample of a random variable uniformly distributed in $[a, b]$. Note that, as the set of points $\{\frac{n-1}{N_k}, \frac{n}{N_k} : n = 1, \dots, N_k\}$ becomes into a dense sequence in $[0, 1]$ when $k \rightarrow \infty$, the set S_{N_k} too. Therefore, for large enough k , there are $t_{1,n}^{N_k}, t_{2,n}^{N_k} \in S_{N_k}$ for some $1 \leq n \leq N_k$ such that

$$t_1^* := \min \{t_{1,n}^{N_k}, t_{2,n}^{N_k}\} \in (t^* - \eta, t^*), \quad t_2^* := \max \{t_{1,n}^{N_k}, t_{2,n}^{N_k}\} \in (t^*, t^* + \eta),$$

and $\alpha_{N_k} \leq \alpha$, where $\alpha > 0$ obeys (4). So, (4) holds as claimed.

We present in Algorithm 2 the pseudo-code of the above described method to find an approximation of some $x^* \in \mathcal{Z}_D(f)$.

Algorithm 2 Bisection method based on α -dense curves

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1: INPUT: A continuous function  $f : D \subset (X, d) \rightarrow \mathbb{R}$  satisfying the conditions of Theorem 1, two sequences
    $(N_k)_{k \geq 1}$  and  $(\alpha_{N_k})_{k \geq 1}$  as above and the  $\alpha_{N_k}$ -dense curves  $\gamma_{N_k}$  in the set  $D$ .
2:
3: INITIALIZATION.
4:
5:  $k \leftarrow 1$ , ExitCondition  $\leftarrow$  false,  $\Sigma, \tilde{\mathcal{Z}}_D(f) \leftarrow \emptyset$ 
6:
7: SEEK  $[a, b] \subset [0, 1]$  where  $f(\gamma_{\alpha_{N_k}}(t))$  has change of sign.
8:
9: while ExitCondition = false do
10:
11:   for  $n = 1, \dots, N_k$  do
12:
13:      $t_{1,n}^{N_k} \sim \mathcal{U}[\frac{n-1}{N_k}, \frac{n}{N_k}]$ ,  $t_{2,n}^{N_k} \sim \mathcal{U}[\frac{n-1}{N_k}, \frac{n}{N_k}]$ .
14:
15:      $a \leftarrow \min \{t_{1,n}^{N_k}, t_{2,n}^{N_k}\}$ ,  $b \leftarrow \max \{t_{1,n}^{N_k}, t_{2,n}^{N_k}\}$ .
16:
17:     if  $f(\gamma_{\alpha_{N_k}}(a))f(\gamma_{\alpha_{N_k}}(b)) < 0$  then
18:
19:        $\Sigma \leftarrow \Sigma \cup [a, b]$ 
20:
21:     end if
22:
23:   end for
24:
25:   if  $\Sigma \neq \emptyset$  then
26:
27:     ExitCondition  $\leftarrow$  true.
28:
29:   else
30:
31:      $k \leftarrow k + 1$ .
32:
33:   end if
34:
35: end while
36:
37: Use bisection method (Algorithm 1) to find approximations of some zeros of  $f$ .
38:
39: for  $[a, b] \in \Sigma$  do
40:
41:   Compute an approximation of a zero  $f(\gamma_{\alpha_{N_k}}(t^*))$  of the function  $f(\gamma_{\alpha_{N_k}}(t)) : [a, b] \rightarrow \mathbb{R}$  by using the
   bisection method,  $\tilde{\mathcal{Z}}_D(f) \leftarrow \tilde{\mathcal{Z}}_D(f) \cup \{f(\gamma_{\alpha_{N_k}}(t^*))\}$ .
42:
43: end for
44:
45: OUTPUT: The set  $\tilde{\mathcal{Z}}_D(f) \subset \mathcal{Z}_D(f)$ , of approximations of zeros of  $F$  in  $f$ .
46:
47: Return  $\tilde{\mathcal{Z}}_D(f)$ .
48:

```

Remark 1. Note that we can define $S_{N_k} := \{\frac{n-1}{N_k}, \frac{n}{N_k} : n = 1, \dots, N_k\}$, because of this set becomes into a dense sequence in $[0, 1]$ if $N_k \rightarrow \infty$. The reason to define S_{N_k} as above is because, after some numerical experiments, with this set Algorithm 2 finds more zeros of f in the set D .

Assume that the *while* loop in Algorithm 2 ends after k^* iterations. Then, if we denote by $\#\Sigma$ the number of intervals of the set Σ and by β_k the number of function evaluations (F.E.) that uses bisection method (Algorithm 1) to approximate a zero of $f(\gamma_{\alpha_{N_k}}(t))$ in each interval of Σ , the number of F.E. in Algorithm 2 is

$$N^* := \underbrace{2 \sum_{k=1}^{k^*} N_k}_{\text{F.E. in the for loop of lines 6-12 of Algorithm 2}} + \underbrace{\beta_1 + \dots + \beta_{\#\Sigma}}_{\text{F.E. of Algorithm 1 to find the desired approximation}}. \quad (5)$$

In a completely analogous way, we can find an upper bound for the computational time of Algorithm 2 in terms of computational time of the *for* loop of lines 6-12 and the computational time of the bisection method. Also, if we assume that β is an upper bound for $\beta_1, \dots, \beta_{\#\Sigma}$ and ν_{k^*} is the first sum in (5), we find that $N^* \leq \nu_{k^*} + \#\Sigma\beta$. So, roughly speaking the computational complexity of Algorithm 2 is, in the worst case, an $\mathcal{O}(\beta)$.

4. Some numerical examples

In this section, by some numerical examples, we will illustrate the feasibility and reliability of Algorithm 2. For the Bisection method (Algorithm 1), we take $\varepsilon = \delta = 10^{-10}$ in all the examples. Also, the notation used in the tables is the following:

- (i) For the indicated values of k , N_k is the number of subintervals of $[0, 1]$ (of the form indicated in Algorithm 2) where we seek the zeros of the function f .
- (ii) The number $\alpha_{N_k} > 0$ is the density of the used α -dense curve.
- (iii) $\#\tilde{\mathcal{Z}}_D(f)$ and Avg. Err. $\tilde{\mathcal{Z}}_D(f)$ are, respectively, the cardinality and the average of the error approximations of the set $\tilde{\mathcal{Z}}_D(f)$ found by Algorithm 2, that is, these sets are

$$\#\tilde{\mathcal{Z}}_D(f) := \#\{\gamma_\alpha(t^*) \in D : t^* \in [0, 1] \text{ is found by Algorithm 2}\},$$

and

$$\text{Avg. Err. } \tilde{\mathcal{Z}}_D(f) := \frac{1}{\#\tilde{\mathcal{Z}}_D(f)} \sum_{\gamma_\alpha(t^*) \in \tilde{\mathcal{Z}}_D(f)} |f(\gamma_\alpha(t^*))|,$$

respectively, where γ_α is the α -dense curve in Algorithm 2 for which the algorithm meets the exit condition.

- (iv) F.E. and C.T. denote, respectively, the number of function evaluations and the computational time, in seconds, in Algorithm 2.

Example 6. Let $f : D := [-1, 1]^2 \rightarrow \mathbb{R}$ be the function defined as

$$f(x, y) := (x^2 + y^2)^3 - 4x^2y^2, \quad \text{for all } (x, y) \in D. \quad (6)$$

In this example, Algorithm 2 can be used to obtain some points on the plane curve defined by $f(x, y) = 0$. The α -dense curve used in Algorithm 2 is that of Example 2. We present in Table 1 the obtained results, and in Figure 2 the graphs of $f(\gamma_{0.04}(t))$ and the set $\tilde{\mathcal{Z}}_D(f)$ returned by Algorithm 2 for the indicated values of k , N_k and α_{N_k} .

Table 1. Results obtained by Algorithm 2 for the function defined in (6)

k	N_k	α_{N_k}	$\#\tilde{\mathcal{Z}}_D(f)$	Avg. Err. $\tilde{\mathcal{Z}}_D(f)$	F.E.	C.T.
1	10	0.040	5	5.6E-11	215	0.06
2	50	0.020	15	3.9E-11	1017	0.44
3	100	0.013	32	5.0E-11	2184	0.93
4	200	0.010	64	5.2E-11	4236	1.38

Example 7. For computation reasons we consider the space $\mathbb{R}^{15}$ as an *approximation* of the space $\mathbb{R}^{\mathbb{N}}$ of Example 2, endowed with the metric

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \sum_{n=1}^{15} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \text{for all } \mathbf{x} := (x_n)_{n \geq 1}, \mathbf{y} := (y_n)_{n \geq 1} \in \mathbb{R}^{15}.$$

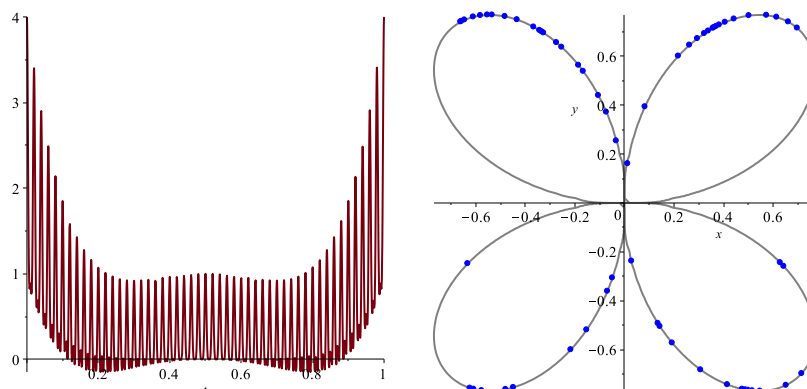


Figure 2. Left: Graph of the function $f(\gamma_{0.04}(t))$. Right: Graphs of $f(x,y) = 0$ and the 64 zeros found with Algorithm 2 (4-th row of Table 1)

Let $\mathbf{x}_0 := (-1, \frac{1}{2}, -\frac{1}{3}, \dots, \frac{1}{10}) \in \mathbb{R}^{15}$ and $f : D := [0,1]^{15} \rightarrow \mathbb{R}$ the function

$$f(\mathbf{x}) := (1 + \tilde{d}(\mathbf{x}, \mathbf{x}_0))^2 \sin(2\pi\tilde{d}(\mathbf{x}, \mathbf{x}_0)) - \frac{\sqrt{2}}{2}, \quad \text{for all } \mathbf{x} \in [0,1]^{15}. \quad (7)$$

Following Example 3, we will consider the α -dense curve

$$\gamma_\alpha(t) := \begin{cases} \gamma_{\alpha_0}^n(t), & \text{for } n \leq 10 \\ t, & \text{for } 11 \leq n \leq 15 \end{cases} \quad \text{for all } t \in [0,1],$$

where $(\gamma_{\alpha_0}^1(t), \dots, \gamma_{\alpha_0}^{10}(t))$ stands for the α_0 -dense curve in $[0,1]^{10}$ of Example 2. The obtained results are presented in Table 2.

Table 2. Results obtained by Algorithm 2 for the function defined in (7)

k	N_k	α_{N_k}	$\#\tilde{\mathcal{Z}}_D(f)$	Avg. Err. $\tilde{\mathcal{Z}}_D(f)$	F.E.	C.T.
1	10	0.380	6	2.5E-11	974	11.5
2	50	0.026	16	2.1E-11	2754	33.8
3	100	0.003	45	4.7E-11	7049	83.62
4	200	0.002	68	4.5E-11	11756	143.8

Example 8. Let the set

$$D := \left\{ \sum_{n=0}^6 a_n T_n(x) : |a_n| \leq \frac{1}{n^2} \text{ for each } n \geq 0 \right\} \subset C[-1,1].$$

So, as in the previous example, for computational reasons, D is a *truncated* subset of that given in Example 4. Also, let $D := \prod_{i=0}^6 [-\frac{1}{n^3}, \frac{1}{n^3}]$ and, γ_{α_0} an α_0 -dense curve in D , for some $\alpha_0 > 0$, as given in Example 2. Following Example 3, we will consider the α -dense curve

$$\gamma_\alpha(t) := \begin{cases} \gamma_{\alpha_0}^n(t), & \text{for } 0 \leq n \leq 5 \\ \frac{1}{216} \sin(2\pi t), & \text{for } n = 6 \end{cases} \quad \text{for all } t \in [0,1],$$

with $\alpha = \alpha_{N_k}$ for all $k \geq 1$ is indicated in Table 3. Let $f : D \rightarrow \mathbb{R}$ defined as

$$f(h)(x) := 2 \ln(1 + |\int_{-1}^1 h(x) dx|) + \left[\int_{-1}^1 h(x) dx \right]^2 - \frac{1}{2} \quad (8)$$

for each $h \in D$. We show in Table 3 the results returned by Algorithm 2.

Table 3. Results obtained by Algorithm 2 for the function defined in (8)

k	N_k	α_{N_k}	$\#\tilde{Z}_D(f)$	Avg. Err. $\tilde{Z}_D(f)$	F.E.	C.T.
1	10	0.031	1	2.3E-11	149	0.5
2	50	0.031	3	8.0E-11	461	1.1
3	100	0.031	2	5.4E-11	444	1.1
4	200	0.031	7	6.6E-11	1221	1.9

In Figure 3 we show the graph of the function $f(\gamma_{0.031}(t))$ and two functions in the set $\tilde{Z}_D(f)$ returned by Algorithm 2.

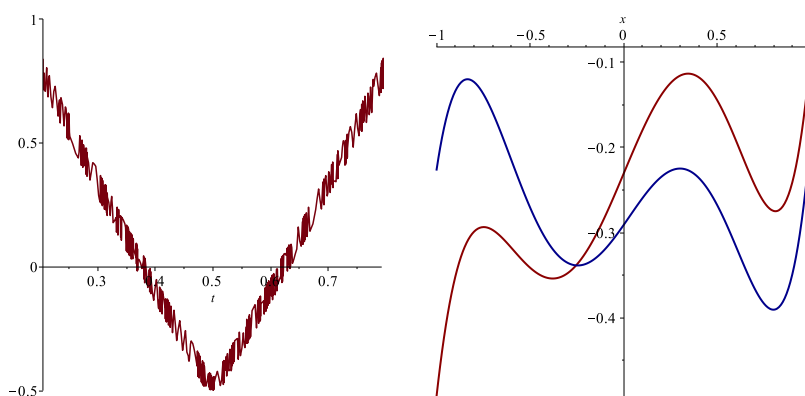


Figure 3. Left: Graph of the function $f(\gamma_{0.031}(t))$. Right: Graphs of two functions in $\tilde{Z}_D(f)$ found with Algorithm 2 (4-th row of Table 3)

5. Final remarks and conclusions

In the present paper we have proved, under suitable conditions and by using α -dense curves, the existence of zeros of a given continuous function $f : D \subseteq (X, d) \rightarrow \mathbb{R}$. Moreover, from our main result we provide a numerical method based on α -dense curves and the bisection method, the pseudo-code shown in Algorithm 2 to approximate some zeros of such function in the set D . The feasibility and reliability of our method are illustrated with numerical examples in §4.

As we have pointed out in §1, the bisection method has many generalizations and improvements, so in future works could be interesting to study and generalize some of these methods by using α -dense curves.

Although Algorithm 2 is a generalization of the bisection method, it is not free of drawbacks. For instance, the proposed method in the present paper has the same limitations that the bisection method. That is to say, a change of sign of the function is necessary to guarantee that Algorithm 2 works. Indeed, consider the function $f(x, y) := (x - 0.5)^2 + (y - 0.5)^2$ defined on $[0, 1]^2$ and γ_α the α -dense curve of Example 2, for any $\alpha > 0$. Clearly, $Z_{[0,1]^2}(f) = \{(0.5, 0.5)\}$ but $f(\gamma_\alpha(t^*)) \neq 0$ for each $t^* \in [0, 1]$ as $\gamma_\alpha(t) \neq (0.5, 0.5)$ for each $t \in [0, 1]$. In other words, Algorithm 2 is not able to approximate all the zeros of f in the densifiable set D .

Conflicts of Interest: The author declares that he has no conflict of interest.

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