

Article

An improved Hardy-Hilbert integral inequality using a modern approach

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Abstract: In recent decades, a wide range of Hardy-Hilbert-type integral inequalities have been established. This article focuses on a one-parameter result introduced by Waadallah Tawfeeq Sulaiman in 2010, which has a unique structure: the double integral involves a power-sum of the variables, as well as a technical power-minimum. The sharp constant factor is also elegantly expressed in terms of the beta function. However, the parameter involved is subject to restrictions on its values. In this article, we refine the inequality by removing this restriction and addressing a theoretical gap in the original proof to yield a sharper result. We provide a thorough, step-by-step proof and demonstrate how this new result can be used to derive additional variants and extensions.

Keywords: integral inequalities, minimum, Hardy-Hilbert-type integral inequalities, beta function

MSC: 26D15, 33E20.

1. Introduction

The inequalities based on the Hardy-Hilbert integral setting play a key role in mathematics. They provide precise and robust integral bounds, offering deeper insights into the behavior of various classes of operators and functions. A comprehensive discussion of the related theory and applications can be found in [1–3]. In their classical form, some modified Hardy-Hilbert-type integral inequalities involve the maximum of the variables, enabling a precise measure of their interaction. These foundational ideas have inspired a growing body of literature that continues to expand and refine the theory, yielding new variants and more general formulations, see [4–14].

For the purpose of this article, we consider a specific one-parameter integral inequality introduced by Waadallah Tawfeeq Sulaiman in 2010 (see [13, Theorem 5]). Let $p > 1$, $q = p/(p-1)$ and $\theta \in (0, 2)$ be the parameter. Let $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions. Then the following inequality holds:

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1}} f(x)g(y) dx dy \\ \leq B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q}, \end{aligned} \quad (1)$$

where $B(\theta/2, \theta/2) = \int_0^1 t^{\theta/2-1} (1-t)^{\theta/2-1} dt$, which corresponds to the standard beta function taken at the parameters $\theta/2$ and $\theta/2$, provided that the integrals involved converge. This inequality is notable for its distinctive structure, incorporating a power of the sum $(x+y)^\theta$, combined with a term $[\min(x/y, y/x)]^{\theta/2-1}$ that captures the interplay between x and y . Its sharp constant is expressed elegantly in terms of the beta function, and the weighted integral norms of f and g are defined with the simple power-weight functions $x^{1-\theta}$ and $y^{1-\theta}$, respectively.

However, this result also raises two questions:

Parameter range: Can we relax the restriction that $\theta \in (0, 2)$ and extend the validity of this inequality to a wider range of θ ?

Refinements: Is it possible to further sharpen or generalize the inequality?

In this article, we provide affirmative answers to both questions by adapting the approach in [6]. More specifically, we construct an auxiliary bivariate function $\varphi(x, y)$, which enables us to extend the range of θ and to obtain the following generalized inequality:

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} \varphi(x, y)} f(x)g(y) dx dy \\ & \leq B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q}, \end{aligned}$$

provided that the integrals involved converge. The introduced φ satisfies the condition, for any $x, y > 0$, $\varphi(x, y) \leq 1$ when $\theta \in (0, 2)$, yielding an improvement of the original result in Eq. (1). Thus, we generalize and refine the existing theory, providing a robust framework for future developments and applications.

The article is organized as follows: §2 is devoted to our refined Hardy-Hilbert integral inequality. A conclusion is given in §3.

2. Result

The improved Hardy-Hilbert integral inequality, along with its proof and a discussion, is given below.

Theorem 1. Let $p > 1$, $q = p/(p-1)$ and $\theta > 0$. Let $f, g: \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions. Then we have

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} \varphi(x, y)} f(x)g(y) dx dy \\ & \leq B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q}, \end{aligned}$$

where

$$\varphi(x, y) = \begin{cases} \left\{ \min \left[\left(\frac{y}{x} \right)^{2-\theta}, 1 \right] \right\}^{1/p} \left\{ \min \left[\left(\frac{x}{y} \right)^{2-\theta}, 1 \right] \right\}^{1/q} & \text{if } \theta \in (0, 2), \\ \left\{ \max \left[\left(\frac{y}{x} \right)^{2-\theta}, 1 \right] \right\}^{1/p} \left\{ \max \left[\left(\frac{x}{y} \right)^{2-\theta}, 1 \right] \right\}^{1/q} & \text{if } \theta \geq 2, \end{cases} \quad (2)$$

provided that the integrals involved converge.

Proof. To simplify future developments, let us rewrite the function in Eq. (2) as follows:

$$\varphi(x, y) = \left\{ h_\theta \left[\left(\frac{y}{x} \right)^{2-\theta}, 1 \right] \right\}^{1/p} \left\{ h_\theta \left[\left(\frac{x}{y} \right)^{2-\theta}, 1 \right] \right\}^{1/q},$$

where

$$h_\theta(x, y) = \begin{cases} \min(x, y) & \text{if } \theta \in (0, 2), \\ \max(x, y) & \text{if } \theta \geq 2, \end{cases}$$

Decomposing the integrand using this expression and $1/p + 1/q = 1$, and applying the Hölder integral inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} \varphi(x, y)} f(x)g(y) dx dy \\ & = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^{\theta/p} \left\{ [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(y/x)^{2-\theta}, 1] \right\}^{1/p}} f(x) \end{aligned}$$

$$\begin{aligned} & \times \frac{1}{(x+y)^{\theta/q} \left\{ [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(x/y)^{2-\theta}, 1] \right\}^{1/q}} g(y) dx dy \\ & \leq \mathfrak{U}^{1/p} \mathfrak{V}^{1/q}, \end{aligned} \quad (3)$$

where

$$\mathfrak{U} = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(y/x)^{2-\theta}, 1]} f^p(x) dx dy,$$

and

$$\mathfrak{V} = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(x/y)^{2-\theta}, 1]} g^q(y) dx dy.$$

Let us express $\mathfrak{U}$ and $\mathfrak{V}$ in turn.

For $\mathfrak{U}$, applying the Fubini-Tonelli integral theorem, we get

$$\mathfrak{U} = \int_{\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(y/x)^{2-\theta}, 1]} dy \right\} f^p(x) dx. \quad (4)$$

Let us focus on the central integral term. Using the definitions of $[\min(x/y, y/x)]^{\theta/2-1}$ and $h_{\theta}[(y/x)^{2-\theta}, 1]$, in particular, for any $x, y, \theta > 0$,

$$h_{\theta} \left[\left(\frac{y}{x} \right)^{2-\theta}, 1 \right] = \begin{cases} \left(\frac{y}{x} \right)^{2-\theta} & \text{if } y < x, \\ 1 & \text{if } x \leq y, \end{cases}$$

(where we have implicitly distinguished between the cases $\theta \in (0, 2)$ and $\theta \geq 2$), and applying the Chasles integral relation, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+} \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(y/x)^{2-\theta}, 1]} dy \\ &= \int_0^x \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(y/x)^{2-\theta}, 1]} dy \\ & \quad + \int_x^{+\infty} \frac{1}{(x+y)^{\theta} [\min(x/y, y/x)]^{\theta/2-1} h_{\theta}[(y/x)^{2-\theta}, 1]} dy \\ &= \int_0^x \frac{1}{(x+y)^{\theta} (y/x)^{\theta/2-1} (y/x)^{2-\theta}} dy + \int_x^{+\infty} \frac{1}{(x+y)^{\theta} (x/y)^{\theta/2-1} \times 1} dy \\ &= x^{1-\theta} \left[\int_0^x \frac{(y/x)^{\theta/2-1}}{(1+y/x)^{\theta}} \frac{1}{x} dy + \int_x^{+\infty} \frac{(y/x)^{\theta/2-1}}{(1+y/x)^{\theta}} \frac{1}{x} dy \right] \\ &= x^{1-\theta} \int_{\mathbb{R}_+} \frac{(y/x)^{\theta/2-1}}{(1+y/x)^{\theta}} \frac{1}{x} dy. \end{aligned} \quad (5)$$

Performing the changes of variables $u = (y/x)/(1+y/x)$ so that $y/x = u/(1-u)$, and recognizing the beta function $B(\theta/2, \theta/2)$, we get

$$\begin{aligned} \int_{\mathbb{R}_+} \frac{(y/x)^{\theta/2-1}}{(1+y/x)^{\theta}} \frac{1}{x} dy &= \int_{\mathbb{R}_+} \left(\frac{y/x}{1+y/x} \right)^{\theta/2-1} \left(1 - \frac{y/x}{1+y/x} \right)^{\theta/2-1} \frac{1}{(1+y/x)^2} \frac{1}{x} dy \\ &= \int_0^1 u^{\theta/2-1} (1-u)^{\theta/2-1} du = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right). \end{aligned} \quad (6)$$

Joining Eqs. (5) and (6), we find that

$$\int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(y/x)^{2-\theta}, 1]} dy = x^{1-\theta} B\left(\frac{\theta}{2}, \frac{\theta}{2}\right). \quad (7)$$

It follows from Eqs. (4) and (7) that

$$\mathfrak{U} = \int_{\mathbb{R}_+} x^{1-\theta} B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) f^p(x) dx = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx. \quad (8)$$

For $\mathfrak{V}$, the steps are the same; we just need to take important details into account. According to the Fubini-Tonelli integral theorem, we have

$$\mathfrak{V} = \int_{\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(x/y)^{2-\theta}, 1]} dx \right\} g^q(y) dy. \quad (9)$$

Let us focus on the central integral term. Using the definitions of $[\min(x/y, y/x)]^{\theta/2-1}$ and $h_\theta[(x/y)^{2-\theta}, 1]$, in particular, for any $x, y, \theta > 0$,

$$h_\theta \left[\left(\frac{x}{y} \right)^{2-\theta}, 1 \right] = \begin{cases} \left(\frac{x}{y} \right)^{2-\theta} & \text{if } x < y, \\ 1 & \text{if } y \leq x, \end{cases}$$

(where we have implicitly distinguished between the cases $\theta \in (0, 2)$ and $\theta \geq 2$), applying the Chasles integral relation and proceeding as in Eq. (6), we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(x/y)^{2-\theta}, 1]} dx \\ &= \int_0^y \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(x/y)^{2-\theta}, 1]} dx \\ & \quad + \int_y^{+\infty} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} h_\theta[(x/y)^{2-\theta}, 1]} dx \\ &= \int_0^y \frac{1}{(x+y)^\theta (x/y)^{\theta/2-1} (x/y)^{2-\theta}} dx + \int_y^{+\infty} \frac{1}{(x+y)^\theta (y/x)^{\theta/2-1} \times 1} dx \\ &= y^{1-\theta} \left[\int_0^y \frac{(x/y)^{\theta/2-1}}{(1+x/y)^\theta} \frac{1}{y} dx + \int_y^{+\infty} \frac{(x/y)^{\theta/2-1}}{(1+x/y)^\theta} \frac{1}{y} dx \right] \\ &= y^{1-\theta} \int_{\mathbb{R}_+} \frac{(x/y)^{\theta/2-1}}{(1+x/y)^\theta} \frac{1}{y} dx = y^{1-\theta} B\left(\frac{\theta}{2}, \frac{\theta}{2}\right). \end{aligned} \quad (10)$$

It follows from Eqs. (9) and (10) that

$$\mathfrak{V} = \int_{\mathbb{R}_+} y^{1-\theta} B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) g^q(y) dy = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy. \quad (11)$$

Combining Eqs. (3), (8) and (11) and using $1/p + 1/q = 1$, we get

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} \varphi(x, y)} f(x) g(y) dx dy \\ & \leq \left[B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q} \\ & = B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q}. \end{aligned}$$

This completes the proof of Theorem 1. $\square$

As a first remark, the parameter θ in Theorem 1 must satisfy $\theta > 0$, not only $\theta \in (0, 2)$, which is a significant improvement of [13, Theorem 5], as recalled in Eq. (1). Furthermore, for $\theta \in (0, 2)$, based on Eq. (2), we clearly have

$$\varphi(x, y) = \left\{ \min \left[\left(\frac{y}{x} \right)^{2-\theta}, 1 \right] \right\}^{1/p} \left\{ \min \left[\left(\frac{x}{y} \right)^{2-\theta}, 1 \right] \right\}^{1/q} \leq 1,$$

so that

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1}} f(x)g(y) dx dy \\ & \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^\theta [\min(x/y, y/x)]^{\theta/2-1} \varphi(x, y)} f(x)g(y) dx dy \\ & \leq B\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_{\mathbb{R}_+} x^{1-\theta} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{1-\theta} g^q(y) dy \right]^{1/q}. \end{aligned}$$

The inequality in Theorem 1 implies that in [13, Theorem 5]. It is an improvement in this sense. For the case $\theta \geq 2$, the result in Theorem 1 is unique and cannot be compared with others.

As a particular example, if we set $\theta = 1$, using $B(1/2, 1/2) = \pi$, then we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\sqrt{\min(x/y, y/x)}}{(x+y)\varphi(x, y)} f(x)g(y) dx dy \leq \pi \left[\int_{\mathbb{R}_+} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} g^q(y) dy \right]^{1/q},$$

with

$$\varphi(x, y) = \left[\min \left(\frac{y}{x}, 1 \right) \right]^{1/p} \left[\min \left(\frac{x}{y}, 1 \right) \right]^{1/q}.$$

This is a new variant of the Hardy-Hilbert integral inequality, which is sharper than the analogue derived from [13, Theorem 5].

If we set $\theta = 2$, using $B(1, 1) = 1$ and noting that $\varphi(x, y) = 1$, then we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{(x+y)^2} f(x)g(y) dx dy \leq \left[\int_{\mathbb{R}_+} x^{-1} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{-1} g^q(y) dy \right]^{1/q}.$$

This is a well-known variant of the Hardy-Hilbert integral inequality. Thus, Theorem 1 includes some existing results. Using different values for θ can lead to other new variants.

3. Conclusion

This article builds upon and generalizes a significant Hardy-Hilbert-type integral inequality established in [13]. By addressing the parameter restriction and filling a theoretical gap, it provides a more robust and versatile framework. The detailed proof presented here provides a solid foundation for future research and facilitates the derivation of further extensions. Thus, this article contributes to the ongoing evolution of integral inequality theory, with potential applications in analysis, operator theory, and related fields.

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