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Classification of 3-dimensional Bihom-Associative and Bihom-Bialgebras

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Abstract: BiHom-associative and BiHom-Hopf algebras (BiHom-bialgebras with antipode structure) have many applications in various areas of mathematics and physics. Based on a significant and growing topic, the current paper aims to investigate the structure of (non-)unital BiHom-associative algebras and explore the algebraic varieties of BiHom-associative (bi-)algebras up to dimension 3. We use the correlations between the structural constants to provide the desired classification results. These results further enable us to differentiate between each isomorphism type of the 3-dimensional BiHom-associative and BiHom-bialgebras inside some equivalence classes. Finally based on our findings, we classified BiHom-Hopf algebras up to dimension 3. These results are useful for understanding related algebraic structures and present a substantial advancement.

Keywords: BiHom-associative algebra, BiHom-Bialgebra, BiHom-Hopf algebra, classification

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1. Introduction

BiHom-associative algebra is a type of algebraic structure useful in studying various areas of mathematics including category theory, representation theory, and Hom-algebra structures. The motivation to study BiHom-associative algebra lies in its applications across many domains of mathematics. Although the history of BiHom-associative algebra is not very new it was first introduced in [1], where the main goal was to create a BiHom-associative bialgebra using Hom-associative algebras. A BiHom-associative algebra (A, μ, α, β) consists of a vector space, a multiplication, and two linear commuting maps called twist maps or structural maps satisfying the following identity

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \beta(z)) \quad \textit{for all } x, y, z \in A. \tag{1}$$

It may be viewed as a deformation of an associative algebra, in which the associativity condition is twisted by the linear maps α and β in a certain way such that when $\alpha = id$ and $\beta = id$, the BiHom-associative algebras degenerate to exactly associative algebras. Later on, this subject was further developed through various research studies. In [2], BiHom-Novikov algebra and quadratic BiHom-Novikov algebras were explored using commutative BiHom-algebra. BiHom-algebra equipped with the skew-symmetric Lie bracket and Jacobi identity yields BiHom-Lie algebra, which can also be constructed from BiHom-associative algebras, and the classifications of simple multiplicative BiHom-Lie algebra were given in [3]. The BiHom-algebra structure is an extension of Hom-algebra mainly studied in [4–12]. If both structure maps of BiHom-associative algebra are the same, i.e., $\alpha = \beta$, we get a Hom-associative algebra structure. Moreover, representation, derivation central extension, and deformation of BiHom-Lie algebra are introduced in [13]. Many other researchers have made prominent contributions in this field which can be seen in [14–18].

Since the 19*th* century, the classification of algebraic structure has been a very hot topic and studied in various contexts. In [19–21], the classification of associative algebra, Hom-associative algebra, and Hopf algebra have been provided. In the past classification problems of various algebraic structures were only performed manually, which required a lot of manpower, and mostly little mistakes resulted in false results. Sometimes, it becomes challenging to handle the computations with many equations. To avoid this situation, we are interested in computing our calculations by using a computer program, as we have used it in our other research see in [21–24]. With careful use of computer programs, the results obtained are more precise and easy to handle.

The Hopf algebra is also one of the famous and powerful mathematical objects involving multiplication, comultiplication, unit, counit, and antipode maps. Specifically for associative algebra context: suppose that (A, m, u) is an associative algebra with multiplication $m : A \times A \to A$ and unit element $u : K \to A$. And (C, Δ, ϵ) is a coassociative coalgebra with comultiplication $\Delta : C \to C \times C$ and counit $\epsilon : C \to K$, where K is the base field. Then there exists a compatibility condition between A and C. This compatibility condition forms (A, C) a bialgebra,

- The compatibility between multiplication and comultiplication is given by:

$$(m \times id_C) \circ (id_A \times \Delta) \circ \Delta = m \circ \Delta, \quad (id_C \times m) \circ (\Delta \times id_A) \circ \Delta = m \circ \Delta. \tag{2}$$

- The compatibility between unit and counit is given by :

$$(u \times id_C) \circ \Delta = id_C = (id_C \times \epsilon) \circ \Delta. \tag{3}$$

The above equations show that the pairs (m, Δ) and (u, ϵ) work in a well-defined way. Hence, A and C together form a bialgebra, that further extends to Hopf algebra with a notion of antipode. More detail on bialgebra and Hopf algebra can be found in [1,25–27].

In this paper, we attempt to combine the above three topics, i.e., BiHom-associative algebra, Hopf algebra, and classification of algebras. We first presented a detailed description of BiHom-associative algebra. In particular, we discussed multiplicative, non-multiplicative, unital, and non-unital BiHom-associative algebra. We then provided the complete classification of algebraic varieties of BiHom-associative algebra and classified multiplicative and unital multiplicative BiHom-associative algebras up to dimension 3. Let A be an n-dimensional $\mathbb{K}$-linear space and $\{e_1, e_2, \cdots, e_n\}$ be a basis of A. A BiHom-algebra structure on A with product μ is determined by n^3 structure constants $\mathcal{C}_{ij}^k$, in particular $\mu(e_i, e_j) = \sum_{k=1}^n \mathcal{C}_{ij}^k e_k$. Moreover, α, β are given by $2n^2$ structure constants a_{ij} and b_{ij}, such that $\alpha(e_i) = \sum\limits_{j=1}^n a_{ji} e_j$ and $\beta(e_j) = \sum\limits_{k=1}^n b_{kj} e_k$. If we impose the conditions of BiHom-associativity and unity on the algebraic structure, we obtain a sub-variety denoted as $\mathcal{H}_n$ (or $\mathcal{H}u_n$) which exists within $k^{n^3+2n^2}$. Base changes in A result in the natural transport of $GL_n(k)$ structure action on $\mathcal{H}_n$. Thus the isomorphism classes of n-dimensional BiHom-algebras are in one-to-one correspondence with the orbits of the action of $GL_n(k)$ on $\mathcal{H}_n$ (rep. $\mathcal{H}u_n$). Furthermore, we introduced the dual concept of BiHom-associative algebra, which yields BiHom-coassociative algebra. BiHom-associative and BiHom-coassociative algebra with a compatibility condition form a BiHom-bialgebra. We present the class of non-isomorphic BiHom-bialgebras of dimensions 2 and 3. We introduced the antipode structure for the BiHom-bialgebra, which significantly yields 2 and 3 dimensional Hopf algebra. This paper is organized as follows:

In §2, we give the basics about BiHom-associative algebras and provide some new properties. Moreover, we discuss unital BiHom-associative algebras. §3 is dedicated to describing algebraic varieties of BiHom-associative algebras and providing classifications, up to isomorphism, of $2-$ and 3-dimensional BiHom-associative algebras. In §4, we study BiHom-bialgebra and BiHom-Hopf algebra.

Bialgebra and Hopf algebra are important mathematical structures that arise in representation theory, algebraic geometry, mathematical physics, and other areas of mathematics. These structure encodes the concept of a noncommutative or quantum version of a symmetric algebra and arises naturally in problems in quantum mechanics and gauge theory. Overall our study about the classification of bialgebra and Hopf algebra provides deep insights into the nature of algebraic structures and their related structures in geometry and physics.

2. Structure of BiHom-associative algebras

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and A be a linear space over $\mathbb{K}$. By a BiHom-algebra, we refer to a 4-tuple (A, μ, α, β), where $\mu : A \times A \to A$ is a bilinear map (multiplication) and α and β are structural maps, also known as twist maps of A.

Definition 1. [9] A BiHom-associative algebra is a 4-tuple (A, μ, α, β) consisting of a linear space A, a bilinear map $\mu : A \times A \to A$ and two linear space homomorphism $\alpha, \beta : A \to A$ satisfying the following equations:

$$\begin{aligned} \alpha \circ \beta &= \beta \circ \alpha. \\ \mu(\alpha(x), \mu(y, z)) &= \mu(\mu(x, y), \beta(z)). \end{aligned} \tag{4}$$

For all $x, y, z \in A$.

Moreover, if we have

$$\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)) \quad and \quad \beta(\mu(x, y)) = \mu(\beta(x), \beta(y)), \tag{5}$$

we call (A, μ, α, β) a multiplicative BiHom-associative algebra. Throughout the paper, we deal with the multiplicative BiHom-associative algebra, but for simplicity, we call it a BiHom-associative algebra. We denote the set of all BiHom-associative algebras by $\mathcal{H}$. In the language of Hopf algebras, the multiplication of a BiHom-associative algebra over A consists of a linear map $\mu : A \otimes A \to A$, and condition (4) can be written as

$$\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \beta(z)). \tag{6}$$

Definition 2. [1] Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ are two BiHom-associative algebras. A linear map $\varphi : A \to B$ is called a morphism of the BiHom-associative algebras if the following identities hold

$$\varphi \circ \mu_A = \mu_B \circ (\varphi \otimes \varphi), \quad \alpha_B \circ \varphi = \varphi \circ \alpha_A \quad and \quad \beta_B \circ \varphi = \varphi \circ \beta_A. \tag{7}$$

In particular, BiHom-associative algebras $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ are isomorphic, if φ is also bijective.

2.1. Non-multiplicative BiHom-associative algebra

In this subsection, we present some properties of the non-multiplicative BiHom-associative algebra's structure.

Proposition 1 ([12])**.** *Let (A, μ, α, β) be a BiHom-associative algebra and $\gamma : A \to A$ be a BiHom-associative algebra morphism. Then $(A, \gamma\mu, \gamma\alpha, \gamma\beta)$ is a BiHom-associative algebra.*

Proof. To show $(A, \gamma\mu, \gamma\alpha, \gamma\beta)$ is a BiHom-associative algebra under the multiplication $\gamma\mu$, assume that

$$\begin{aligned} \gamma\mu(\gamma\alpha(x), \gamma\mu(y, z)) &= \gamma\mu\gamma(\alpha(x), \mu(y, z)) \\ &= \gamma^2\mu(\alpha(x), \mu(y, z)) \\ &= \gamma^2\mu(\mu(x, y), \beta(z)) \\ &= \gamma\mu(\gamma\mu(x, y), \gamma\beta(z)). \end{aligned}$$

Above identity is proved by considering γ is a homomorphism and $(A, \mu_A, \alpha_A, \beta_A)$ is a BiHom-associative algebra. □

Proposition 2. *Let (A, μ, α, β) be an n-dimensional BiHom-associative algebra and $\phi : A \to A$ be an invertible linear map. Then there is an isomorphism with an n-dimensional BiHom-associative algebra $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$ where $\mu' = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})$. Furthermore, if $\left\{C_{ij}^k\right\}$ are the structure constants of μ with respect to the basis $\{e_1, \ldots, e_n\}$, then μ' has the same structure constants with respect to the basis $\{\Lambda(e_1), \ldots, \Lambda(e_n)\}$.*

Proof. We prove for any invertible linear map $\Lambda : A \to A$, the quardruple $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$ is a BiHom-associative algebra.

$$\begin{aligned}
\mu'(\mu'(x,y), \Lambda\beta\Lambda^{-1}(z)) &= \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(\Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(x,y), \Lambda\beta\Lambda^{-1}(z)) \\
&= \Lambda\mu(\mu(\Lambda^{-1}(x), \Lambda^{-1}(y)), \beta\Lambda^{-1}(z)) \\
&= \Lambda\mu(\alpha\Lambda^{-1}(x), \mu(\Lambda^{-1}(y), \Lambda^{-1}(z))) \\
&= \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(\Lambda\otimes\Lambda)(\alpha\Lambda^{-1}(x), \mu(\Lambda^{-1}\otimes\Lambda^{-1})(y,z))) \\
&= \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(\Lambda\alpha\Lambda^{-1}(x), \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(y,z))) \\
&= \mu'(\Lambda\alpha\Lambda^{-1}(x), \mu'(y,z)).
\end{aligned}$$

So $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$ is a BiHom-associative algebra. It is also multiplicative.
Indeed, for α, we have

$$\begin{aligned}
\Lambda\alpha\Lambda^{-1}\mu'(x,y) &= \Lambda\alpha\Lambda^{-1}\Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(x,y) \\
&= \Lambda\alpha\mu(\Lambda^{-1}\otimes\Lambda^{-1})(x,y) \\
&= \Lambda\mu(\alpha\Lambda^{-1}(x), \alpha\Lambda^{-1}(y)) \\
&= \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(\Lambda\otimes\Lambda)(\alpha\Lambda^{-1}(x), \alpha\Lambda^{-1}(y)) \\
&= \mu'(\Lambda\alpha\Lambda^{-1}(x), \Lambda\alpha\Lambda^{-1}(y)).
\end{aligned}$$

For β, we have

$$\begin{aligned}
\Lambda\beta\Lambda^{-1}\mu'(x,y) &= \Lambda\beta\Lambda^{-1}\Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(x,y) \\
&= \Lambda\beta\mu(\Lambda^{-1}\otimes\Lambda^{-1})(x,y) \\
&= \Lambda\mu(\beta\Lambda^{-1}(x), \beta\Lambda^{-1}(y)) \\
&= \Lambda\mu(\Lambda^{-1}\otimes\Lambda^{-1})(\Lambda\otimes\Lambda)(\beta\Lambda^{-1}(x), \beta\Lambda^{-1}(y)) \\
&= \mu'(\Lambda\beta\Lambda^{-1}(x), \Lambda\beta\Lambda^{-1}(y)).
\end{aligned}$$

Therefore $\Lambda : (A, \mu, \alpha, \beta) \to (A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$ is a BiHom-associative algebras morphism. Since

$$\Lambda \circ \mu' = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ (\Lambda \otimes \Lambda) = \mu' \circ (\Lambda \otimes \Lambda),$$

and

$$(\Lambda\alpha\Lambda^{-1}) \circ \Lambda = \Lambda \circ \alpha \text{ and } (\Lambda\beta\Lambda^{-1}) \circ \Lambda = \Lambda \circ \beta,$$

it is easy to see that $\{\Lambda(e_i), \cdots, \Lambda(e_n)\}$ is a set of basis for A. For $i, j = 1, \cdots, n$, we have

$$\mu'(\Lambda(e_i), \Lambda(e_j)) = \Lambda\mu(\Lambda^{-1}(e_i), \quad \Lambda^{-1}(e_j)) = \Lambda\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^{k}\Lambda(e_k).$$

It completes the proof. □

Remark 1. A BiHom-associative algebra (A, μ, α, β) is isomorphic to an associative algebra if and only if $\alpha = \beta = id$. Indeed, $\Lambda \circ \alpha \circ \Lambda^{-1} = \Lambda \circ \beta \circ \Lambda^{-1} = id$ is equivalent to $\alpha = \beta = id$.

Remark 2. Proposition 2 is useful for the classification of BiHom-associative algebras. Indeed, we need to consider the class of morphisms that are conjugate. Representations of these classes are given by Jordan forms of the matrix. Any $n \times n$ matrix over $\mathbb{K}$ is equivalent up to basis change to Jordan's canonical form. That is why, we choose Λ such that the matrix of $\Lambda\alpha\Lambda^{-1} = \gamma$ and $\lambda = \Lambda\beta\Lambda^{-1}$, where γ and λ are Jordan canonical forms.

Hence, to obtain the classification, we consider only Jordan forms for the structure map of BiHom-associative algebras.

Proposition 3. *Let* (A, μ, α, β) *be a BiHom-associative algebra over* $\mathbb{K}$. *Let* $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$ *be its isomorphic BiHom-associative algebra described in Proposition 2. If* ψ *is an automorphism of* (A, μ, α, β), *then* $\Lambda\psi\Lambda^{-1}$ *is an automorphism of* $(A, \mu, \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$.

Proof. Note that $\gamma = \Lambda\alpha\Lambda^{-1}$. We have

$$\Lambda\psi\Lambda^{-1}\gamma = \Lambda\psi\Lambda^{-1}\Lambda\alpha\Lambda^{-1} = \Lambda\psi\alpha\Lambda^{-1} = \Lambda\alpha\psi\Lambda^{-1} = \Lambda\alpha\Lambda^{-1}\Lambda\psi\Lambda^{-1} = \gamma\Lambda\psi\Lambda^{-1}.$$

For β, we consider $\lambda = \Lambda\beta\Lambda^{-1}$, we have

$$\Lambda\psi\Lambda^{-1}\lambda = \Lambda\psi\Lambda^{-1}\Lambda\beta\Lambda^{-1} = \Lambda\psi\beta\Lambda^{-1} = \Lambda\beta\psi\Lambda^{-1} = \Lambda\beta\Lambda^{-1}\Lambda\psi\Lambda^{-1} = \lambda\Lambda\psi\Lambda^{-1}.$$

For any $x, y \in A$,

$$\begin{aligned}\Lambda\psi\Lambda^{-1}\mu'(\Lambda(x), \Lambda(y)) &= \Lambda\psi\Lambda^{-1}\Lambda\mu(x,y) = \Lambda\psi\mu(x,y) = \Lambda\mu(\psi(x), \psi(y)) \\ &= \mu'(\Lambda\psi(x), \Lambda\psi(y)) = \mu'(\Lambda\psi\Lambda^{-1}(\Lambda(x)), \Lambda\psi\Lambda^{-1}(\Lambda(y))).\end{aligned}$$

By Definition, $\Lambda\psi\Lambda^{-1}$ is an automorphism of $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1})$. □

Proposition 4. *Given two BiHom-associative algebras* $(A, \mu_A, \alpha_A, \beta_A)$ *and* $(B, \mu_B, \alpha_B, \beta_B)$ *over field* $\mathbb{K}$, *there is a BiHom-associative algebra* $(A \oplus B, \mu_{A\oplus B}, \alpha_A + \alpha_B, \beta_A + \beta_B)$, *where* $\mu_{A\oplus B}$ *gien by the usual multiplication* $((a + b), (a' + b')) = (a, a') + (b, b')$.

$\mu_{A\oplus B}(-,-) : (A \oplus B) \times (A \oplus B) \to (A \oplus B)$ *is given by*

$$\mu_{A\oplus B}(a + b, a' + b') = (\mu_A(a, a'), \mu_B(b, b')), \forall\, a, a' \in A, \forall\, b, b' \in B,$$

and the linear map $(\alpha_A + \alpha_B, \beta_A + \beta_B) : A \oplus B \to A \oplus B$ *is given by*

$$(\alpha_A + \alpha_B, \beta_A + \beta_B)(a, b) = ((\alpha_A + \beta_A)(a), (\alpha_B + \beta_B)(b))\, \forall a \in A, b \in B.$$

Proof. For any $a, a', a'' \in A$, $b, b', b'' \in B$, by direct computation, we get

$$\begin{aligned}&\mu_{A\oplus B}((\alpha_A + \beta_A, \alpha_B + \beta_B)(a, b), \mu_{A\oplus B}(a' + b', a'' + b'')) \\ &\quad= \mu_{A\oplus B}(((\alpha_A + \beta_A)a, (\alpha_B + \beta_B)b), (\mu_A(a', a''), \mu_B(b', b''))) \\ &\quad= (\mu_A(((\alpha_A + \beta_A)a), \mu_A(a', a'')), \mu_B((\alpha_B + \beta_B)b, \mu_B(b', b''))) \\ &\quad= (\mu_A(\mu_A(a, a'), (\alpha_A + \beta_A)a''), \mu_B(\mu_B(b, b'), (\alpha + \beta)b'')) \\ &\quad= (\mu_{A\oplus B}(\mu_{A\oplus B}(a + b, a' + b'), (\alpha_A + \beta_A, \alpha_B + \beta_B)(a'', b''))).\end{aligned}$$

This ends the proof. □

Proposition 5. *Let* $(A, \mu_A, \alpha_A, \beta_A)$ *and* $(B, \mu_B, , \alpha_B, \beta_B)$ *be two BiHom-associative algebras. Then, there exists a BiHom-associative algebra structure on* $A \oplus B$ *with the bilinear map* $* : (A \oplus B)^{\otimes 2} \to A \oplus B$ *given by*

$$\mu((a_1 + b_2), (a_2 + b_2)) := \mu_A(a_1, a_2) + \mu_B(b_2, b_2),$$

and the linear maps $\alpha = \alpha_A + \alpha_B$, $\beta = \beta_A + \beta_B : A \oplus B \to A \oplus B$ *given by*

$$(\alpha_A + \alpha_B)(a + b) := \alpha_A(a) + \alpha_B(b),\ (\beta_A + \beta_B)(a + b) := \beta_A(a) + \beta_B(b),\ \forall (a, b) \in A \times B.$$

Moreover, if $\Lambda : A \to B$ *is a linear map. Then,*

$$\Lambda : (A, \mu_A, \alpha_A, \beta_A) \to (B, \mu_B, \alpha_B, \beta_B),$$

is a morphism if and only if its graph $\Gamma_\Lambda = \{(x, \Lambda(x)), x \in A\}$ *is a BiHom-subalgebra of* $(A \oplus B, *, \alpha, \beta)$.

Proof. The proof of the first part comes from a direct computation, so we omit it. Now, let us suppose that $\Lambda : (A, \mu_A, \alpha_A, \beta_A) \to (B, \mu_B, \alpha_B, \beta_B)$ is a morphism of BiHom-associative algebras. Then

$$\mu((u + \Lambda(u)), (v + \Lambda(v))) = (\mu_A(u,v) + \mu_B(\Lambda(u), \Lambda(v))) = \mu_A(u,v) + \mu_A(\Lambda((u,v)).$$

Thus the graph Γ_Λ is closed under the operation μ.

Furthermore, since $\Lambda \circ \alpha_A = \alpha_B \circ \Lambda$, and $\Lambda \circ \beta_A = \beta_B \circ \Lambda$, we have

$$(\alpha_A \oplus \alpha_B)(u, \Lambda(u)) = (\alpha_A(u), \alpha_B \circ \Lambda(u)) = (\alpha_A(u), \Lambda \circ \alpha_A(u)),$$

and

$$(\beta_A \oplus \beta_B)(u, \Lambda(u)) = (\beta_A(u), \beta_B \circ \Lambda(u)) = (\beta_A(u), \Lambda \circ \beta_A(u)).$$

Which implie that Γ_Λ is closed $\alpha_A \oplus \alpha_B$ and $\beta_A \oplus \beta_B$. Thus, Γ_Λ is a BiHom-subalgebra of $(A \oplus B, \mu, \alpha, \beta)$.

Conversely, if the graph $\Gamma_\Lambda \subset A \oplus B$ is a BiHom-subalgebra of $(A \oplus B, \mu, \alpha, \beta)$ then we

$$\mu((u + \Lambda(u)), (v + \Lambda(v))) = \mu_A(u,v) + \mu_B(\Lambda(u), \Lambda(v)) \in \Gamma_\Lambda.$$

Furthermore, $(\alpha_A \oplus \alpha_B)(\Gamma_\Lambda) \subset \Gamma_\Lambda$, $(\beta_A \oplus \beta_B)(\Gamma_\Lambda) \subset \Gamma_\Lambda$, implies

$$(\alpha_A \oplus \alpha_B)(u, \Lambda(u)) = (\alpha_A(u), \alpha_B \circ \Lambda(u)) \in \Gamma_\Lambda,\ (\beta_A \oplus \beta_B)(u, \Lambda(u)) = (\beta_A(u), \beta_B \circ \xi(u)) \in \Gamma_\Lambda,$$

which is equivalent to the condition $\alpha_B \circ \Lambda(u) = \Lambda \circ \alpha_A(u)$, i.e. $\alpha_B \circ \Lambda = \Lambda \circ \alpha_A$. Similary, $\beta_B \circ \Lambda = \Lambda \circ \beta_A$. Therefore, Λ is a morphism of BiHom-associative algebras. □

2.2. Unital BiHom-associative algebra

In this section, we discuss unital BiHom-associative algebra. We denote by $\mathcal{H}u_n$ the set of n-dimensional unital BiHom-associative algebras.

Definition 3. A BiHom-associative algebra (A, μ, α, β) is called unital if there exists an element $\mathbf{1} \in A$ such that $\mu(x, \mathbf{1}) = \alpha(x)$ and $\mu(\mathbf{1}, x) = \beta(x)$ for all $x \in A$.

A morphism of unital BiHom-associative algebras $\phi : A \longrightarrow B$ is called unital if $\phi(\mathbf{1}_A) = \mathbf{1}_B$.

Proposition 6. *Let (A, μ, α, β) be a BiHom-associative algebra. We set $\tilde{A} = span(A, \mathbf{1})$ the vector space generated by elements of A and $\mathbf{1}$. Assume $\mu(x, \mathbf{1}) = \alpha(x)$, $\mu(\mathbf{1}, x) = \beta(x)$, $\alpha(\mathbf{1}) = \mathbf{1}$ and $\beta(\mathbf{1}) = \mathbf{1}$, for all $x \in A$. Then $(\tilde{A}, \mu, \alpha, \beta, \mathbf{1})$ is a unital BiHom-associative algebra.*

Proof. It is straightforward to check the BiHom-associativity. For example

$$\mu(\mu(x,y), \beta(\mathbf{1})) = \mu(\mu(x,y), \mathbf{1}) = \alpha(\mu(x,y)) = \mu(\alpha(x), \alpha(y)) = \mu(\alpha(x), \mu(y, \mathbf{1})).$$

This completes the proof. □

Remark 3. Some unital BiHom-associative cannot be obtained as an extension of a non-unital BiHom-associative algebra.

Remark 4. Let $(A, \mu, \alpha, \beta, \mathbf{1})$ be an n-dimensional unital BiHom-associative algebra and $\Lambda : A \to A$ be an invertible linear map such that $\Lambda(\mathbf{1}) = \mathbf{1}$. Then it is isomorphic to a n-dimensional BiHom-associative algebra $(A, \mu', \Lambda\alpha\Lambda^{-1}, \Lambda\beta\Lambda^{-1}, \mathbf{1})$ where $\mu' = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})$. Moreover, if $\left\{C_{ij}^k\right\}$ are the structure constants of μ with respect to the basis $\{e_1, \ldots, e_n\}$ with $e_1 = \mathbf{1}$ being the unit, then μ' has the same structure constants with respect to the basis $\{\Lambda(e_1), \ldots, \Lambda(e_n)\}$ with $\mathbf{1}$ the unit element. Indeed, we use Proposition 2 and Definition 3. The unit has been preserved since

$$\mu'(x, e_1) = \Lambda \circ \mu(\Lambda^{-1}(x), \Lambda^{-1}(e_1)) = \Lambda \circ \alpha \circ \Lambda^{-1}(x),$$

and
$$\mu'(e_1,x) = \Lambda\circ\mu(\Lambda^{-1}(e_1),\Lambda^{-1}(x)) = \Lambda\circ\beta\circ\Lambda^{-1}(x).$$

Proposition 7. *Let $(A,\mu_A,\alpha_A,\beta_A,\mathbf{1}_A)$ and $(B,\mu_B,\alpha_B,\beta_B,\mathbf{1}_B)$ are two unital BiHom-associative algebras with $\phi(\mathbf{1}_A)=\mathbf{1}_B$. Suppose there exists a BiHom-associative algebra morphism $\phi: A\to B$. If $(A,\mu'_A=\alpha_A\circ\beta_A\circ\mu_A,\mathbf{1}'_A)$ is a morphism of $(A,\mu_A,\alpha_A,\beta_A,\mathbf{1}_A)$ then there exists an untwist of $(B,\mu_B,\alpha_B,\beta_B,\mathbf{1}_B)$ such that*
$$\phi:(A,\mu'_A,\mathbf{1}'_A)\to(B,\mu'_B,\mathbf{1}'_B),$$
is an algebra morphism.

Proof. Because ϕ is a homomorphism from $(A,\mu_A,\alpha_A,\beta_A,\mathbf{1}_A)$ to $(B,\mu_B,\alpha_B,\beta_B,\mathbf{1}_B)$. Then by
$$\alpha_B\circ\phi=\phi\circ\alpha_A \text{ and } \beta_B\circ\phi=\phi\circ\beta_A,$$
we have
$$\mu_B(\phi(x),\phi(\mathbf{1}_A)) = \mu_B(\phi(x),\mathbf{1}_B) = \alpha_B\circ\phi(x),$$
and
$$\mu_B(\phi(\mathbf{1}_A),\phi(x)) = \mu_B(\mathbf{1}_B,\phi(x)) = \beta_B\circ\phi(x),$$
for all $x\in A$. Additionally, we possess
$$\phi\circ\mu_A(x,\mathbf{1}_A)=\phi\circ\alpha_A(x) \text{ and } \phi\circ\mu_A(\mathbf{1}_A,x)=\phi\circ\beta_A(x).$$

By Proposition 6, we can see that $(B,\mu_B,\mathbf{1}_B)$ is also an associative algebra. Furthermore
$$\begin{aligned}\mu'_B(\phi(x),\phi(\mathbf{1}_A)) &= \mu'_B(\phi(x),\mathbf{1}_B)\\ &= \phi\circ\alpha'_A\circ\phi(x)\\ &= \phi\circ\alpha_A\circ\mu_A(x,\mathbf{1}_A)\\ &= \alpha_B\circ\phi\circ\mu_A(x,\mathbf{1}_A)\\ &= \alpha_B\circ\mu_B(\phi(x),\mathbf{1}_B)\end{aligned}$$
and
$$\begin{aligned}\mu'_B(\phi(\mathbf{1}_A),\phi(x)) &= \mu'_B(\mathbf{1}_B,\phi(x))\\ &= \phi\circ\beta'_A\circ\phi(x)\\ &= \phi\circ\alpha_A\circ\mu_A(\mathbf{1}_A,x)\\ &= \beta_B\circ\phi\circ\mu_A(\mathbf{1}_A,x)\\ &= \beta_B\circ\mu_B(\mathbf{1}_B,\phi(x)).\end{aligned}$$

This completes the proof. □

3. Classifications of algebraic varieties of BiHom-associative algebra

In this section, we deal with algebraic varieties of BiHom-associative algebras with a fixed dimension. A BiHom-associative algebra is identified with its structure constants concerning a fixed basis. Their set corresponds to an algebraic variety where the ideals are generated by polynomials corresponding to the BiHom-associativity condition.

3.1. Algebraic varieties $\mathcal{H}_n$ and their action of linear group

Let A be a n-dimensional $\mathbb{K}$-linear space and $\{e_1,\cdots,e_n\}$ be the basis of A. A BiHom-algebra structure on A with product μ and a structure map α and β is determined by n^3 structure constants $\mathcal{C}_{ij}^k$ where $\mu(e_i,e_j)=\sum\limits_{k=1}^{n}\mathcal{C}_{ij}^k e_k$ and by $2n^2$ structure constants a_{ji} and b_{kj}, where $\alpha(e_i)=\sum\limits_{j=1}^{n}a_{ji}e_j$ and $\beta(e_j)=\sum\limits_{k=1}^{n}b_{kj}e_k$. If we require this algebra structure to be BiHom-associative, then this limits the set of structure constants $(\mathcal{C}_{ij}^k, a_{ij}, b_{jk})$ to a cubic sub-variety of the affine algebraic variety $\mathbb{K}^{n^3+2n^2}$ defined by the following polynomial equations system :

By using the above definition of ψ and basis multiplication of BiHom-associative algebra, we get the following equations of structure constants :

$$\sum_{p=1}^{n}(b_{pi}a_{qp}-a_{pi}b_{qp})=0, \tag{8}$$

$$\sum_{l=1}^{n}\sum_{m=1}^{n}(a_{li}\mathcal{C}_{jk}^{m}\mathcal{C}_{lm}^{s}-b_{mk}\mathcal{C}_{ij}^{l}\mathcal{C}_{lm}^{s})=0, \tag{9}$$

$$\sum_{p=1}^{n}a_{sp}\mathcal{C}_{ij}^{p}-\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pi}a_{qj}\mathcal{C}_{pq}^{s}=0, \tag{10}$$

$$\sum_{p=1}^{n}b_{rp}\mathcal{C}_{jk}^{p}-\sum_{p=1}^{n}\sum_{q=1}^{n}b_{pj}b_{qk}\mathcal{C}_{pq}^{r}=0. \tag{11}$$

Moreover, if μ is commutative, we have $\mathcal{C}_{ij}^k=\mathcal{C}_{ji}^k, \quad i,j,k=1,\cdots,n$. The first set of equations corresponds to the BiHom-associative condition $\mu(\alpha(e_i),\mu(e_j,e_k))=\mu(\mu(e_i,e_j),\beta(e_k))$ and the second set to multiplicativity condition $\alpha\circ\mu(e_i,e_j)=\mu(\alpha(e_i),\alpha(e_j))$ and $\beta\circ\mu(e_j,e_k)=\mu(\beta(e_j),\beta(e_k))$. We denote by $\mathcal{H}_n$ the set of all n-dimensional multiplicative BiHom-associative algebras.

The group $GL_n(\mathbb{K})$ acts on the algebraic varieties of BiHom-structures by the so-called transport of structure action defined as follows. Let $A=(A,\mu,\alpha,\beta)$ be a n-dimensional BiHom-associative algebra defined by multiplication μ and a linear map α and β. Given $\Lambda\in GL_n(\mathbb{K})$, the action $\Lambda\cdot A$ transports the structure,

$$\begin{aligned}\Theta: \quad & GL_n(\mathbb{K})\times\mathcal{H}_n && \longrightarrow \mathcal{H}_n\\ & (\Lambda,(A,\mu,\alpha,\beta)) && \longmapsto (A,\Lambda^{-1}\circ\mu\circ(\Lambda\otimes\Lambda),\Lambda\circ\alpha\circ\Lambda^{-1},\Lambda\circ\beta\circ\Lambda^{-1})\end{aligned}$$

defined for $x,y\in A$ by

$$\begin{aligned}\Lambda\cdot\mu(x,y)&=\Lambda^{-1}\mu(\Lambda(x),\Lambda(y)),\\ \Lambda\cdot\alpha(x)&=\Lambda^{-1}\alpha(\Lambda(x)),\\ \Lambda\cdot\beta(x)&=\Lambda^{-1}\beta(\Lambda(x)).\end{aligned}$$

The conjugate class is given by

$$\Theta(\Lambda,(A,\mu,\alpha,\beta))=(A,\Lambda^{-1}\circ\mu\circ(\Lambda\otimes\Lambda),\Lambda\circ\alpha\circ\Lambda^{-1},\Lambda\circ\beta\circ\Lambda^{-1})),$$

for $\Lambda\in GL_n(\mathbb{K})$.

The orbit of a BiHom-associative algebra A of $\mathcal{BH}as_n$ is given by

$$\vartheta(A)=\left\{A'=\Lambda\cdot A,\ \Lambda\in GL_n(\mathbb{K})\right\}.$$

The orbits are in **1-1 correspondence** with the isomorphism classes of n-dimensional BiHom-associative algebras. The stabilizer is

$$Stab((A,\mu,\alpha,\beta))=\left\{\Lambda\in GL_n(\mathbb{K})|(\Lambda^{-1}\circ\mu\circ(\Lambda\otimes\Lambda)=\mu \text{ and } \Lambda\circ\alpha=\alpha\circ\Lambda \text{ and } \Lambda\circ\beta=\beta\circ\Lambda\right\}.$$

Let $\Lambda(e_i)=\sum\limits_{j=1}^{n}\lambda_{ij}e_j\in GL_n(\mathbb{K})$ and $\Lambda^{-1}(e_i)=\sum\limits_{j=1}^{n}\gamma_{ij}e_j$. Then we can describe the action of $GL_n(\mathbb{K})$ on $\mathcal{BH}as_n$ as follows.

$$\Lambda\cdot(C_{ij}^k,\lambda_{ij})=(\sum_{l,p,q=1}^{n}\gamma_{lk}C_{pq}^{l}\lambda_{ij}\lambda_{jq},\sum_{k,l=1}^{n}\gamma_{kj}\gamma_{lk}\lambda_{il})=(\tilde{C}_{ij}^k,\tilde{\lambda}_{ij}).$$

We characterize the properties of structure constants, that determine whether two BiHom-associative algebras are in the same orbit (or isomorphic). Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ be to n-dimensional BiHom-associative algebras. They are isomorphic if there exists $\varphi \in GL_n(\mathbb{K})$ such that

$$\Lambda \circ \mu_A = \mu_B(\Lambda \otimes \Lambda), \quad \alpha_B \circ \Lambda = \Lambda \circ \alpha_A \quad and \quad \beta_B \circ \Lambda = \Lambda \circ \beta_A. \tag{12}$$

Remark 5. Conditions (12) are equivalent to $\mu_A = \Lambda^{-1} \circ \mu_B \circ \Lambda \otimes \Lambda$, $\alpha_A = \Lambda^{-1} \circ \alpha_B \circ \Lambda$ and $\beta_A = \Lambda^{-1} \circ \beta_B \circ \Lambda$.

Proposition 8. *The transport of structure action on $\mathcal{H}_n$ is well defined.*

Proof. First, we check that this action is well-defined. Let $\mu' = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})$ and $\sigma' = \Lambda \circ \sigma \circ \Lambda^{-1}$. To To demonstrate that the action given by $\Lambda \cdot (\mu, \sigma) = (\mu', \sigma')$ is well-defined, it is sufficient to prove that for any (μ, σ) which satisfies certain equations and $\sigma(\mathbf{1}) = \mathbf{1}$, then the pair (μ', σ') also satisfies the same conditions. Additionally, since $\Lambda \in GL_n(\mathbb{K})$, then $\Lambda\mathbf{1} = \mathbf{1}$ and $\Lambda^{-1}\mathbf{1} = \mathbf{1}$ too.

$$\begin{aligned}
\sigma(\mathbf{1}) &= (\Lambda \circ \Lambda^{-1})(\mathbf{1}) \\
&= \Lambda(\sigma(\Lambda^{-1}(\mathbf{1})) \\
&= \Lambda(\sigma(\mathbf{1})) \\
&= \Lambda(\mathbf{1}) \\
&= \mathbf{1}.
\end{aligned}$$

$$\begin{aligned}
\sigma' \circ \sigma' &= (\Lambda \circ \sigma \circ \Lambda^{-1}) \circ (\Lambda \circ \sigma \Lambda^{-1}) \\
&= \Lambda \circ \sigma \circ \sigma \circ \Lambda^{-1} \\
&= \Lambda \circ id \circ \Lambda^{-1} \\
&= \Lambda \circ \Lambda^{-1} \\
&= id.
\end{aligned}$$

$$\begin{aligned}
\mu'(\mathbf{1} \otimes e_i) &= (\Lambda \circ \mu(\Lambda^{-1} \otimes \Lambda^{-1}))(\mathbf{1} \otimes e_i) \\
&= (\Lambda \circ \mu)(\Lambda^{-1}\mathbf{1} \otimes \Lambda^{-1} e_i) \\
&= \Lambda(\mu(\mathbf{1} \otimes \Lambda^{-1} e_i)) \\
&= \Lambda(\Lambda^{-1} e_i) \\
&= e_i.
\end{aligned}$$

$$\begin{aligned}
\mu'(e_i \otimes \mathbf{1}) &= (\Lambda \circ \mu(\Lambda^{-1} \otimes \Lambda^{-1}))(e_i \otimes \mathbf{1}) \\
&= (\Lambda \circ \mu)(\Lambda^{-1}\mathbf{1} \otimes \Lambda^{-1} e_1) \\
&= \Lambda(\mu(e_i \otimes \Lambda^{-1}\mathbf{1})) \\
&= \Lambda(\Lambda^{-1} e_i) \\
&= e_i.
\end{aligned}$$

$$\begin{aligned}
\mu'(\mu' \otimes id) &= (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ ((\Lambda \circ \mu(\Lambda^{-1} \otimes \Lambda^{-1})) \otimes id) \\
&= \Lambda \circ \mu \circ ((\Lambda^{-1} \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})) \otimes (\Lambda^{-1} \circ id)) \\
&= \Lambda \circ \mu \circ ((\mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})) \otimes (id \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ (\mu \otimes id) \circ (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \mu \circ (id \otimes \mu) \circ (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes (\mu(\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes (\Lambda^{-1} \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ (id \otimes (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \mu'(id \otimes \mu').
\end{aligned}
$$

$$
\begin{aligned}
\mu' \circ (\sigma' \otimes \sigma') &= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ ((\Lambda \circ \sigma \circ \Lambda^{-1}) \otimes (\Lambda \circ \sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ ((\Lambda^{-1} \circ \Lambda \circ \sigma \circ \Lambda^{-1}) \otimes (\Lambda^{-1} \otimes \Lambda \circ \sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ ((\sigma \circ \Lambda^{-1}) \otimes (\sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ (\sigma \otimes \sigma) \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \sigma \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \sigma' \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \sigma' \circ \mu'.
\end{aligned}
$$

This shows that the action is well-defined. Finally, we show that this is indeed an action of $GL_n(\mathbb{K})$ on $\mathcal{H}_n$. Note, I_n the $n \times n$ identity matrix of the group $GL_n(\mathbb{K})$.

$$
\begin{aligned}
I_n(\mu, \sigma) &= (I_n^{-1} \circ (I_n \otimes I_n), I_n^{-1} \circ \sigma \circ I_n) \\
&= \Lambda \circ \sigma \circ \sigma \circ \Lambda^{-1} \\
&= (I_n \circ \mu \circ (I_n \otimes I_n), I_n \circ \sigma \circ I_n) \\
&= (\mu, \sigma).
\end{aligned}
$$

Let $\Gamma, \Delta \in GL_n(\mathbb{K})$,

$$
\begin{aligned}
\Gamma \cdot (\Delta \cdot (\mu, \sigma)) \quad &= \Gamma \cdot (\Delta^{-1} \circ \mu \circ (\Delta \otimes \Delta), \Delta^{-1} \circ \sigma \circ \Delta) \\
&= (\Gamma \cdot (\Delta^{-1} \circ \mu \circ (\Delta \otimes \Delta)) \circ (\Gamma^{-1} \otimes \Gamma^{-1}), \Gamma \circ (\Delta^{-1} \circ \sigma \circ \Delta) \circ \Gamma^{-1}) \\
&= (\Gamma \circ \Lambda\mu \circ (\Lambda \otimes \Lambda) \circ (\Gamma^{-1} \otimes \Gamma^{-1}), (\Gamma \circ \Delta^{-1} \circ \sigma \circ \Delta \circ \Gamma^{-1})) \\
&= (\Gamma \circ \Lambda^{-1} \circ \mu \circ ((\Delta \circ \Lambda^{-1}) \otimes (\Delta \circ \Lambda)), (\Gamma \circ \Delta^{-1} \circ \sigma \circ \Delta \circ \Gamma)) \\
&= ((\Gamma \circ \Lambda)^{-1} \circ \mu \circ ((\Gamma \circ \Delta \circ) \otimes (\Gamma \circ \Lambda), \otimes (\Delta \circ \Lambda)), (\Gamma \circ \Lambda) \circ \sigma \circ (\Gamma \circ \Lambda)^{-1}) \\
&= ((\Gamma \circ \Lambda) \cdot (\mu, \sigma)).
\end{aligned}
$$

This completes the proof. □

Theorem 1. *Any 2-dimensional real BiHom-associative algebra is either associative or isomorphic to one of the following pairwise non-isomorphic BiHom-associative algebras :*

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}_2^1$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_1,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^2$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^3$	$e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^4$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_2 = e_1,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^5$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^6$	$e_1 \cdot e_1 = ae_1 + e_2, \quad e_2 \cdot e_1 = be_1,$ $e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^7$	$e_1 \cdot e_1 = -e_1, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_1 = e_2, \quad e_2 \cdot e_1 = e_1,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = -e_2, \quad \beta(e_2) = -e_2.$

$\mathcal{H}_2^8$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^9$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}_2^{10}$	$e_1 \cdot e_1 = \frac{1}{b}e_1,$ $e_1 \cdot e_2 = e_2,$ $\quad e_2 \cdot e_1 = e_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$

Proof. Let $\mathcal{A}$ be a two-dimensional vector space. To determine a BiHom-associative algebra structure on $\mathcal{A}$, we consider that $\mathcal{A}' = (\mathcal{H}, \cdot, \alpha, \beta)$ with the multiplication operation $'' \cdot ''$ is a BiHom-associative algebra given by the following relation

$$\begin{array}{lll} e_1 \cdot e_1 = a_1e_1 + a_2e_2, & & \\ e_1 \cdot e_2 = a_3e_1 + a_4e_2, & \alpha(e_1) = x_1e_1 + x_2e_2, & \beta(e_1) = y_1e_1 + y_2e_2, \\ e_2 \cdot e_1 = a_5e_1 + a_6e_2, & \alpha(e_2) = x_3e_1 + x_4e_2, & \beta(e_2) = y_3e_1 + y_4e_2. \\ e_2 \cdot e_2 = a_7e_1 + a_8e_2, & & \end{array}$$

By verifying BiHom-associative algebra axioms, we get several constraints for the coefficients a_i, x_j, y_j, where $1 \le i \le 8$ and $1 \le j \le 4$.
We obtain $a_2 = a_4 = a_6 = a_8 = 0, \quad a_1 = a_3 = a_5 = a_7 = 1, \quad x_2 = x_3 = y_2 = y_3 = 0$ and $x_1 = x_4 = y_1 = y_4 = 1$. Hence, $\mathcal{A}' = (\mathcal{H}, \cdot, \alpha, \beta)$ is isomorphic to $\mathcal{H}_2^1$, given by

$$\begin{array}{llll} e_1 \cdot e_1 = e_1, & e_2 \cdot e_1 = e_1, & \alpha(e_1) = e_1, & \beta(e_1) = e_1, \\ e_1 \cdot e_2 = e_1, & e_2 \cdot e_2 = e_1, & \alpha(e_2) = e_2, & \beta(e_2) = e_2. \end{array}$$

The other BiHom-associative algebras of the list in Theorem 1 can be obtained through minor modifications of the observation above. □

Theorem 2. *Any 2-dimensional unital BiHom-associative algebra is either associative or isomorphic to one of the following pairwise non-isomorphic BiHom-associative algebras :*

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}u_2^1$	$e_1 \cdot e_1 = ae_1, \quad e_2 \cdot e_1 = ce_2,$ $e_1 \cdot e_2 = be_2, \quad e_2 \cdot e_2 = de_1 + fe_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}u_2^2$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = be_2,$ $\quad e_2 \cdot e_1 = ce_2,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$
$\mathcal{H}u_2^3$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = -e_2,$ $e_1 \cdot e_2 = -e_2, \quad e_2 \cdot e_2 = e_1,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = -e_2, \quad \beta(e_2) = -e_2.$
$\mathcal{H}u_2^4$	$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_2 = e_1,$	$\alpha(e_1) = e_1, \quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2, \quad \beta(e_2) = e_2.$

Proof. Let $\mathcal{B}$ be a two-dimensional vector space. To determine a unital BiHom-associative algebra structure on $\mathcal{B}$, we consider that $\mathcal{B}' = (\mathcal{H}u, \cdot, \alpha, \beta)$ with the multiplication operation $'' \cdot ''$ is a unital BiHom-associative algebra given by the following relation

$$\begin{array}{lll} e_1 \cdot e_1 = b_1e_1 + b_2e_2, & & \\ e_1 \cdot e_2 = b_3e_1 + b_4e_2, & \alpha(e_1) = y_1e_1 + y_2e_2, & \beta(e_1) = z_1e_1 + z_2e_2, \\ e_2 \cdot e_1 = b_5e_1 + b_6e_2, & \alpha(e_2) = y_3e_1 + y_4e_2, & \beta(e_2) = z_3e_1 + z_4e_2. \\ e_2 \cdot e_2 = b_7e_1 + b_8e_2, & & \end{array}$$

By verifying unital BiHom-associative algebra axioms, we get several constraints for the coefficients b_i, y_j, z_j, where $1 \le i \le 8$ and $1 \le j \le 4$.

We obtain $b_2 = b_3 = b_5 = 0, \quad b_1 = a, \quad b_4 = b, \quad b_7 = d, \quad b_8 = f, \quad y_2 = y_3 = z_2 = z_3 = 0$ and $y_1 = y_4 = z_1 = z_4 = 1$. Hence, $\mathcal{B}' = (\mathcal{H}u, \cdot, \alpha, \beta)$ is isomorphic to $\mathcal{H}u_2^1$, given by

$$\begin{array}{llll} e_1 \cdot e_1 = ae_1, & e_2 \cdot e_1 = ce_1, & \alpha(e_1) = e_1, & \beta(e_1) = e_1, \\ e_1 \cdot e_2 = be_1, & e_2 \cdot e_2 = de_1 + fe_2, & \alpha(e_2) = e_2, & \beta(e_2) = e_2. \end{array}$$

The other unital BiHom-associative algebras of the list in Theorem 2 can be obtained through minor modifications of the observation above. □

Theorem 3. *Any 3-dimensional BiHom-associative algebra is either associative or isomorphic to one of the following pairwise non-isomorphic BiHom-associative algebras :*

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}_3^1$	$e_1 \cdot e_1 = e_1 + e_3$, $e_2 \cdot e_1 = e_3$, $e_1 \cdot e_2 = e_2$, $e_2 \cdot e_2 = e_3$, $e_1 \cdot e_3 = e_2$, $e_3 \cdot e_3 = e_3$,	$\alpha(e_2) = e_2$, $\beta(e_1) = e_1$, $\beta(e_2) = e_2$
$\mathcal{H}_3^2$	$e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_1 \cdot e_3 = e_2$, $e_2 \cdot e_1 = e_3$, $e_2 \cdot e_2 = e_3$, $e_3 \cdot e_2 = e_1$, $e_3 \cdot e_3 = e_3$,	$\alpha(e_2) = e_2$, $\beta(e_1) = e_1$, $\beta(e_1) = e_1$, $\beta(e_3) = e_3$.
$\mathcal{H}_3^3$	$e_1 \cdot e_1 = e_1 + e_2$, $e_1 \cdot e_2 = e_1 + e_2$, $e_2 \cdot e_1 = e_2 + e_3$, $e_3 \cdot e_2 = e_1 + e_2$, $e_3 \cdot e_3 = e_1 + e_3$,	$\alpha(e_1) = e_1$, $\beta(e_1) = e_1$, $\alpha(e_2) = e_2$, $\beta(e_3) = e_3$.
$\mathcal{H}_3^4$	$e_1 \cdot e_2 = e_1 + e_2$, $e_2 \cdot e_2 = e_1 + e_2$, $e_1 \cdot e_3 = e_1 + e_2$, $e_3 \cdot e_2 = e_1 + e_2$, $e_2 \cdot e_1 = e_2 + e_3$, $e_3 \cdot e_3 = e_1 + e_3$	$\alpha(e_1) = e_1$, $\beta(e_1) = e_1$, $\beta(e_3) = e_3$.
$\mathcal{H}_3^5$	$e_1 \cdot e_2 = e_1 - e_2$, $e_2 \cdot e_1 = ae_1 + e_3$, $e_2 \cdot e_3 = e_1 + e_2$, $e_3 \cdot e_2 = e_1 + be_2$ $e_3 \cdot e_3 = ce_1 + e_3$,	$\alpha(e_1) = e_1$, $\alpha(e_2) = e_2$, $\beta(e_2) = e_1$.
$\mathcal{H}_3^6$	$e_1 \cdot e_1 = e_1$, $e_2 \cdot e_2 = e_2$, $e_1 \cdot e_2 = e_1$, $e_2 \cdot e_3 = e_3$, $e_1 \cdot e_3 = e_3$, $e_3 \cdot e_1 = e_3$, $e_2 \cdot e_1 = e_2$, $e_3 \cdot e_3 = e_3$,	$\alpha(e_1) = e_1$, $\alpha(e_2) = e_2$, $\alpha(e_3) = e_3$, $\beta(e_1) = e_1$, $\beta(e_2) = e_1$
$\mathcal{H}_3^7$	$e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = -e_3$, $e_2 \cdot e_2 = e_2$, $e_3 \cdot e_1 = -e_3$, $e_3 \cdot e_3 = e_1$,	$\alpha(e_1) = e_1$, $\beta(e_2) = e_2$, $\alpha(e_3) = e_3$, $\beta(e_3) = e_3$
$\mathcal{H}_3^8$	$e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_2 \cdot e_2 = e_2$, $e_3 \cdot e_2 = e_3$, $e_3 \cdot e_3 = e_3$,	$\alpha(e_1) = e_1$, $\alpha(e_2) = e_2$, $\beta(e_1) = e_1$, $\beta(e_2) = e_1 + e_2$, $\beta(e_3) = e_3$.
$\mathcal{H}_3^9$	$e_1 \cdot e_3 = e_1 - e_2$, $e_3 \cdot e_1 = e_1 - e_2$, $e_2 \cdot e_3 = e_1 - e_2$, $e_3 \cdot e_3 = e_1 - e_2$,	$\alpha(e_1) = e_1$, $\alpha(e_2) = e_1 + e_2$, $\alpha(e_3) = e_2 + e_3$. $\beta(e_2) = e_1$, $\beta(e_3) = e_2$.
$\mathcal{H}_3^{10}$	$e_1 \cdot e_2 = e_1 + e_3$, $e_2 \cdot e_2 = e_2 + e_3$, $e_2 \cdot e_1 = e_2 + e_3$, $e_3 \cdot e_3 = e_2 + e_3$,	$\alpha(e_2) = e_1$, $\beta(e_2) = e_1$, $\alpha(e_3) = e_2$, $\beta(e_3) = e_2$.
$\mathcal{H}_3^{11}$	$e_1 \cdot e_1 = e_1 + e_2$, $e_2 \cdot e_3 = e_2 + e_3$, $e_2 \cdot e_2 = e_1 + e_2$, $e_3 \cdot e_3 = e_2 + e_3$,	$\alpha(e_2) = e_1$, $\beta(e_2) = e_1$, $\alpha(e_3) = e_2$, $\beta(e_3) = e_2$.
$\mathcal{H}_3^{12}$	$e_1 \cdot e_3 = e_1 + e_2$, $e_2 \cdot e_2 = e_1 + e_2$, $e_2 \cdot e_3 = e_1 - e_2$, $e_3 \cdot e_2 = e_1 - e_2$, $e_3 \cdot e_3 = e_1 - e_2$,	$\alpha(e_1) = -e_1$, $\alpha(e_2) = e_1 - e_2$, $\alpha(e_3) = e_2 - e_3$. $\beta(e_2) = e_1$, $\beta(e_3) = e_2$.
$\mathcal{H}_3^{13}$	$e_1 \cdot e_3 = e_1 - e_2$, $e_2 \cdot e_3 = e_1 + e_2$, $e_2 \cdot e_1 = e_1 - e_2$, $e_3 \cdot e_2 = e_1 + e_2$, $e_2 \cdot e_2 = e_1 + e_2$, $e_3 \cdot e_3 = e_1 - e_2$,	$\alpha(e_1) = e_1$, $\beta(e_2) = e_2$, $\alpha(e_2) = -e_2$, $\beta(e_3) = e_3$.
$\mathcal{H}_3^{14}$	$e_1 \cdot e_3 = e_1 - e_2$, $e_2 \cdot e_3 = e_1 - e_2$, $e_3 \cdot e_1 = e_1 - e_2$, $e_3 \cdot e_2 = e_1 - e_2$, $e_3 \cdot e_3 = e_1 - e_2$,	$\alpha(e_1) = e_1$, $\alpha(e_2) = e_1$, $\beta(e_1) = e_1$, $\beta(e_2) = e_1$, $\beta(e_3) = e_2$.
$\mathcal{H}_3^{15}$	$e_1 \cdot e_2 = e_3$, $e_2 \cdot e_3 = e_2 + e_3$, $e_1 \cdot e_3 = e_1 + e_2$, $e_3 \cdot e_1 = e_1 + e_2$, $e_2 \cdot e_2 = e_2 + e_3$, $e_3 \cdot e_3 = e_1$,	$\alpha(e_2) = e_2$, $\beta(e_1) = e_1$, $\alpha(e_3) = e_3$, $\beta(e_2) = e_2$,

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}_3^{16}$	$e_1 \cdot e_3 = e_2 + e_3,\quad e_2 \cdot e_2 = e_2 - e_3,$ $e_2 \cdot e_1 = e_2 + e_3,\quad e_3 \cdot e_3 = e_2 - e_3,$	$\alpha(e_1) = -e_1,\quad \beta(e_1) = e_1,$ $\alpha(e_2) = -e_2,\quad \beta(e_3) = e_3.$

Proof. We give the proof of one case. Suppose that BiHom-associative algebra $\mathcal{B} = (\mathcal{H}, \cdot, \alpha, \beta)$ has the following multiplication table :

$$\begin{array}{lll} e_1 \cdot e_1 = e_1 + e_3, & e_2 \cdot e_1 = e_3, & \alpha(e_2) = e_2, \\ e_1 \cdot e_2 = e_2, & e_2 \cdot e_2 = e_3, & \beta(e_1) = e_1, \\ e_1 \cdot e_3 = e_2, & e_3 \cdot e_3 = e_3, & \beta(e_2) = e_2. \end{array}$$

Now let us define $\mathcal{C} = (\mathcal{H}, \cdot, \alpha, \underline{)}$ by the multiplication table :

$$\begin{array}{ll} e_1 \cdot e_1 = a_1e_1 + a_2e_2 + a_3e_3, & \\ e_1 \cdot e_2 = a_4e_1 + a_5e_2 + a_6e_3, & \alpha(e_1) = x_1e_1 + x_2e_2 + x_3e_3, \\ e_1 \cdot e_3 = a_7e_1 + a_8e_2 + a_9e_3, & \alpha(e_2) = x_4e_1 + x_5e_2 + x_6e_3, \\ e_2 \cdot e_1 = b_1e_1 + b_2e_2 + b_3e_3, & \alpha(e_3) = x_7e_1 + x_8e_2 + x_9e_3, \\ e_2 \cdot e_2 = b_4e_1 + b_5e_2 + b_6e_3, & \beta(e_1) = y_1e_1 + y_2e_2 + y_3e_3, \\ e_2 \cdot e_3 = b_7e_1 + b_8e_2 + b_9e_3, & \beta(e_2) = y_4e_1 + y_5e_2 + y_6e_3, \\ e_3 \cdot e_1 = c_1e_1 + c_2e_2 + c_3e_3, & \beta(e_3) = y_7e_1 + y_8e_2 + y_9e_3, \\ e_3 \cdot e_2 = c_4e_1 + c_5e_2 + c_6e_3, & \\ e_3 \cdot e_3 = c_7e_1 + c_8e_2 + c_9e_3, & \end{array}$$

where a_i, b_i, c_i, x_i, y_i and z_i are unknowns $(i = 1, 2, ..., 9)$. Verifying the BiHom-associative algebras axioms, we get the following constraints for the structure :

$$\begin{array}{ll} a_2 = a_4 = a_6 = a_7 = a_9 = 0, & \\ b_1 = b_2 = b_4 = b_5 = b_7 = b_8 = b_9 = 0, & x_1 = x_2 = x_3 = x_4 = x_6 = x_7 = x_8 = x_9 = 0, \\ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0, & y_2 = y_3 = y_4 = y_6 = y_7 = y_8 = y_9 = 0, \\ a_1 = a_3 = a_5 = a_8 = 1 = b_1 = b_6 = b_9 = c_1 = 1. & x_1 = y_1 = y_5 = 1. \end{array}$$

This is equivelant to BiHom-assossiative algebra $\mathcal{H}_3^1$. Similar observations can be applied to other cases. □

Theorem 4. *Any 3-dimensional unital BiHom-associative algebra is either associative or isomorphic to one of the following pairwise non-isomorphic BiHom-associative algebras :*

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}u_3^1$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_2,$ $e_2 \cdot e_1 = e_2,$ $e_2 \cdot e_2 = -e_1,$ $e_3 \cdot e_3 = -e_3,$	$\alpha(e_1) = e_1,\quad \beta(e_1) = -e_1,$ $\alpha(e_2) = e_2,\quad \beta(e_2) = e_1 - e_2.$
$\mathcal{H}u_3^2$	$e_1 \cdot e_1 = e_1,\quad e_2 \cdot e_3 = e_1 - e_2,$ $e_1 \cdot e_2 = e_1,\quad e_3 \cdot e_1 = e_3,$ $e_2 \cdot e_1 = e_1,\quad e_3 \cdot e_2 = e_3,$ $e_2 \cdot e_1 = e_2,\quad e_3 \cdot e_3 = e_1 - e_2,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_1.$
$\mathcal{H}u_3^3$	$e_1 \cdot e_1 = e_1,\quad e_2 \cdot e_2 = e_2,$ $e_1 \cdot e_2 = e_1,\quad e_3 \cdot e_1 = e_3,$ $e_2 \cdot e_1 = e_2,\quad e_3 \cdot e_2 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_1.$
$\mathcal{H}u_3^4$	$e_1 \cdot e_1 = e_1,\quad e_2 \cdot e_2 = e_2,$ $e_1 \cdot e_2 = e_1,\quad e_3 \cdot e_1 = e_3,$ $e_2 \cdot e_1 = e_2,\quad e_3 \cdot e_2 = e_3,$	$\alpha(e_1) = e_1,\quad \beta(e_1) = e_1,$ $\alpha(e_2) = e_2,\quad \beta(e_2) = e_1.$
$\mathcal{H}u_3^5$	$e_1 \cdot e_1 = e_1,\quad e_2 \cdot e_1 = e_2,$ $e_1 \cdot e_2 = e_1 - e_2,\quad e_3 \cdot e_1 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = -e_1,$ $\beta(e_2) = e_1 - e_2.$

Algebras	*Multiplications*	*Morphisms* α, β.
$\mathcal{H}u_3^6$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_1,$ $e_2 \cdot e_1 = e_2,$ $e_2 \cdot e_2 = e_2,$ $e_2 \cdot e_3 = e_1 - e_2,$ $e_3 \cdot e_1 = e_3,$ $e_3 \cdot e_2 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_1.$
$\mathcal{H}u_3^7$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_1,$ $e_2 \cdot e_1 = e_2,$ $e_2 \cdot e_2 = e_2 + e_3,$ $e_3 \cdot e_1 = e_3,$ $e_3 \cdot e_2 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_1.$
$\mathcal{H}u_3^8$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_1 - e_2,$ $e_2 \cdot e_1 = e_2,$ $e_3 \cdot e_1 = e_3,$ $e_3 \cdot e_2 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3$ $\beta(e_1) = -e_1,$ $\beta(e_2) = e_1 - e_2.$
$\mathcal{H}u_3^9$	$e_1 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_3,$ $e_2 \cdot e_3 = e_3,$ $e_3 \cdot e_3 = e_3,$ $e_3 \cdot e_3 = e_2 + e_3,$	$\alpha(e_1) = e_1,$ $\beta(e_1) = e_1.$
$\mathcal{H}u_3^{10}$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_2,$ $e_1 \cdot e_3 = e_3,$ $e_2 \cdot e_1 = e_2,$ $e_2 \cdot e_3 = e_2,$ $e_3 \cdot e_1 = e_3,$ $e_3 \cdot e_3 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_2,$ $\beta(e_3) = e_3.$
$\mathcal{H}u_3^{11}$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = -e_2,$ $e_2 \cdot e_3 = e_3,$ $e_3 \cdot e_3 = -e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_2) = e_2,$ $\beta(e_2) = -e_2.$
$\mathcal{H}u_3^{12}$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_3 = e_3,$ $e_2 \cdot e_2 = e_2,$ $e_3 \cdot e_2 = e_2,$	$\alpha(e_1) = e_1,$ $\beta(e_1) = e_1,$ $\beta(e_3) = e_3.$
$\mathcal{H}u_3^{13}$	$e_1 \cdot e_1 = e_1,$ $e_1 \cdot e_2 = e_2,$ $e_2 \cdot e_3 = e_3,$ $e_3 \cdot e_3 = e_3,$	$\alpha(e_1) = e_1,$ $\beta(e_1) = e_1,$ $\beta(e_2) = e_2.$
$\mathcal{H}u_3^{14}$	$e_1 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2,$ $e_2 \cdot e_3 = e_2,$ $e_3 \cdot e_1 = e_3,$	$\alpha(e_1) = e_1,$ $\alpha(e_3) = e_3,$ $\beta(e_1) = e_1.$

Proof. We give the proof of one case. Suppose that unital BiHom-associative algebra $\mathcal{D} = (\mathcal{H}u, \cdot, \alpha, \beta)$ has the following multiplication table :

$$\begin{array}{lll} e_1 \cdot e_1 = e_1, & & \\ e_1 \cdot e_2 = e_2, & e_2 \cdot e_2 = -e_1, & \alpha(e_1) = e_1, \quad \beta(e_1) = -e_1, \\ e_2 \cdot e_1 = e_2, & e_3 \cdot e_3 = -e_3, & \alpha(e_2) = e_2, \quad \beta(e_2) = e_1 - e_2. \end{array}$$

Now let us define $\mathcal{D} = (\mathcal{H}u, \cdot, \alpha, \underline{)}$ by the multiplication table :

$$\begin{array}{ll} e_1 \cdot e_1 = a_1e_1 + a_2e_2 + a_3e_3, & \\ e_1 \cdot e_2 = a_4e_1 + a_5e_2 + a_6e_3, & \\ e_1 \cdot e_3 = a_7e_1 + a_8e_2 + a_9e_3, & \alpha(e_1) = x_1e_1 + x_2e_2 + x_3e_3, \\ e_2 \cdot e_1 = b_1e_1 + b_2e_2 + b_3e_3, & \alpha(e_2) = x_4e_1 + x_5e_2 + x_6e_3, \\ e_2 \cdot e_2 = b_4e_1 + b_5e_2 + b_6e_3, & \alpha(e_3) = x_7e_1 + x_8e_2 + x_9e_3, \\ e_2 \cdot e_3 = b_7e_1 + b_8e_2 + b_9e_3, & \beta(e_1) = y_1e_1 + y_2e_2 + y_3e_3, \\ e_3 \cdot e_1 = c_1e_1 + c_2e_2 + c_3e_3, & \beta(e_2) = y_4e_1 + y_5e_2 + y_6e_3, \\ e_3 \cdot e_2 = c_4e_1 + c_5e_2 + c_6e_3, & \beta(e_3) = y_7e_1 + y_8e_2 + y_9e_3, \\ e_3 \cdot e_3 = c_7e_1 + c_8e_2 + c_9e_3, & \end{array}$$

where a_i, b_i, c_i, x_i, y_i and z_i are unknowns $(i = 1, 2, ..., 9)$. Verifying the unital BiHom-associative algebras axioms, we get the following constraints for the structure:

$$\begin{array}{ll} a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = a_9 = 0, & \\ b_1 = b_3 = b_5 = b_6 = b_7 = b_8 = b_9 = 0, & x_2 = x_3 = x_3 = x_4 = x_6 = x_7 = x_8 = x_9 = 0, \\ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0, & y_2 = y_3 = y_4 = y_6 = y_7 = y_8 = y_9 = 0, \\ a_1 = a_5 = a_5 = b_2 = 1, \quad b_4 = c_9 = -1. & x_1 = x_5 = y_1 = 1, \quad x_1 = y_5 = -1. \end{array}$$

This is equivelant to unital BiHom-assossiative algebra $\mathcal{H}u_3^1$. Similar observations can be applied to other cases. □

4. BiHom-coassociative coalgebras and BiHom-bialgebras

In this Section, we show that for a fixed dimension n, the set of BiHom-bialgebras is endowed with a structure of an algebraic variety and a natural structure transport action which describes the set of isomorphic BiHom-algebras. Solving such systems of polynomial equations leads to the classification of such structures. We shall now introduce the dual concept to BiHom-associative algebras:

Definition 4. A BiHom-coassociative coalgebra is 4-tuple $(A, \Delta, \psi, \omega)$ in which A is a linear space, $\psi, \omega : A \longrightarrow A$ and $\Delta : A \longrightarrow A \otimes A$ are linear maps, such that

$$\psi \circ \omega = \omega \circ \psi, \quad (\psi \otimes \psi) \circ \Delta = \Delta \circ \psi, \quad (\omega \otimes \omega) \circ \Delta = \Delta \circ \omega, \quad (\Delta \otimes \psi) \circ \Delta = (\omega \otimes \Delta) \circ \Delta. \tag{13}$$

We call ψ and ω (in this order) the structure maps of A.

Let V be an n-dimensional vector space over $\mathbb{K}$. Fixing a basis $\{e_i\}_{i=\{1,\dots,n\}}$ of V, a multiplication μ (resp. linear maps α, β, ψ, ω and a comultiplication Δ) is identified with its n^3, $2n^2$ structure constants $\mathcal{C}_{ij}^k \in \mathbb{K}$ (resp. a_{ji}, b_{ji}, ξ_{ji}, D_i^{jk} and γ_{ji}) where $\mu(e_i \otimes e_j) = \sum\limits_{k=1}^n \mathcal{C}_{ij}^k e_k$, $\alpha(e_i) = \sum\limits_{j=1}^n a_{ji} e_j$, $\beta(e_i) = \sum\limits_{j=1}^n b_{ji} e_j$, $\psi(e_i) = \sum\limits_{j=1}^n \xi_{ji} e_j$, $\omega(e_i) = \sum_{j=1}^n \gamma_{ji} e_j$ and $\Delta(e_i) = \sum\limits_{j,k=1}^n D_i^{jk} e_j \otimes e_k$. The counit ε is identified with its n structure constants ζ_i. We assume that e_1 is the unit.

A family $\left\{(\mathcal{C}_{ij}^k, a_{ji}, b_{ji}, \xi_{ji}, \gamma_{ji}, D_i^{jk}), \dots, i, j, k \in \{1, \dots, n\}\right\}$ represents a BiHom-coassociative coalgebra if the underlying family satisfies the appropriate conditions which translate to the following polynomial equations:

$$\begin{cases} \sum\limits_{j=1}^n (\gamma_{ji}\xi_{kj} - \xi_{ji}\gamma_{kj}) = 0, \quad \forall\, i, k \in \{1, \dots, n\}, \\ \sum\limits_{j=1}^n \sum\limits_{k=1}^n D_i^{jk} \xi_{pj} \xi_{qk} - \sum\limits_{j=1}^n \xi_{ji} D_j^{qp} = 0, \quad \forall\, i, p, q \in \{1, \dots, n\}, \\ \sum\limits_{j=1}^n \sum\limits_{k=1}^n D_i^{jk} \gamma_{pj} \gamma_{qk} - \sum\limits_{j=1}^n \gamma_{ji} D_j^{pq} = 0, \quad \forall\, i, p, q \in \{1, \dots, n\}, \\ \sum\limits_{j=1}^n \sum\limits_{k=1}^n (D_i^{jk} \xi_{pj} D_k^{qr} - D_i^{jk} D_j^{pq} \gamma_{rk}) \quad = 0, \quad \forall\, i, p, q, r \in \{1, \dots, n\}. \end{cases} \tag{14}$$

A morphism $f : (A, \Delta_A, \psi_A, \omega_A) \longrightarrow (B, \Delta_B, \psi_B, \omega_B)$ of BiHom-coassociative coalgebras is a linear map $f : A \longrightarrow B$ such that $\psi_B \circ f = f \circ \psi_A$, $\quad \omega_B \circ f = f \circ \omega_A$ and $(f \otimes f) \circ \Delta_A = \Delta_B \circ f$.

$$\begin{cases} \sum\limits_{j=1}^n (d_{ji}\xi_{kj} - \xi_{ji} d_{kj}) = 0, \quad \forall\, i, k \in \{1, \dots, n\}, \\ \sum\limits_{j=1}^n (d_{ji}\gamma_{kj} - \gamma_{ji} d_{kj}) = 0, \quad \forall\, i, k \in \{1, \dots, n\}, \\ \sum\limits_{j=1}^n \sum\limits_{k=1}^n D_i^{jk} d_{pj} d_{qk} - \sum\limits_{j=1}^n d_{ji} D_j^{qp} = 0, \quad \forall\, i, p, q \in \{1, \dots, n\}. \end{cases} \tag{15}$$

A BiHom-coassociative coalgebra $(A, \Delta, \psi, \omega)$ is called counital if there exists a linear map $\varepsilon : A \to \mathbb{K}$ (called a counit) such that

$$\varepsilon \circ \psi = \varepsilon, \quad \varepsilon \circ \omega = \varepsilon, \quad (id_A \otimes \varepsilon) \circ \Delta = \omega \quad \text{and} \quad (\varepsilon \otimes id_A) \circ \Delta = \psi. \tag{16}$$

In structure constant form , we can write it as

$$\sum_{j=1}^n \xi_{ji}\zeta_j = \zeta_i, \quad \sum_{j=1}^n \gamma_{ji}\zeta_j = \zeta_i, \quad \sum_{j=1}^n \sum_{k=1}^n D_i^{jk}\zeta_k = \sum_{j=1}^n \gamma_{ji}, \quad \sum_{j=1}^n \sum_{k=1}^n D_i^{jk}\zeta_j = \sum_{k=1}^n \xi_{ki}.$$

A morphism of counital BiHom-coassociative coalgebras $f : A \longrightarrow B$ is called counital if $f = \varepsilon_A$, where ε_A and ε_B are the counits of A and B, respectively.

Definition 5. A BiHom-bialgebra is a 7-tuple $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$, with the property that (H, μ, α, β) is a BiHom-associative algebra, $(H, \Delta, \psi, \omega)$ is a BiHom-coassociative coalgebra and the following relations are satisfied, for $h, h' \in H$:

$$\begin{aligned} \Delta(hh') = h_1h'_1 \otimes h_2h'_2, \quad \alpha \circ \psi = \psi \circ \alpha, \quad \alpha \circ \omega = \omega \circ \alpha, \\ \beta \circ \psi = \psi \circ \beta, \quad \beta \circ \omega = \omega \circ \beta, \quad (\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, \\ (\beta \otimes \beta) \circ \Delta = \Delta \circ \beta, \quad \psi(hh') = \psi(h)\psi(h'), \quad \omega(hh') = \omega(h)\omega(h'). \end{aligned} \tag{17}$$

$$\begin{cases} \sum_{k=1}^{n} \mathcal{C}_{i,j}^{k} D_{k}^{pq} - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n} D_{i}^{rs} D_{j}^{uv} \mathcal{C}_{ru}^{p} \mathcal{C}_{sv}^{q} = 0, \quad \forall\, i,j,p,q \in \{1,\dots,n\}, \\ \sum_{j=1}^{n} (\xi_{ji}a_{kj} - a_{ji}\xi_{kj}) = 0, \quad \forall\, i,k \in \{1,\dots,n\}, \\ \sum_{j=1}^{n} (\gamma_{ji}a_{kj} - a_{ji}\gamma_{kj}) = 0, \quad \forall\, i,k \in \{1,\dots,n\}, \\ \sum_{j=1}^{n} (\xi_{ji}b_{kj} - b_{ji}\xi_{kj}) = 0, \quad \forall\, i,k \in \{1,\dots,n\}, \\ \sum_{j=1}^{n} (\gamma_{ji}b_{kj} - b_{ji}\gamma_{kj}) = 0, \quad \forall\, i,k \in \{1,\dots,n\}, \\ \sum_{p=1}^{n}\sum_{q=1}^{n} D_{i}^{pq} a_{rp}a_{sq} - \sum_{j=1}^{n} a_{ji} D_{j}^{rs} = 0, \quad \forall\, i,r,s \in \{1,\dots,n\}, \\ \sum_{p=1}^{n}\sum_{q=1}^{n} D_{i}^{pq} b_{rp}b_{sq} - \sum_{j=1}^{n} b_{ji} D_{j}^{rs} = 0, \quad \forall\, i,r,s \in \{1,\dots,n\}, \\ \sum_{k=1}^{n} \mathcal{C}_{ij}^{k}\xi_{qk} - \sum_{k=1}^{n}\sum_{p=1}^{n} \xi_{ki}\xi_{pj}\mathcal{C}_{kp}^{q} = 0, \quad \forall\, i,j,q \in \{1,\dots,n\}, \\ \sum_{k=1}^{n} \mathcal{C}_{ij}^{k}\gamma_{qk} - \sum_{k=1}^{n}\sum_{p=1}^{n} \gamma_{ki}\gamma_{pj}\mathcal{C}_{kp}^{q} = 0, \quad \forall\, i,j,q \in \{1,\dots,n\}. \end{cases} \tag{18}$$

We say that H is an unital and counital BiHom-bialgebra if, in addition, it admits a unit u_H and a counit ε_H such that

$$\begin{aligned} \Delta(u_H) = u_H \otimes u_H, \quad \varepsilon_H(u_H) = 1, \quad \psi(u_H) = u_H, \quad \omega(u_H) = u_H, \\ \varepsilon_H \circ \alpha = \varepsilon_H, \quad \varepsilon_H \circ \beta = \varepsilon_H, \quad \varepsilon_H(hh') = \varepsilon_H(h)\varepsilon_H(h'), \quad \forall h, h' \in H. \end{aligned} \tag{19}$$

We have $\Delta(e_1) = e_1 \otimes e_1$, $\varepsilon(e_1) = 1$, $\psi(e_1) = e_1$, $\omega(e_1) = e_1$, $\sum_{j=1}^{n} a_{ji}\zeta_j = \zeta_i$, and $\sum_{j=1}^{n} b_{ji}\eta_j = \eta_i$.

Let us record the formula expressing the BiHom-coassociativity of Δ :

$$\Delta(h_1) \otimes \psi(h_2) = \omega(h_1) \otimes \Delta(h_2), \quad h \in H. \tag{20}$$

4.1. Classification of 2-Dimensional BiHom-bialgebras

Theorem 5. *The set of 2-dimensional unital BiHom-bialgebras yields two non-isomorphic algebras. Let $\{e_1, e_2\}$ be a basis of $\mathbb{K}^2$, then the unital BiHom-bialgebras are given by the following non-trivial comultiplications.*

1. $\Delta^2_{1,1}(e_1) = e_1 \otimes e_1$, $\Delta^2_{1,1}(e_2) = -e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 2$;
2. $\Delta^2_{1,2}(e_1) = e_1 \otimes e_1$, $\Delta^2_{1,2}(e_2) = -e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 2$;
4. $\Delta^2_{1,4}(e_1) = e_1 \otimes e_1$, $\Delta^2_{1,4}(e_2) = e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = -e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = -e_1$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = -1$;
6. $\Delta^2_{2,2}(e_1) = e_1 \otimes e_1$, $\Delta^2_{2,2}(e_2) = -e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;
7. $\Delta^2_{2,3}(e_1) = e_1 \otimes e_1$, $\Delta^2_{2,3}(e_2) = e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;
9. $\Delta^2_{4,1}(e_1) = e_1 \otimes e_1$, $\Delta^2_{3,1}(e_2) = e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;
10. $\Delta^2_{4,2}(e_1) = e_1 \otimes e_1$, $\Delta^2_{4,2}(e_2) = e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;

11. $\Delta^2_{4,3}(e_1) = e_1 \otimes e_1$, $\Delta^2_{4,3}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;
12. $\Delta^2_{4,4}(e_1) = e_1 \otimes e_1$, $\Delta^2_{4,4}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\varepsilon(e_1) = 1$;
13. $\Delta^2_{4,5}(e_1) = e_1 \otimes e_1$, $\Delta^2_{4,5}(e_2) = e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$;
14. $\Delta^2_{4,6}(e_1) = e_1 \otimes e_1$, $\Delta^2_{4,6}(e_2) = e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + 2e_2 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$.

Proof. The proof is straightforward, see [1,26] for more detail. □

Remark 6. There is no BiHom-Bialgabra whose underlying BiHom-associative algebra is given by $\mathcal{H}u^3_2$.

4.2. Classification of 3-dimensional BiHom-biallgebras

Theorem 6. *The set of 3-dimensional unital BiHom-bialgebras yields two non-isomorphic algebras. Let $\{e_1, e_2, e_3\}$ be a basis of $\mathbb{K}^3$, then the unital BiHom-bialgebras are given by the following non-trivial comultiplications.*

1. $\Delta^3_{1,2}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = -2e_1 \otimes e_1 + 2e_1 \otimes e_2 + 2e_2 \otimes e_1 - e_2 \otimes e_2$, $\Delta(e_3) = 2e_1 \otimes e_3 - e_2 \otimes e_3 + 2e_3 \otimes e_1 - e_3 \otimes e_2 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\psi(e_3) = e_1 - e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\omega(e_2) = e_1 - e_2$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;
2. $\Delta^3_{1,4}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = -2e_1 \otimes e_1 + 2e_1 \otimes e_2 + 2e_2 \otimes e_1 - e_2 \otimes e_2$, $\Delta(e_3) = 2e_1 \otimes e_3 - e_2 \otimes e_3 + 2e_3 \otimes e_1 - e_3 \otimes e_2 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\psi(e_3) = e_1 - e_2 + e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\omega(e_2) = e_1 - e_2 + e_3$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;
3. $\Delta^3_{1,5}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = -e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\Delta(e_3) = e_1 \otimes e_1 - e_1 \otimes e_2 + 2e_1 \otimes e_3 - e_2 \otimes e_1 + e_2 \otimes e_2 - e_2 \otimes e_2 + 2e_3 \otimes e_1 - e_3 \otimes e_2 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;
4. $\Delta^3_{2,3}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = -e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\Delta(e_3) = e_1 \otimes e_1 - e_1 \otimes e_2 + 2e_1 \otimes e_3 - e_2 \otimes e_1 + e_2 \otimes e_2 - e_2 \otimes e_2 + 2e_3 \otimes e_1 - e_3 \otimes e_2 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;
5. $\Delta^3_{2,4}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = 2e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_3) = -e_1 \otimes e_1 + e_2 \otimes e_3 - e_3 \otimes e_1 + e_3 \otimes e_2 - e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\psi(e_3) = e_1 - e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\omega(e_2) = e_1 - e_2\, \varepsilon(e_1) = \varepsilon(e_2) = 2$;
6. $\Delta^3_{2,5}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = 2e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_3) = -e_1 \otimes e_1 + e_2 \otimes e_3 - e_3 \otimes e_1 + e_3 \otimes e_2 - e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_1$, $\psi(e_3) = e_1 - e_2 + e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_1$, $\omega(e_2) = e_1 - e_2 + e_3$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 2$;
7. $\Delta^3_{2,6}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_1 + e_1 \otimes e_2 + e_1 \otimes e_3 + e_2 \otimes e_1 - e_2 \otimes e_2 - e_2 \otimes e_3 - e_3 \otimes e_1 + e_3 \otimes e_2$, $\Delta(e_3) = -ae_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3\, \varepsilon(e_1) = \varepsilon(e_2) = 1$;
8. $\Delta^3_{4,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;
9. $\Delta^3_{4,2}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1$, $\Delta(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1$, $\psi(e_1) = e_1$, $\omega(e_1) = e_1$, $\varepsilon(e_1) = 1$;
10. $\Delta^3_{4,3}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_1 + e_1 \otimes e_2 - e_1 \otimes e_3 + e_2 \otimes e_1 - e_2 \otimes e_3 - e_3 \otimes e_1 - e_3 \otimes e_2 + 2e_3 \otimes e_3$, $\Delta(e_3) = -e_2 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_2$, $\psi(e_1) = e_1$, $\psi(e_3) = e_1$, $\omega(e_1) = e_1$, $\omega(e_3) = e_1$, $\varepsilon(e_1) = \varepsilon(e_3) = 1$;
11. $\Delta^3_{4,4}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = ae_2 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_2$, $\Delta(e_3) = be_2 \otimes e_2 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_3) = ce_2 + e_3$, $\omega(e_1) = e_1$, $\omega(e_3) = ce_2 + e_3$, $\varepsilon(e_1) = \varepsilon(e_3) = 1$;
12. $\Delta^3_{6,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - ae_2 \otimes e_2 - e_2 \otimes e_3 - e_3 \otimes e_2$, $\Delta(e_3) = e_1 \otimes e_3 + be_2 \otimes e_2 - e_3 \otimes e_1 - e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = ce_2$, $\psi(e_3) = de_2 + e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = ce_2$, $\omega(e_3) = de_2 + e_3$, $\varepsilon(e_1) = 1$;
13. $\Delta^3_{6,2}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + 2ae_2 \otimes e_2 - e_2 \otimes e_3 - e_3 \otimes e_2$, $\Delta(e_3) = e_1 \otimes e_3 + be_2 \otimes e_2 + e_3 \otimes e_1 + 2ce_3 \otimes e_2 - 2e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = 1$;
14. $\Delta^3_{8,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, , $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$;

15. $\Delta^3_{8,2}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_3) = ae_3 \otimes e_3$, $\psi(e_1) = e_1$, , $\psi(e_2) = e_2$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = \frac{1}{a}$;
16. $\Delta^3_{10,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1$, $\Delta(e_3) = e_1 + \otimes e_3 + e_3 \otimes e_1$, $\psi(e_1) = e_1$, $\omega(e_1) = e_1$, $\varepsilon(e_1) = 1$;
17. $\Delta^3_{11,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes e_2$, $\Delta(e_3) = e_1 + \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = 1$;
25. $\Delta^3_{12,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_3) = e_1 + \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\omega(e_1) = e_1$, $\varepsilon(e_1) = 1$;
26. $\Delta^3_{13,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_3) = e_1 + \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_3) = e_3\, \omega(e_1) = e_1$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = 1$;
27. $\Delta^3_{14,1}(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_3) = e_1 + \otimes e_3 + e_3 \otimes e_1 + e_3 \otimes e_3$, $\psi(e_1) = e_1$, $\psi(e_2) = e_2$, $\psi(e_3) = e_3$, $\omega(e_1) = e_1$, $\omega(e_2) = e_2$, $\omega(e_3) = e_3$, $\varepsilon(e_1) = \varepsilon(e_2) = 1$.

Proof. The proof is straightforward, see [1,26] for more detail. □

Remark 7. There is no unital BiHom-bialgebra whose underlying unital BiHom-associative algebra is given by $\mathcal{H}u^3_1$, $\mathcal{H}u^3_5$, $\mathcal{H}u^3_7$, $\mathcal{H}u^3_8$.

We introduce the concept of BiHom-Hopf algebras.

Definition 6. Let $(H, \mu, \Delta, \alpha, \beta)$ be an unital and counital BiHom-bialgebra with a unit 1_H and a co-unit ε_H. A linear map $S : H \to H$ is called an antipode if it commutes with all the maps $\alpha, \beta, \psi, \omega$ and it satisfies the relation :

$$\psi\omega(S(h_1))\alpha\beta(h_2) = \varepsilon_H(h)1_H = \beta\psi(h_1)\alpha\omega(S(h_2)), \quad \forall\ h \in H.$$

A BiHom-Hopf algebra is a unital and counital BiHom-bialgebra with an antipode.

$$\sum_{s=1}^{n}\sum_{q,r=1}^{n}\sum_{l,p=1}^{n}\sum_{j,k=1}^{n} D_i^{jk} S_{lj} g_{pk} f_{ql} a_{rp} b_{qr} C^t_{sr} - \xi_i = 0, \quad \forall\, i,t \in \{1,n\}\,;$$

$$\sum_{s=1}^{n}\sum_{q,r=1}^{n}\sum_{l,p=1}^{n}\sum_{j,k=1}^{n} D_i^{jk} f_{lj} S_{lj} b_{ql} g_{rp} a_{sr} C^t_{qr} - \xi_i = 0, \quad \forall\, i,t \in \{1,n\}\,.$$

Corollary 1. *The BiHom-bialgebra structures on $\mathbb{K}^2$ which are BiHom-Hopf algebras are given by the following pairs of multiplication and comultiplication with the appropriate unit and counits :*

$\mathcal{H}^n u_m$	Δ^m_{ij}	ψ	ω	ε	S
$\mathcal{H}u^1_2$	$\Delta^2_{1,1}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_2$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_2$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 2$,	$S(e_1) = e_1$.
$\mathcal{H}u^2_2$	$\Delta^2_{1,2}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_2$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_2$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 2$,	$S(e_2) = e_2$.
$\mathcal{H}u^3_2$	$\Delta^2_{1,4}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = -e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = -1$,	$S(e_2) = -e_1$.
$\mathcal{H}u^4_2$	$\Delta^2_{2,2}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_1) = e_1$, $S(e_2) = e_2$.
$\mathcal{H}u^5_2$	$\Delta^2_{2,3}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_2$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_2$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_1) = e_1$, $S(e_2) = e_2$.
$\mathcal{H}u^6_2$	$\Delta^2_{4,2}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_2) = e_1$.
$\mathcal{H}u^7_2$	$\Delta^2_{4,3}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_2) = e_2$.
$\mathcal{H}u^8_2$	$\Delta^2_{4,4}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_2) = -e_1$.
$\mathcal{H}u^9_2$	$\Delta^2_{4,5}$	$\psi(e_1) = e_1$, $\psi(e_2) = e_1$,	$\omega(e_1) = e_1$, $\omega(e_2) = e_1$,	$\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 1$,	$S(e_2) = e_1$.

$\mathcal{H}^n u_m$	Δ_{ij}^m	ψ	ω	ε	S
$\mathcal{H}u_2^{10}$	$\Delta_{4,6}^2$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1,$	$S(e_1) = e_1,$ $S(e_2) = e_2.$

Corollary 2. *The BiHom-bialgebra structures on $\mathbb{K}^3$ which are BiHom-Hopf algebras are given by the following pairs of multiplication and comultiplication with the appropriate unit and conits :*

$\mathcal{H}^n u_m$	Δ_{ij}^m	ψ	ω	ε	S
$\mathcal{H}u_3^1$	$\Delta_{1,2}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$ $\psi(e_3) = e_1 - e_2,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$ $\omega(e_3) = e_1 - e_2,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1,$	$S(e_1) = e_2$ $S(e_2) = e_2,$
$\mathcal{H}u_3^2$	$\Delta_{1,4}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$ $\psi(e_2) = e_1 - e_2 + e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$ $\omega(e_2) = e_1 - e_2 + e_3,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1$	$S(e_2) = e_1 + e_2.$
$\mathcal{H}u_3^3$	$\Delta_{1,5}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_2,$ $\psi(e_2) = e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_2,$ $\omega(e_2) = e_3,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1,$	$S(e_1) = e_1,$ $S(e_2) = e_2,$
$\mathcal{H}u_3^4$	$\Delta_{2,3}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1,$	$S(e_1) = e_1 + e_2,$ $S(e_2) = e_2.$
$\mathcal{H}u_3^5$	$\Delta_{2,4}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$ $\psi(e_3) = e_1 - e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$ $\omega(e_2) = e_1 - e_2,$	$\varepsilon(e_1) = 2,$ $\varepsilon(e_2) = 2,$	$S(e_1) = e_1,$ $S(e_2) = e_2.$

$\mathcal{H}^n u_m$	Δ_{ij}^m	ψ	ω	ε	S
$\mathcal{H}u_3^6$	$\Delta_{2,5}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_1,$ $\psi(e_3) = e_1 - e_2 + e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_1,$ $\omega(e_2) = e_1 - e_2 + e_3,$	$\varepsilon(e_1) = 2,$ $\varepsilon(e_2) = 2,$	$S(e_2) = e_1,$ $S(e_2) = e_2,$ $S(e_3) = e_3..$
$\mathcal{H}u_3^7$	$\Delta_{2,6}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_2,$ $\psi(e_3) = e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_2,$ $\omega(e_3) = e_3,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = 1,$	$S(e_2) = e_1,$ $S(e_3) = e_3.$
$\mathcal{H}u_3^8$	$\Delta_{4,4}^3$	$\psi(e_1) = e_1,$ $\psi(e_3) = ae_2 + e_3,$	$\omega(e_1) = e_1,$ $\omega(e_3) = ae_2 + e_3,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_3) = 1,$	$S(e_1) = e_1,$ $S(e_3) = e_3.$
$\mathcal{H}u_3^9$	$\Delta_{4,5}^3$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_2,$ $\psi(e_3) = e_3,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_2,$ $\omega(e_3) = e_3,$	$\varepsilon(e_1) = 1,$	$S(e_2) = e_1 + e_2.$
$\mathcal{H}u_3^{10}$	$\Delta_{6,2}^2$	$\psi(e_1) = e_1,$ $\psi(e_2) = e_2,$	$\omega(e_1) = e_1,$ $\omega(e_2) = e_2,$	$\varepsilon(e_1) = 1,$ $\varepsilon(e_2) = \frac{1}{a},$	$S(e_1) = e_1,$ $S(e_2) = e_2 + e_3.$
$\mathcal{H}u_3^{11}$	$\Delta_{10,1}^3$	$\psi(e_1) = e_1,$	$\omega(e_1) = e_1,$	$\varepsilon(e_1) = 1,$	$S(e_2) = e_1,$
$\mathcal{H}u_3^{12}$	$\Delta_{11,1}^3$	$\psi(e_1) = e_1,$ $\psi(e_3) = e_3,$	$\omega(e_1) = e_1,$ $\omega(e_3) = e_3,$	$\varepsilon(e_1) = 1,$	$S(e_1) = e_1,$ $S(e_2) = e_2,$ $S(e_3) = e_3.$

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