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Fixed point results of hardy and rogers mappings in bipolar metric spaces with application to integral equations

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Abstract: Several fixed point results of mappings on bipolar metric space have been discussed in the literature, and this has become an interesting area to many researchers because of its theoretical tool to allow diverse uses in various disciplines, such as biology, game theory, engineering and so on. In this paper, we make further extension of some existing mappings in the literature to bipolar metric space and also introduce Hardy and Roger mappings on bipolar metric spaces with applications to integral equations. The results obtain generalized and complements some existing works in literature.

Keywords: bipolar metric space, Hardy and Rogers maps, covariant and contravariant maps, fixed point

MSC: 35K57, 37N25, 92D30

1. Introduction

he rising of metric space by Frechet in 1906 [1] has materialised into many generalizations and explored in the literature by different researchers. Most recently, Mutlu and Gurdal [2] made a modification of metric space by introducing a space which allows distances to be taking between elements of two different sets rather than points of a unique set. This space generalized metric space formation by switching the product of the same metric spaces with two different metric space and called this space bipolar metric space.

Some of the basic examples of such spaces are existing distance amidst points and sets in a metric space, occurring distance amidst lines and points in an Euclidean space, connection between a class of students and a set of activities, suitability measurement of habitats to species, rates of reaction of pairs from disjoint sets of chemical substances and lots more.

Bipolar metric space, have and still continue to receive lots of improvement and extension from different investigators since its inception in 2016. For example, common fixed point Theorem in bipolar metric spaces with application to integral equations was proved by Srinuvasa and Kishore [3]. Kishore et al. [4] studied common coupled fixed point results for three covariant mapping in bipolar metric spaces in which the existence and uniqueness of these theorems were proved. Gaba et al. [1] established (α, Bk) -contractions in bipolar metric spaces and Mutlu [5] made an extension on certain coupled fixed point theorems, which can be considered as generalizations of Banach fixed-point theorem. For more review on this, please refer to [2,4-15] and the references therein.

Several generalization of contraction mapping have been discovered and different applications made in literature. It remain a crucial source of metric fixed point theory which never stop receiving improvement from different researchers because of its ability to allow application in some branches of mathematics particularly, in analysis. The following are the list of some existing mappings in literature. Kannan [7] introduced a map in which continuity is not assumed and proved the existence of unique fixed point on metric spaces. Reich [14] made an extension of Kannan mapping in which some theorems on unique fixed point were proved, while Hardy and Rogers [6] introduced a generalization of a fixed point Theorem thof Banach, Kannan and Reich

contraction. Previously, Olaleru [11] introduced approximation of common fixed points of weakly compatible pairs using the Jungck iteration by considering Hardy and Rogers mapping.

Bipolar metric space has an edge over a metric space in terms of applications by giving two products of two different metric spaces rather than product of one metric space. This makes the idea of exploring Bipolar metric space to proving fixed point Theorem to be seen as a subsequent flourishing domain that remain an active research area.

Motivated by some of the existing results mentioned above, we introduce Hardy and Rogers mappings on bipolar metric spaces. We proceed with the extension of existing mappings on bipolar metric space in literature, by proving the covariant part of this mappings. We give applications of our work to integral equations.

2. Basic definitions

Some elementary definitions as related to bipolar metric spaces are stated, see [2], $\mathbb{R}^+$ is a positive real number.

Definition 1. Suppose X and V are two nonempty sets and $d : X \times V \rightarrow \mathbb{R}^+$ be the set of nonnegative real numbers satisfying the following properties:

- (1) If $d(x, v) = 0$ then $x = v$ for all $(x, v) \in X \times V$.
- (2) If $x = v$, then $d(x, v) = 0$ for all $(x, v) \in X \times V$.
- (3) $d(x, v) = d(v, x)$ for all $x, v \in X \cap V$.
- (4) $d(x_1, v_2) \leq d(x_1, v_1) + d(x_2, v_1) + d(x_2, v_2)$ for all $x_1, x_2 \in X$ and $v_1, v_2 \in V$.

The pair (X, V) is called bipolar metric and the triple (X, V, d) is called bipolar metric space.

Remark 1. (i) If axioms (2) and (3) hold in Definition 1, then d is called a bipolar pseudo-semimetric on the pair (X, V) [2].

(ii) If axioms (2),(3) and (4) hold in Definition 1, then d is called a bipolar pseudo-metric [2].

Definition 2. If (X, V, d) is a bipolar metric space, then

- (i) X is called the set of left points.
- (ii) V is called the set of right points.
- (iii) $X \cap V$ is called the set of central points.

Definition 3. A sequence $\{x_n\}$ and $\{v_n\}$ in the sets X and V respectively is called a left and right sequence.

Definition 4. Let (X_1, V_1, d_1) and (X_2, V_2, d_2) be a bipolar pseudo-semimetric spaces and $f : X_1 \cup V_1 \rightarrow X_2 \cup V_2$ be a function. If $f(X_1) \subseteq X_2$ and $f(V_1) \subseteq V_2$, then f is called covariant map, or a map from (X_1, V_1, d_1) to (X_2, V_2, d_2) and this is illustrated as $f : (X_1, V_1, d_1) \Rightarrow (X_2, V_2, d_2)$.

Definition 5. Let (X_1, V_1, d_1) and (X_2, V_2, d_2) be a bipolar pseudo-semimetric spaces and $f : X_1 \cup V_1 \rightarrow X_2 \cup V_2$ be a function. If $f(X_1) \subseteq V_2$ and $f(V_1) \subseteq X_2$, and $f : (X_1, V_1, d_1) \Rightarrow (V_2, X_2, d_2)$ then f is called contravariant map from (X_1, V_1, d_1) to (X_2, V_2, d_2) , this is illustrated as $f : (X_1, V_1, d_1) \Leftarrow (X_2, V_2, d_2)$.

Definition 6. Let (X, V, d) be a bipolar pseudo-metric space and $d_X : X \times X \rightarrow \mathbb{R}^+$ and $d_V : V \times V \rightarrow \mathbb{R}^+$ be two functions defined as

$$d_X(x_1, x_2) = \sup_{v \in V} |d(x_1, v) - d(x_2, v)| \text{ for all } x_1, x_2 \in X \text{ and}$$

$$d_V(v_1, v_2) = \sup_{x \in X} |d(x, v_1) - d(x, v_2)| \text{ for all } v_1, v_2 \in V,$$

respectively. They are called inner pseudo-metrics generated by (X, V, d) .

Definition 7. Let (X, V, d) be a bipolar metric space. If the inner pseudo-metric d_X is a metric on X , then we say V characterizes X , and if d_V is a metric, we say X characterizes V but, if X and V characterize each other, then the space (X, V, d) is called bicharacterized.

Definition 8. Let (X, V, d) be a bipolar pseudo-semimetric space. A left sequence $\{x_n\}$ converges to a right point v (that is, $(x_n \rightarrow v)$) if and only if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, v) < \varepsilon$ for all $n \geq n_0$. Also, A right sequence $\{v_n\}$ converges to a left point x (that is, $(v_n \rightarrow x)$) if and only if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x, v_n) < \varepsilon$ for all $n \geq n_0$.

It is obvious that, in a bipolar pseudo-metric space, a left sequence $\{x_n\}$ converges to a right point v if and only if $d(x_n, v) \rightarrow 0$ on $\mathbb{R}^+$ and a right sequence $\{v_n\}$ converges to a left point x if and only if $d(x, v_n) \rightarrow 0$ on $\mathbb{R}^+$.

Remark 2. Convergence in bipolar metric space differs from convergence in metric space, in a bipolar metric space, a convergent sequence may necessarily not have a unique limit.

Definition 9. Let (X_1, V_1, d_1) and (X_2, V_2, d_2) be a bipolar pseudo-semimetric spaces then,

(i) A map $f : (X_1, V_1, d_1) \rightrightarrows (X_2, V_2, d_2)$ is said to be continuous at a point $x_0 \in X_1$, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $v \in V_1$ and $d_1(x_0, v) < \delta$, $d_2(f(x_0), f(v)) < \varepsilon$. Also continuous at a point $v_0 \in V_1$, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X_1$ and $d_1(x, v_0) < \delta$, $d_2(f(x), f(v_0)) < \varepsilon$.

(ii) A contravariant map $f : (X_1, V_1, d_1) \leftrightsquigarrow (X_2, V_2, d_2)$ is said to be continuous if and only if it is continuous as covariant map.

In clear terms, (i) and (ii) implies:

A covariant or contravariant map from (X_1, V_1, d_1) to (X_2, V_2, d_2) is called continuous if and only if $\{u_n\} \rightarrow b$ on (X_1, V_1, d_1) implies $\{f(u_n)\} \rightarrow f(b)$ on (X_2, V_2, d_2) .

Remark 3. Continuity in bipolar metric space is the same as continuity in metric spaces.

Definition 10. Let (X_1, V_1, d_1) and (X_2, V_2, d_2) be bipolar metric spaces and $\lambda > 0$. A covariant map $f : (X_1, V_1, d_1) \rightrightarrows (X_2, V_2, d_2)$ such that $d(f(x), f(v)) \leq \lambda d(x, v)$ for all $x \in X_1, v \in V_1$ or a contravariant map $f : (X_1, V_1, d_1) \leftrightsquigarrow (X_2, V_2, d_2)$ such that $d(f(v), f(x)) \leq \lambda d(x, v)$ for all $x \in X_1, v \in V_1$ is called Lipschitz continuous.

Remark 4. (i) If in Definition 10, $\lambda = 1$, then the covariant or contravariant map is called non-expansive (see [2]).

(ii) If in Definition 10, $\lambda \in (0, 1)$, then the covariant or contravariant map is called a contraction (see [2]).

Definition 11. A left or right sequence in a bipolar metric space, is simply refer to as sequence.

Definition 12. Let (X, V, d) be a bipolar pseudo-semimetric space.

(i) A sequence $(\{x_n\}, \{v_n\})$ on the set $X \times V$ is called a bisequence on (X, V, d) .

(ii) If both $\{x_n\}$ and $\{v_n\}$ converge, then the bisequence $\{x_n, v_n\}$ is said to be convergent.

(iii) If $\{x_n\}$ and $\{v_n\}$ both converge to same point $u \in X \cap V$, then the bisequence is said to be biconvergent.

(iv) A bisequence $\{x_n, v_n\}$ on (X, V, d) is called Cauchy bisequence, if for each $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$, so that for each positive integers $n, m \geq n_0$, $d(x_n, v_m) < \varepsilon$.

In a bipolar metric space, every convergent Cauchy bisequence is biconvergent and in a bipolar pseudo-metric space, every biconvergent bisequence is a Cauchy bisequence.

Definition 13. A bipolar metric space is known to be complete if all Cauchy bisequence in this space is convergent.

Remark 5. Completeness in a bipolar metric spaces differs from completeness in metric spaces via bisequences, if it is equivalent, then a bipolar metric space (X, V, d) will represents a metric space, that is $X = V$ [2].

Definition 14. A bipolar metric space (X, V, d) is known to be bicomplete, if the pair of inner pseudo-metric spaces (X, d_X) and (V, d_V) are complete metric spaces.

Proposition 1. Suppose a central point is the limit of a sequence, it implies that, the sequence has a unique limit.

Example 1. [2]. Assume that $X = \{\{x\} : x \in \mathbb{R}\}$ and $V = \{B \subseteq \mathbb{R} : B \text{ is nonempty, compact}\}$ and define $d : X \times V \rightarrow \mathbb{R}_+$ as

$$d(x, B) = |x - \inf(B)| + |x - \sup(B)|.$$

It implies that, the triple (X, V, d) is a complete bipolar metric space.

Proposition 2. . Suppose the point in which a covariant or contravariant map is left and right continuous is central point, it implies that the map is continuous at this point.

Lemma 1. [2]. Assume (X, V, d) is a bipolar metric space.

(i) Suppose (X, V, d) is bicharacterized, then it follows that every convergent sequence has a unique limit.

(ii) Suppose a central point is a limit of a sequence, then it follows that, it is the unique limit of the sequence.

Proof. The proof will be justify based on left sequences which follows same approach as the right sequence.

(i) Assume $\{x_n\}$ to be the left sequence in such that both $\{x_n\} \rightarrow v_1 \in V$ and $\{x_n\} \rightarrow v_2 \in V$. Then for each $x \in X$, it implies that $d(x, v_2) \leq d(x, v_1) + d(x_n, v_1) + d(x_n, v_2)$ and $d(x, v_1) \leq d(x, v_2) + d(x_n, v_2) + d(x_n, v_1)$. But, $\lim_n d(x_n, v_1) = \lim_n d(x_n, v_2) = 0$ on $\mathbb{R}^+$, $d(x, v_1) = d(x, v_2)$ for every $x \in X$. Hence $d_V(v_1, v_2) = \sup_{x \in X} |d(x, v_1) - d(x, v_2)| = 0$ this with the fact that X characterizes V , implies that d_V is a metric so that $v_1 = v_2$.

(ii) Assume that $\{x_n\}$ is a left sequence and $\{x_n\} \rightarrow u \in X \cap V$. Also, suppose $\{x_n\} \rightarrow v \in V$, then $d(u, v) \leq d(u, u) + d(x_n, u) + d(x_n, v)$ and the fact that $\lim_n d(x_n, u) = \lim_n d(x_n, v) = 0$, it therefore implies that $d(u, v) = 0$, hence $u = v$. $\square$

3. Main results

Main purpose of this section is to prove fixed point theorems of Hardy and Rogers mapping in complete bipolar metric space for both covariant and contravariant maps. Its also show that, the obtained results generalize and modified some existing ones in literature.

Theorem 1. Let (X, V, d) be a complete bipolar metric space, and let $f : (X, V, d) \rightleftharpoons (X, V, d)$ be a self contravariant map satisfying,

$$d(f(v), f(x)) \leq a_1 d(x, v) + a_2 d(x, f(x)) + a_3 d(f(v), v) + a_4 d(x, f(v)) + a_5 d(v, f(x)), \quad (1)$$

for all $(x, v) \in X \times V$, and $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1$, where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$. Then the function $f : X \cup V \rightarrow X \cup V$ has a unique fixed point.

Proof. Let $x_0 \in X$ and $v_0 \in V$, for each nonnegative integer n , we define $v_n = f(x_n)$ and $x_{n+1} = f(v_n) = v_n$, to have

$$\begin{aligned} d(x_n, v_n) &= d(f(v_{n-1}), f(x_n)) \\ &\leq a_1 d(x_n, v_{n-1}) + a_2 d(x_n, f(x_n)) + a_3 d(f(v_{n-1}), v_{n-1}) + a_4 d(x_n, f(v_{n-1})) + a_5 d(v_{n-1}, f(x_n)) \\ &= a_1 d(x_n, v_{n-1}) + a_2 d(x_n, v_n) + a_3 d(x_n, v_{n-1}) + a_4 d(x_n, v_{n-1}) + a_5 d(x_n, v_n) \\ &= (a_1 + a_3 + a_4) d(x_n, v_{n-1}) + (a_2 + a_5) d(x_n, v_n). \end{aligned} \quad (2)$$

That is, (2) implies

$$(1 - a_2 - a_5) d(x_n, v_n) \leq (a_1 + a_3 + a_4) d(x_n, v_{n-1}).$$

Then

$$d(x_n, v_n) \leq \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_5)} d(x_n, v_{n-1}). \quad (3)$$

Also, we have

$$\begin{aligned} d(x_n, v_{n-1}) &= d(f(v_{n-1}), f(x_{n-1})) \\ &\leq a_1 d(x_{n-1}, v_{n-1}) + a_2 d(x_{n-1}, f(x_{n-1})) + a_3 d(f(v_{n-1}), v_{n-1}) + a_4 d(x_{n-1}, f(v_{n-1})) \\ &\quad + a_5 d(v_{n-1}, f(x_{n-1})) \\ &= a_1 d(x_{n-1}, v_{n-1}) + a_2 d(x_{n-1}, v_{n-1}) + a_3 d(x_n, v_{n-1}) + a_4 d(x_{n-1}, v_{n-1}) + a_5 d(x_n, v_{n-1}) \\ &= (a_1 + a_2 + a_4) d(x_{n-1}, v_{n-1}) + (a_3 + a_5) d(x_n, v_{n-1}), \end{aligned} \quad (4)$$

that is, (4) implies

$$(1 - a_3 - a_5) d(x_n, v_{n-1}) \leq (a_1 + a_2 + a_4) d(x_{n-1}, v_{n-1}).$$

Then

$$d(x_n, v_{n-1}) \leq \frac{(a_1 + a_2 + a_4)}{1 - a_3 - a_5} d(x_{n-1}, v_{n-1}). \quad (5)$$

Setting

$$\lambda = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_5)}, \quad (6)$$

$$\beta = \frac{(a_1 + a_2 + a_4)}{1 - a_3 - a_5}, \quad (7)$$

thus $\lambda < 1$ also $\beta < 1$ seeing that $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

It is easy to note from (3) and (5) that for any $n \in \mathbb{N}$, we have

$$d(x_n, v_n) \leq \lambda^{2n} d(x_0, v_0),$$

and

$$d(x_n, v_{n-1}) \leq \beta^{2n-1} d(x_0, v_0). \quad (8)$$

Thus, for any positive integers m and n , the following cases hold:

Case I: If $m > n$

$$\begin{aligned} d(x_n, v_m) &\leq d(x_n, v_n) + d(x_{n+1}, v_n) + d(x_{n+1}, v_m) \\ &\leq (\lambda^{2n} + \beta^{2n+1}) d(x_0, v_0) + d(x_{n+1}, v_m) \\ &\vdots \\ &\leq (\lambda^{2n} + \beta^{2n+1} + \dots + \beta^{2m}) d(x_0, v_0). \end{aligned}$$

Since $\lambda, \beta < 1$, it follows easily that

$$\lim_{n,m} \rightarrow \infty d(x_n, v_m) = 0.$$

Case II: If $m < n$

$$d(x_n, v_m) \leq d(x_{m+1}, v_m) + d(x_{m+1}, v_{m+1}) + d(x_n, v_{m+1})$$

$$\begin{aligned}
&\leq (\beta^{2m+1} + \lambda^{2m+2}) d(x_0, v_0) + d(x_n, v_{m+1}) \\
&\vdots \\
&\leq (\beta^{2m+1} + \lambda^{2m+2} + \beta^{2m+1} + \dots + \beta^{2n}) d(x_0, v_0) + d(x_n, v_n) \\
&\leq (\beta^{2m+1} + \lambda^{2m+2} + \beta^{2m+1} + \dots + \beta^{2n} + \lambda^{2n}) d(x_0, v_0).
\end{aligned}$$

Also, since $\lambda, \beta < 1$, it follows easily that $\lim_{n,m} d(x_n, v_m) = 0$.

Given $\lambda, \beta \in (0, 1)$, it implies that $d(x_n, v_m)$ can be taken arbitrarily small for large m and n , and hence, $\{x_n, v_m\}$ is a Cauchy bisequence in (X, V) .

Taking u to be the point which $(\{x_n\}, \{v_m\})$ biconverges to and following Lemma 1, then $\{x_n\} \rightarrow u$, $\{v_n\} \rightarrow u$ and $u \in X \cap V$. More also, $fx_n = (x_n) \rightarrow u$, $d(fu, fx_n) \rightarrow d(fu, u)$, and we have from Theorem 1 that, (X, V, d) is complete, it implies $(\{x_n\}, \{v_m\})$ converges and as a convergence Cauchy bisequence, implying that it is biconverges. Suppose $\{x_n\} \rightarrow u$, $\{v_n\} \rightarrow u$, where $u \in X \cap V$. Then it follows that there exist a unique limit in $\{x_n\}$ and $\{v_n\}$. Also, the fact that the contravariant map is continuous means $\{x_n\} \rightarrow u$ and this implies that $\{v_n\} = (f(x_n)) \rightarrow f(u)$, this together with $\{v_n\} \rightarrow u$ gives $f(u) = u$.

Suppose also that v is a fixed point of f , then $f(v) = v$ and this implies $v \in X \cap V$ and hence

$$d(fu, fx_n) \leq \lambda d(x_n, fx_n) + d(fu, u) = \lambda(d(x_n, v_n)) + d(fu, u),$$

and this in turn implies $d(fu, u) \leq \lambda d(fu, u)$. Therefore $fu = u$, it follows that a unique fixed point holds on f .

If b is taking to be any fixed point of f , then $fb = b$ implies b is in $X \cap V$. Then

$$d(u, b) = d(fu, fb) \leq \lambda(d(u, fu)) + d(fb, b) = \lambda(d(u, b) + d(b, b)) = 0. \quad (9)$$

Hence $u = b$, and this implies a unique fixed point exists. $\square$

Remark 6. In respect to Lemma 1, the following holds

(i) If $X = V$ in Theorem 1, then we have the results of Hardy and Rogers [6].

(ii) If $X = V$ and $a_4 = a_5 = 0$ in Theorem 1, then we have the results of Reich [14].

(iii) If $X = V$ and $a_1 = a_4 = a_5 = 0$ in Theorem 1, then we have the results of Kannan [7].

(iv) Following all the hypothesis of Theorem 1, and letting $a_4 = a_5 = 0$. Inequality (1) reduces to the results obtained in Theorem 1 of [5].

(v) Following all the hypothesis of Theorem 1, and letting $a_1 = a_4 = a_5 = 0$. Inequality (1) reduces to the results obtained in Theorem 5.6 of [2].

Theorem 2. Let (X, V, d) be a complete bipolar metric space, and let $f : (X, V, d) \rightrightarrows (X, V, d)$ be a self covariant map satisfying,

$$d(f(x), f(v)) \leq a_1 d(x, v) + a_2 d(x, f(x)) + a_3 d(v, f(v)) + a_4 d(x, f(v)) + a_5 d(v, f(x)), \quad (10)$$

for all $(x, v) \in X \times V$, and $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1$, where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$. Then a unique fixed point exists on the function $f : X \cup V \rightarrow X \cup V$.

Proof. Let $x_0 \in X$ and $v_0 \in V$ for each nonnegative integer n . Assume $a_1 = c, a_2 + a_3 = 2a$ and $a_4 + a_5 = 2b$, then Eq. (10) can be of the form

$$d(f(x), f(v)) \leq cd(x, v) + ad(x, f(x)) + ad(v, f(v)) + bd(x, f(v)) + bd(v, f(x)),$$

where $2a + 2b + c < 1$, we define $x_{n+1} = f(x_n)$ and $v_{n+1} = f(v_n)$. Then we have

$$\begin{aligned}
d(x_n, v_n) &= d(f(x_{n-1}), f(v_{n-1})) \\
&\leq cd(x_{n-1}, v_{n-1}) + ad(x_{n-1}, f(x_{n-1})) + ad(v_{n-1}, f(v_{n-1})) + bd(x_{n-1}, f(v_{n-1})) + bd(v_{n-1}, f(x_{n-1})) \\
&= cd(x_{n-1}, v_{n-1}) + ad(x_{n-1}, x_n) + ad(v_{n-1}, v_n) + bd(x_{n-1}, v_n) + bd(v_{n-1}, x_n)
\end{aligned}$$

$$=cd(x_{n-1}, v_{n-1}) + ad(x_{n-1}, v_{n-1}) + ad(x_n, v_n) + bd(x_{n-1}, v_{n-1}) + bd(x_n, v_n). \quad (11)$$

That is, (11) implies

$$d(x_n, v_n) \leq \frac{(c+a+b)}{1-a-b} d(x_{n-1}, v_{n-1}). \quad (12)$$

Also, we have

$$\begin{aligned} d(x_n, v_{n+1}) &= d(f(x_{n-1}), f(v_n)) \\ &\leq cd(x_{n-1}, v_n) + ad(x_{n-1}, f(x_{n-1})) + ad(v_n, f(v_n)) + bd(x_{n-1}, f(v_n)) + bd(v_n, f(x_{n-1})) \\ &= cd(x_{n-1}, v_n) + ad(x_{n-1}, x_n) + ad(v_n, v_{n+1}) + bd(x_{n-1}, v_{n+1}) + bd(v_n, x_n) \\ &= cd(x_{n-1}, v_n) + ad(x_{n-1}, v_n) + ad(x_n, v_{n+1}) + bd((x_{n-1}, v_n)) + bd(x_n, v_{n+1}). \end{aligned} \quad (13)$$

Then, (13) implies

$$d(x_n, v_{n+1}) \leq \frac{(c+a+b)}{1-a-b} d(x_{n-1}, v_n). \quad (14)$$

Setting

$$\lambda' = \frac{(a_1 + a_2 + a_5)}{(1 - a_3 - a_4)}, \quad (15)$$

$$\beta' = \frac{(a_1 + a_2 + a_4)}{1 - a_3 - a_5}. \quad (16)$$

Then $\lambda' < 1$ and $\beta' < 1$ since $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

It is easy to see from (12) and (14) that for any $n \in \mathbb{N}$, we have

$$d(x_n, v_n) \leq \lambda'^{2n} d(x_0, v_0),$$

and

$$d(x_n, v_{n+1}) \leq \beta'^{2n+1} d(x_0, v_1). \quad (17)$$

Thus, for every positive integers m and n , the following cases are satisfied:

Case I: If $m > n$

$$\begin{aligned} d(x_n, v_m) &\leq d(x_n, v_n) + d(x_{n+1}, v_n) + d(x_{n+1}, v_m) \\ &\leq (\lambda'^{2n} + \beta'^{2n+1})d(x_0, v_1) + d(x_{n+1}, v_m) \\ &\vdots \\ &\leq (\lambda'^{2n} + \beta'^{2n+1} + \dots + \beta'^{2m})d(x_0, v_1). \end{aligned}$$

Since $\lambda', \beta' < 1$, it follows easily that

$$\lim_{n,m} \rightarrow \infty d(x_n, v_m) = 0.$$

Case II: If $m < n$

$$\begin{aligned} d(x_n, v_m) &\leq d(x_{m+1}, v_m) + d(x_{m+1}, v_{m+1}) + d(x_n, v_{m+1}) \\ &\leq (\beta'^{2m+1} + \lambda'^{2m+2})d(x_0, v_1) + d(x_n, v_{m+1}) \\ &\vdots \\ &\leq (\beta'^{2m+1} + \lambda'^{2m+2} + \beta'^{2m+1} + \dots + \beta'^{2n})d(x_0, v_1) + d(x_n, v_n) \end{aligned}$$

$$\leq (\beta^{2m+1} + \lambda^{2m+2} + \beta^{2m+1} + \cdots + \beta^{2n} + \lambda^{2n})d(x_0, v_1).$$

Also, since $\lambda', \beta' < 1$, it follows easily that

$$\lim_{n,m} \rightarrow \infty d(x_n, v_m) = 0.$$

Given $\lambda', \beta' \in (0, 1)$, it implies that $d(x_n, v_m)$ can be taken arbitrarily small for large m and n , and hence, $(\{x_n\}, \{v_m\})$ is a Cauchy bisequence in (X, V) , and we have from Theorem 2 that, (X, V, d) is complete, it implies $(\{x_n\}, \{v_m\})$ converges and hence biconvergent, since it is convergent Cauchy bisequence.

Taking u to be the point which $(\{x_n\}, \{v_m\})$ biconverges to and $f(x_n) = x_{n+1} \rightarrow u \in X \cap V$ this together with Lemma 1 justify that $f(x_n)$ has a unique limit. But the map f is continuous, $(f(x_n)) \rightarrow f(u)$, hence $f(u) = u$. And we conclude that u is a fixed point of f .

If b is taking to be any fixed point of f , then $fb = b$ implies b is in $X \cap V$. Then

$$d(u, b) = d(fu, fb) \leq \lambda'(d(u, fu)) + d(fb, b) = \lambda'(d(u, b) + d(b, b)) = 0. \quad (18)$$

Hence $u = b$, this implies a unique fixed point exists. $\square$

Remark 7. Under the consideration of double mappings and taking $X = V$, in such case $(X, V, d) = (X, d)$, in Theorem 2, the obtained result in Theorem 2 reduces to the results obtained in Theorem 2.1 of [11] under metric space.

Corollary 1. Let (X, V, d) be a complete bipolar metric space, and let $f : (X, V, d) \rightrightarrows (X, V, d)$ be a self covariant map satisfying

$$d(f(x), f(v)) \leq a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(v, f(v)), \quad (19)$$

for all $(x, v) \in X \times V$, and $0 \leq a_1 + a_2 + a_3 < 1$, where $a_1, a_2, a_3 \in [0, 1)$. Then the function $f : X \cup V \rightarrow X \cup V$ has a unique fixed point.

Proof. Following the same method as seen in Theorem 2, the required results is obtained. $\square$

Corollary 2. Let (X, V, d) be a complete bipolar metric space and let $f : (X, V, d) \rightrightarrows (X, V, d)$ be a self covariant map satisfying

$$d(f(x), f(v)) \leq \alpha [d(x, f(x)) + d(v, f(v))], \quad (20)$$

for all $(x, v) \in X \times V$, and $\alpha \in (0, \frac{1}{2})$. Then the function $f : X \cup V \rightarrow X \cup V$ has a unique fixed point.

Proof. The proofs follow as seen in Theorem 2. $\square$

Theorem 3. Let (X, V, d) be a complete bipolar metric space and $f : (X, V, d) \leftrightsquigarrow (X, V, d)$ be a self contravariant map. Suppose the following conditions hold:

$$d(f(v), f(x)) \leq \lambda \max \left\{ d(x, v), d(x, f(x)), d(f(v), v), \frac{d(v, f(x)) + d(x, f(v))}{2} \right\}, \quad (21)$$

for all $(x, v) \in X \times V$ and $0 < \lambda < 1$. Then the function $f : X \cup V \rightarrow X \cup V$ has a unique fixed point.

Proof. Let $x_0 \in X$ and $v_0 \in V$, for each nonnegative integer n . We define $v_n = f(x_n)$ and $x_{n+1} = f(v_n) = v_n$, we want to show that

$$d(x_n, v_{n-1}) \leq \lambda d(x_{n-1}, v_{n-1}), \quad n \geq 1. \quad (22)$$

From (21), we obtain

$$\begin{aligned}
 d(x_n, v_{n-1}) &= d(f(v_{n-1}), f(x_{n-1})) \\
 &\leq \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_{n-1}, f(x_{n-1})), d(f(v_{n-1}), v_{n-1}), \frac{d(v_{n-1}, f(x_{n-1})) + d(x_{n-1}, f(v_{n-1}))}{2} \right\} \\
 &= \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_{n-1}, v_{n-1}), d(x_n, v_{n-1}), \frac{d(x_{n-1}, v_{n-1}) + d(x_{n-1}, v_{n-1})}{2} \right\} \\
 &= \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_n, v_{n-1}), \frac{d(x_{n-1}, v_{n-1})}{2} \right\}. \tag{23}
 \end{aligned}$$

Next is to examine each of the possibilities.

Case 1. Suppose,

$$d(x_n, v_{n-1}) = \lambda d(x_{n-1}, v_{n-1}).$$

Then,

$$d(x_n, v_{n-1}) \leq \lambda d(x_{n-1}, v_{n-1}) \leq \lambda^2 d(x_{n-2}, v_{n-2}) \leq \lambda^3 d(x_{n-3}, v_{n-3}).$$

Hence, it follows that

$$d(x_n, v_{n-1}) \leq \lambda^n d(x_0, v_0), \quad n \geq 1.$$

Case 2. Suppose,

$$d(x_n, v_{n-1}) = \lambda d(x_n, v_{n-1}).$$

Then,

$$\begin{aligned}
 (1 - \lambda)d(x_n, v_{n-1}) &= 0 \\
 d(x_n, v_{n-1}) &= 0.
 \end{aligned}$$

Case 3. Suppose,

$$d(x_n, v_{n-1}) = \lambda \frac{d(x_{n-1}, v_{n-1})}{2}.$$

Then,

$$d(x_n, v_{n-1}) = \frac{\lambda}{2} d(x_{n-1}, v_{n-1}). \tag{24}$$

For all $x_0 \in X$ and $n \geq 1$. Following the same argument as used in Theorem 1 its seen that $\{x_n, v_n\}$ is a Cauchy bisequence in (X, V, d) and hence biconvergent, since it is complete. $\square$

Remark 8. Under the consideration of double mappings and taking $X = V$ in Theorem 3, the obtained result in Theorem 3 reduces to the results obtained in Theorem 2.5 of [11] under metric space consideration.

Theorem 4. Let (X, V, d) be a complete bipolar metric space and $f : (X, V, d) \rightrightarrows (X, V, d)$ be a self covariant map. Suppose the following conditions hold:

$$d(f(x), f(v)) \leq \lambda \max \left\{ d(x, v), d(x, f(x)), d(v, f(v)), \frac{d(v, f(x)) + d(x, f(v))}{2} \right\}, \tag{25}$$

for all $(x, v) \in X \times V$ and $0 < \lambda < 1$. Then the function $f : X \cup V \longrightarrow X \cup V$ has a unique fixed point.

Proof. Let $x_0 \in X$ and $v_0 \in V$, for each nonnegative integer n . We define $x_{n+1} = f(x_n) = v_n$ and $v_{n+1} = f(v_n)$ we want to show that

$$d(x_n, v_n) \leq \lambda d(x_{n-1}, v_{n-1}), \quad n \geq 1. \tag{26}$$

From (25), we obtain

$$\begin{aligned}
 d(x_n, v_n) &= d(f(x_{n-1}), f(v_{n-1})) \\
 &\leq \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_{n-1}, f(x_{n-1})), d((v_{n-1}), f(v_{n-1})), \frac{d(v_{n-1}, f(x_{n-1})) + d(x_{n-1}, f(v_{n-1}))}{2} \right\} \\
 &= \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_{n-1}, v_{n-1}), d(x_n, v_n), \frac{d(v_{n-1}, v_{n-1}) + d(x_{n-1}, v_n)}{2} \right\} \\
 &= \lambda \max \left\{ d(x_{n-1}, v_{n-1}), d(x_n, v_n), \frac{d(x_{n-1}, v_n)}{2} \right\}. \tag{27}
 \end{aligned}$$

Next is to examine each of the possibilities.

Case 1. Suppose,

$$d(x_n, v_n) = \lambda d(x_{n-1}, v_{n-1}).$$

Then,

$$d(x_n, v_n) \leq \lambda d(x_{n-1}, v_{n-1}) \leq \lambda^2 d(x_{n-2}, v_{n-2}) \leq \lambda^3 d(x_{n-3}, v_{n-3}).$$

Hence, it follows that

$$d(x_n, v_n) \leq \lambda^n d(x_0, v_0), \quad n \geq 1.$$

Case 2. Suppose,

$$d(x_n, v_n) = \lambda d(x_n, v_n).$$

Then,

$$(1 - \lambda)d(x_n, v_n) = 0.$$

Case 3. Suppose,

$$d(x_n, v_n) = \lambda \frac{d(x_{n-1}, v_n)}{2}.$$

Then,

$$d(x_n, v_n) = \frac{\lambda}{2} [d(x_{n-1}, v_n)].$$

For all $x_0 \in X$ and $n \geq 1$. Following the same argument as used in Theorem 2 its seen that $\{x_n, v_n\}$ is a Cauchy bisequence in (X, V, d) and hence biconvergent, since it is complete. $\square$

Example 2. Let $X = \{\{x\} : x \in \mathbb{R}\}$ be the class of all singleton subsets of $\mathbb{R}$, and V the class of all nonempty compact subsets of $\mathbb{R}$. Define $d : X \times V \rightarrow \mathbb{R}^+$ as

$$d(x, B) = |x - \inf B| + |x - \sup B|.$$

Then (X, V, d) is a complete bipolar metric space. Define a contravariant map $T : (X, V, d) \rightrightarrows (X, V, d)$ as $TB = \left\{ \frac{\inf(B) + \sup(B)}{8} \right\} \subset \mathbb{R}$ for all $B \in X \cup V$, satisfying the inequality $d(Tv, Tx) \leq \gamma(d(x, Tx)) + d(Tv, v)$ for $\gamma = \frac{1}{2}$. That is, for any $x \in X$ and $v \in V$, the following are satisfies $Tx = Tv = \{2\}$, and since $\{2\} \in X \subset V$, we have

$$d(Tv, Tx) = d(\{2\}, \{2\}) = |2 - 2| + |2 - 2| = 0.$$

Hence, for any $0 \leq \gamma < 1$,

$$d(Tv, Tx) = 0 \leq \gamma d(x, Tx) + d(Tv, v).$$

Hence, a unique fixed point must exist in T , which is the set $\{2\} \in X \cap V$.

4. Application

We consider the existence and unique solution of an integral equation as an application of Theorem 1. Consider the integral equation

$$\alpha(x) = f(x) + \int_{U \cup B} (\gamma(x, v, \alpha(v))) dv, x, v \in U \cup B, \quad (28)$$

where $U \cup B$ is a Lebesgue measurable set with $m(U \cup B) < \infty$.

Assume that

(1) $\gamma : (U^2 \cup B^2) \times [0, \infty) \rightarrow [0, \infty)$ and $f \in L^\infty(U) \cup L^\infty(B)$.

(2) There is a continuous function $\varphi : U^2 \cup B^2 \rightarrow [0, \infty)$ and $\lambda \in (0, 1)$ such that for all $(x, v) \in U^2 \cup B^2$.

$$|\gamma(x, v, \alpha(v)) - \gamma(x, v, \beta(v))| \leq a_1|\alpha(x) - \beta(v)| + a_2|\alpha(x) - f\alpha(x)| + a_3|f\beta(v) - \beta v| + a_4|\alpha(x) - f\beta(v)| + a_5|\beta(v) - f\alpha(x)|\varphi(x, v), \quad (29)$$

for all $x, v \in U^2 \cup B^2, \alpha, \beta \in \mathbb{R}$.

(3) $\|\int_{U \cup B} \varphi(x, v) dv\| \leq 1$, that is, $\sup_{n \in U \cup B} \int_{U \cup B} |\varphi(n, m)| dm \leq 1$.

Then, the integral equations have a unique solution in $L^\infty(U) \cup L^\infty(B)$.

Proof. Let $X = L^\infty(U)$ and $V = L^\infty(B)$ be two normed linear spaces, where U, B are Lebesgue measurable sets with $m(U \cup B) < \infty$.

We consider $d : X \times V \rightarrow [0, \infty)$ defined as, $d(h, g) = \|h - g\|_\infty$ for all $h, g \in X \times V$. Then (X, V, d) is a complete bipolar metric spaces. Define the contravariant contraction $H : L^\infty(U) \cup L^\infty(B) \rightarrow L^\infty(U) \cup L^\infty(B)$ by

$$H(\alpha(v)) = \int_{U \cup B} \gamma(x, v, \alpha(v)) dv + f(x)x, v \in U \cup B. \quad (30)$$

Now, we have

$$\begin{aligned} d(H(\alpha(v)), H\beta(v)) &= \|H\alpha(v) - H\beta(v)\| \\ &= \left| \int_{U \cup B} \gamma(x, v, \alpha(v)) dv + f(x) - \left(\int_{U \cup B} \gamma(x, v, \beta(v)) dv + f(x) \right) \right| \\ &= \left| \int_{U \cup B} \gamma(x, v, \alpha(v)) - \int_{U \cup B} \gamma(x, v, \beta(v)) dv \right| \\ &\leq \int_{U \cup B} |\gamma(x, v, \alpha(v)) - \gamma(x, v, \beta(v))| dv \\ &\leq (a_1|\alpha(x) - \beta(v)| + a_2|\alpha(x) - f\alpha(x)| + a_3|f\beta(v) - \beta(v)| \\ &\quad + a_4|\alpha(x) - f\beta(v)| + a_5|\beta(v) - f\alpha(x)|)\varphi(x, v) \\ &\leq a_1\|\alpha(x) - \beta(v)\| + a_2\|\alpha(x) - f\alpha(x)\| + a_3\|f\beta(v) - \beta(v)\| \\ &\quad + a_4\|\alpha(x) - f\beta(v)\| + a_5\|\beta(v) - f\alpha(x)\| \int_{U \cup B} |\varphi(x, v)| dv \\ &\leq a_1\|\alpha(x) - \beta(v)\| + a_2\|\alpha(x) - f\alpha(x)\| + a_3\|f\beta(v) - \beta(v)\| \\ &\quad + a_4\|\alpha(x) - f\beta(v)\| + a_5\|\beta(v) - f\alpha(x)\| \sup_{n \in U \cup B} \int_{U \cup B} |\varphi(n, m)| dm \\ &\leq a_1\|\alpha(x) - \beta(v)\| + a_2\|\alpha(x) - f\alpha(x)\| + a_3\|f\beta(v) - \beta(v)\| \\ &\quad + a_4\|\alpha(x) - f\beta(v)\| + a_5\|\beta(v) - f\alpha(x)\|. \\ &= a_1d(\alpha(x), \beta(v)) + a_2d(\alpha(x), f\alpha(x)) + a_3d(f\beta(v), \beta(v)) + a_4d(\alpha(x), f\beta(v)) + a_5d(\beta(v), f\alpha(x)). \end{aligned}$$

It follows from Theorem 1 that, the contravariant map defined in (30) has a unique solution. $\square$

Conflicts of Interest: The authors declare no conflict of interest.

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