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Laguerre-type 2nd kind hypergeometric Bernoulli polynomials and numbers

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Abstract: We introduce the Laguerre-type 2nd kind hypergeometric Bernoulli polynomials and numbers. After showing their recursive computation, we exploit the Laguerre-type Blissard problem to derive a representation formula of the relevant numbers in terms of Bell’s polynomials.

Keywords: Laguerre-type exponentials, 2nd kind hypergeometric Bernoulli polynomials and numbers, Bell’s polynomials, Blissard’s problem

MSC: Primary 11B83; Secondary 33B10, 11B73, 33C15, 33C99.

1. Introduction

In a preceding article [1], we have considered the 2nd kind hypergeometric Bernoulli polynomials, and the relevant numbers, according to what follows.

Let

$$T_r(x) := \sum_{\ell=0}^{r-1} \frac{x^\ell}{\ell!}, \quad (1)$$

that is the r th partial sum of the exponential series, which is polynomial of degree $r - 1$. Recalling the Stirling numbers of the second kind which can be computed as

$$S(n, k) = \frac{1}{k!} \sum_{m=0}^k (-1)^{k-m} \binom{k}{m} m^n, \quad (2)$$

and the r -associate Stirling numbers of the second kind $S(n, k; r)$, defined by

$$\left(\sum_{\ell=r}^{\infty} \frac{x^\ell}{\ell!} \right)^k = (e^x - T_r(x))^k = k! \sum_{n=rk}^{\infty} S(n, k; r) \frac{x^n}{n!}, \quad (3)$$

we find as a particular case: $S(n, k; 1) = S(n, k)$.

Then the 2nd kind hypergeometric Bernoulli polynomials are defined by the generating function:

$$\frac{\frac{x^r}{r!} e^{tx}}{e^x - T_r(x)} = \frac{e^{tx}}{{}_1F_1(1, r+1, x)} = \sum_{n=0}^{\infty} B_n^{[r-1, 1]}(t) \frac{x^n}{n!}, \quad (4)$$

and, by definition, the relevant numbers are their values at the origin:

$$B_n^{[r-1, 1]} := B_n^{[r-1, 1]}(0). \quad (5)$$

We recall that the study of Bernoulli numbers, together with Stirling numbers, is an important topic in Number theory because they intervene in many fields and are closely related to Riemann's zeta-function and with Fermat's last theorem.

Several extensions of these mathematical entities also appeared in the literature. The interested reader could look at the articles [2–8], where the main generalizations are recalled.

All these extensions were obtained by replacing the denominator $e^x - 1$, in the generating function of Bernoulli polynomials, with the remainder term of the truncated exponential $e^x - T_r(x)$, where $T_r(x)$ is defined in (1).

In [1] we have determined the exponential generating function of the polynomials $B_n^{[r-1,1]}(t)$, providing a recursive computation for the $B_n^{[r-1,1]}$ numbers and their representation in terms of Bell polynomials.

Further information can be found in the above quoted article and in the references therein, where the connection with the preceding literature by several authors [3–17], is reported.

In this article we exploit the Laguerre-type exponential of the first order $e_1(x)$ defined in [18], (see also [19], where further information is given), which is used instead of the ordinary exponential for defining new sets of 2nd kind hypergeometric Bernoulli polynomials and numbers, which are called 2nd kind hypergeometric Laguerre-type Bernoulli polynomials and numbers.

We first give a recursive method for computing the 2nd kind hypergeometric Laguerre-type Bernoulli numbers, denoted as $\mathcal{B}_n^{[(r-1)^2,1]}$. This method avoids the definition of these numbers by using partition of integers (as it is done in [17]), because the use of partitions is a computationally more expensive method. Then, by using the Laguerre-type Blissard problem, we derive a connection of these 2nd kind hypergeometric Laguerre-type Bernoulli numbers with Bell's polynomials.

A Table of the $\mathcal{B}_n^{[(r-1)^2,1]}$ numbers, for rational values of r , derived by the second author by using the Mathematica[®] computer algebra program, is reported in the Appendix.

Of course a further extension is possible using the Laguerre-type exponential of order n , $e_n(x)$, also defined in [18], and a formal extension of this type is recalled in the last Remark, but the relevant numbers are not shown here, as the method to be used is essentially the same.

2. The Laguerre derivative and the relevant exponentials

The Laguerre derivative (shortly L-derivative) of order 1, is defined by [18,19]

$$D_L := Dx D = D + x D^2, \quad (6)$$

where $D = D_x = d/dx$.

An eigenfunction of the operator (6) is given by the Laguerre-type exponential (shortly L-exponential), of order 1

$$e_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}, \quad (7)$$

since it results

$$D_L e_1(ax) = a e_1(ax). \quad (8)$$

Furthermore, for every positive integer n , the function

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}, \quad (9)$$

is an eigenfunction of the operator

$$\begin{aligned} D_{(n-1)L} &:= Dx \cdots Dx Dx D = D (xD + x^2 D^2 + \cdots + x^{n-1} D^{n-1}) = \\ &= S(n,1)D + S(n,2)x D^2 + \cdots + S(n,n)x^{n-1} D^n, \end{aligned} \quad (10)$$

where $S(n, k)$, $k = 1, 2, \dots, n$ denote Stirling numbers of the second kind. In fact, for every constant a , it results

$$D_{nL} e_n(ax) = a e_n(ax). \quad (11)$$

Note that $e_n(x)$ reduces to the exponential function when $n = 0$, so that we put by definition:

$$e_0(x) := e^x, \quad D_{0L} := D, \quad (\text{obviously it results } D_{1L} = D_L).$$

2.1. The Laguerre-type Blissard problem

Consider the formal exponential

$$e_1(at) = \sum_{k=0}^{\infty} \frac{a^k t^k}{[k!]^2} = \sum_{k=0}^{\infty} \frac{a_k t^k}{[k!]^2}, \quad (12)$$

associated with the umbral sequence $a = \{a_k\}$, where

$$a^k := a_k, \quad \forall k \geq 0, \quad a_0 := 1. \quad (13)$$

According to Blissard [20], the solution $b = \{b_n\}$, of the umbral equation

$$e_1(at) e_1(bt) = 1, \quad (14)$$

is expressed by the Bell's polynomials $Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n)$ [20], as follows

$$\begin{cases} b_0 := 1, \\ b_n = Y_n(-1!, a_1; [2!]^2, a_2; -[3!]^2, a_3; \dots; (-1)^n [n!]^2, a_n), \quad (\forall n > 0). \end{cases} \quad (15)$$

At present, the Blissard symbolic method is called the *umbral calculus*, a term coined by J.J. Sylvester.

The modern version of the umbral calculus is due to Rota and Roman [21,22].

3. Basic definitions

In what follows, dealing with hypergeometric functions, and for typographical convenience, we use for the rising factorial the Pochhammer symbol according to the notation:

$$(x)_n = \begin{cases} x(x+1) \cdots (x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}, & n \geq 1, \\ 1, & n = 0. \end{cases} \quad (16)$$

We put, for shortness:

$$\mathcal{T}_r(x) := \sum_{\ell=0}^{r-1} \frac{x^\ell}{[\ell!]^2}, \quad (17)$$

that is the r th partial sum of the exponential series, which is polynomial of degree $r - 1$. Then, the r -associate Laguerre-type Stirling numbers of the second kind $\mathcal{S}(n, k; r)$ are defined by

$$\left(\sum_{\ell=r}^{\infty} \frac{x^\ell}{[\ell!]^2} \right)^k = (e_1(x) - \mathcal{T}_r(x))^k = [k!]^2 \sum_{n=rk}^{\infty} \mathcal{S}(n, k; r) \frac{x^n}{[n!]^2}. \quad (18)$$

Of course, when $k = 1$, we put by definition $\mathcal{S}(n, k; 1) = \mathcal{S}(n, k)$, and the $\mathcal{S}(n, k)$ are the Laguerre-type version (of order 1) of the above recalled $S(n, k)$.

Note that

$$e_1(x) - \mathcal{T}_r(x) = \frac{x^r}{[r!]^2} {}_1F_2(1; r+1; r+1; x) = \frac{x^r}{[r!]^2} \sum_{n=0}^{\infty} \frac{(1)_n}{[(r+1)_n]^2} \frac{x^n}{n!}. \quad (19)$$

4. Hypergeometric Laguerre-type Bernoulli polynomials

In what follows we use the definition by Booth and Hassen [9] for the generalized Bernoulli polynomials, which is different from that introduced by Kurt [9] and in a previous article by Natalini and Bernardini [5].

The hypergeometric Laguerre-type Bernoulli polynomials are defined by the generating function:

$$\frac{\frac{x^r}{[r!]^2} e_1(tx)}{e_1(x) - \mathcal{T}_r(x)} = \frac{e_1(tx)}{{}_1F_2(1; r+1; r+1; x)} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[(r-1)^2, 1]}(t) \frac{x^n}{[n!]^2}. \quad (20)$$

4.1. The exponential generating function

Since

$$e_1(x) - \mathcal{T}_r(x) = \sum_{n=r}^{\infty} \frac{x^n}{[n!]^2} = x^r \sum_{n=0}^{\infty} \frac{1}{[(n+r)!]^2} x^n,$$

and

$$\frac{\frac{x^r}{[r!]^2} e_1(tx)}{e_1(x) - \mathcal{T}_r(x)} = \frac{e_1(tx)}{[r!]^2 \sum_{n=0}^{\infty} \frac{1}{[(n+r)!]^2} x^n}, \quad (21)$$

we have the exponential generating function of the hypergeometric Bernoulli polynomials $\mathcal{B}_n^{[(r-1)^2, 1]}(t)$

$$\frac{e_1(tx)}{[\Gamma(r+1)]^2 \sum_{n=0}^{\infty} \frac{1}{[\Gamma(n+r+1)]^2} x^n} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[(r-1)^2, 1]}(t) \frac{x^n}{[n!]^2}. \quad (22)$$

4.2. Recursive computation of the 2nd kind hypergeometric Laguerre-type Bernoulli numbers

In (19), putting $t = 0$, gives the exponential generating function [23] of the generalized hypergeometric Bernoulli numbers $\mathcal{B}_n^{[(r-1)^2, 1]} := \mathcal{B}_n^{[(r-1)^2, 1]}(0)$:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{[(r-1)^2, 1]} \frac{x^n}{[n!]^2} = \frac{1}{[\Gamma(r+1)]^2 \sum_{n=0}^{\infty} \frac{1}{[\Gamma(n+r+1)]^2} x^n}, \quad (23)$$

which is valid even for non integer (in particular for fractional) values of the parameter r .

From Eq. (20) we find:

$$\sum_{n=0}^{\infty} \sum_{h=0}^n \binom{n}{h} \mathcal{B}_{n-h}^{[(r-1)^2, 1]} \frac{h! [\Gamma(r+1)]^2}{[\Gamma(h+r+1)]^2} \frac{x^n}{[n!]^2} = 1,$$

and therefore

$$\mathcal{B}_0^{[(r-1)^2, 1]} = 1,$$

and for $n = 1, 2, 3, \dots$ we find the $\mathcal{B}_n^{[(r-1)^2, 1]}$ numbers solving by recursion the triangular system:

$$\sum_{h=0}^n \binom{n}{h} \mathcal{B}_{n-h}^{[(r-1)^2, 1]} \frac{h! [\Gamma(r+1)]^2}{[\Gamma(h+r+1)]^2} = 0.$$

5. Representation in terms of Bell polynomials

We use a Laguerre-type version of the Blissard problem [20] showing that the reciprocal of a Laguerre-type Taylor series can be expressed in terms of Bell polynomials.

In fact, consider the sequences $a := \{a_k\} = (1, a_1, a_2, a_3, \dots)$, and $b := \{b_k\} = (b_0, b_1, b_2, b_3, \dots)$, and the function:

$$\frac{1}{\sum_{h=0}^{\infty} a_h \frac{x^h}{[h!]^2}} \quad (t \geq 0). \quad (24)$$

Using the umbral formalism (that is, letting $a_k \equiv a^k$ and $b_k \equiv \mathcal{B}^k$), the solution of the equation,

$$\frac{1}{\sum_{n=0}^{\infty} \frac{a^n t^n}{[n!]^2}} = \sum_{n=0}^{\infty} \frac{\mathcal{B}^n t^n}{[n!]^2}, \quad i.e. \quad e_1(at) e_1(bt) = 1, \quad (25)$$

is given by

$$\begin{cases} b_0 := 1, \\ b_n = Y_n(-1!, a_1; [2!]^2, a_2; -[3!]^2, a_3; \dots; (-1)^n [n!]^2, a_n), \quad (\forall n > 0), \end{cases} \quad (26)$$

where Y_n is the n th Bell polynomial [20].

The Bell polynomials are usually written in the form

$$Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{k=1}^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) f_k.$$

where the $B_{n,h}$ are called Bell polynomials of the second kind [24].

By using this equation, the function (21) can be rewritten as

$$\frac{1}{\sum_{n=0}^{\infty} \frac{a_n t^n}{[n!]^2}} = 1 + \sum_{n=1}^{\infty} \sum_{h=1}^n (-1)^h [h!]^2 B_{n,h}(a_1, a_2, \dots, a_{n-h+1}) \frac{t^n}{[n!]^2}. \quad (27)$$

It is convenient to introduce the definition

$$C_n(a) := \sum_{h=1}^n (-1)^h [h!]^2 B_{n,h}(a_1, a_2, \dots, a_{n-h+1}), \quad C_0(a) := 1, \quad (28)$$

so that Eq. (24) becomes

$$\frac{1}{\sum_{n=0}^{\infty} \frac{a_n t^n}{[n!]^2}} = \sum_{n=0}^{\infty} C_n(a) \frac{t^n}{[n!]^2}. \quad (29)$$

Applying this result to Eq. (20), with

$$a_n = \frac{[n!]^2 [\Gamma(r+1)]^2}{[\Gamma(n+r+1)]^2},$$

we find our result

Theorem 1. The 2nd kind hypergeometric Bernoulli numbers $\mathcal{B}_n^{[(r-1)^2, 1]}$ are expressed in terms of Bell polynomials by the equation

$$\mathcal{B}_n^{[(r-1)^2, 1]} = C_n \left(\frac{[n!]^2 [\Gamma(r+1)]^2}{[\Gamma(n+r+1)]^2} \right),$$

that is

$$\mathcal{B}_n^{[(r-1)^2, 1]} = \sum_{h=1}^n (-1)^h [h!]^2 B_{n,h} \left(\frac{[\Gamma(r+1)]^2}{[\Gamma(r+2)]^2}, \frac{2! [\Gamma(r+1)]^2}{[\Gamma(r+3)]^2}, \dots, \frac{(n-h+1)! [\Gamma(r+1)]^2}{[\Gamma(n-h+r+2)]^2} \right)$$

$$= \sum_{h=1}^n (-1)^h [h!]^2 B_{n,h} \left(\frac{1}{[(r+1)]^2}, \frac{2!}{[(r+1)_2]^2}, \dots, \frac{(n-h+1)!}{[(r+1)_{n-h+1}]^2} \right). \quad (30)$$

Many representation formulas in terms of Bell's polynomials are reported in [25].

Remark 1. Since Laguerre-type exponentials exist for any integer n , the above formulas can be generalized in a straightforward way. It is sufficient to substitute the exponent 2 in the preceding equations (17)–(30).

For example, Eq. (23) becomes

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{[(r-1)^{n+1}, 1]} \frac{x^n}{[n!]^{n+1}} = \frac{1}{[\Gamma(r+1)]^{n+1} \sum_{n=0}^{\infty} \frac{1}{[\Gamma(n+r+1)]^{n+1}} x^n},$$

and Eq. (30) writes

$$\mathcal{B}_n^{[(r-1)^{n+1}, 1]} = \sum_{h=1}^n (-1)^h [h!]^{n+1} B_{n,h} \left(\frac{1}{[(r+1)]^{n+1}}, \frac{2!}{[(r+1)_2]^{n+1}}, \dots, \frac{(n-h+1)!}{[(r+1)_{n-h+1}]^{n+1}} \right).$$

The proofs are obtained in the same way, so it is not necessary to further insist on this subject.

6. Conclusion

Starting from some recent generalizations of Bernoulli polynomials we have introduced in [17] a generalized hypergeometric versions of these polynomials and of the corresponding r -associated Stirling numbers.

In the same article we have introduced values of these mathematical entities for rational values of r , thus expanding the original definitions.

In this article we have computed the Laguerre-type hypergeometric Bernoulli numbers of the second kind $\mathcal{B}_n^{[(r-1)^2, 1]}$ (of order 1), by using the same methodology of the preceding article. As this technique can be generalized to any integer order n , we have provided a method to introduce infinite many sequences of numbers, which extend the classical hypergeometric Bernoulli polynomials and their r -associated Stirling numbers.

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Appendix

Examples of Laguerre-type hypergeometric Bernoulli numbers of the second kind (of order 1) for fractional values of r are reported in the following table.

$$B_n^{[(r-1)^2, 1]}$$

	$r = \frac{1}{2}$	$r = \frac{3}{2}$	$r = \frac{5}{2}$
$n = 0$	1	1	1
$n = 1$	$-\frac{4}{9}$	$-\frac{4}{25}$	$-\frac{4}{49}$
$n = 2$	$\frac{512}{2025}$	$\frac{768}{30625}$	$\frac{1024}{194481}$
$n = 3$	$-\frac{54272}{297675}$	$-\frac{69632}{20671875}$	$-\frac{44032}{384359283}$
$n = 4$	$\frac{10944512}{66976875}$	$\frac{583221248}{3064088671875}$	$\frac{8227422208}{257813217024363}$
$n = 5$	$\frac{2592800768}{14587563375}$	$\frac{10162405376}{95894626953125}$	$\frac{273529372672}{63164238170968935}$
$n = 6$	$\frac{2076128595410944}{9059970923128125}$	$\frac{4831515683323904}{85636299234814453125}$	$\frac{960183899411972096}{2922229493606610586243845}$
$n = 7$	$-\frac{3994590858182656}{116485344044021875}$	$\frac{1265768498802458624}{88388894567362060546875}$	$\frac{1420764981357694681088}{7384473930343904951438196315}$
$n = 8$	$\frac{443310617419493408768}{757445926122522421875}$	$\frac{72766396544266019012608}{92737671507965767730712890625}$	$\frac{6249695099236333124373088}{49532117117991408029131635998445715}$
$n = 9$	$\frac{15043421255400186580041728}{13398460988773309120546875}$	$\frac{3409251665395799255146299392}{1064164780553907184709930419921875}$	$\frac{16424267841858310022449512448}{172616369453000976699737962595675200035}$
$n = 10$	$\frac{174535890043789072451453845504}{72954620083870668161377734375}$	$\frac{45170646204261335263651924364404864}{2330866190884346473961859920501708984375}$	$\frac{23073711488077198082354872998011015584}{85639269294870119960158372035884358617364375}$

