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Mathematical modelling of child mortality with Caputo–Fabrizio fractional derivative

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Abstract: In this article, we proposed a fractional-order mathematical model of Child mortality. We analyzed the existence of a unique solution for our model using the fixed point theory and Picard–Lindelöf technique. We propose a Caputo operator for modeling child mortality in a given population of 1000 susceptible under five children. Our stability analysis was based on the fixed point theory, which was used to prove that our Picard iteration was stable. Using the Julia software and some real world values for our parameters, we numerically simulated the system through graphs. Our findings were that, reducing child mortality rates alone is insufficient to significantly improve survival rates for children under five. To make a real impact, a holistic approach is necessary, including access to healthcare, proper nutrition, vaccination programs, hygiene practices, clean water sources and comprehensive public health campaigns can greatly enhance the survival rates of children under five.

Keywords: Caputo–Fabrizio, fractional-order derivative, mathematical model, compartmental model, child mortality

MSC: 34A08

1. Introduction

The death of children under five years is known as child mortality. Child mortality rate is defined as the number of deaths of children under the age of five per one thousand (1,000) live births in a given population [1,2]. It is an important indicator of society's well-being and overall health, reflecting issues like nutrition, access to healthcare, sanitation, and socioeconomic conditions. There has been significant progress in reducing child mortality rates worldwide, over the past few decades [2]. However, despite these positive trends, child mortality rates still vary greatly between different regions and countries, highlighting the ongoing need for targeted interventions and investments in healthcare systems [2]. For example, Sub-Saharan Africa and some Asian countries have the highest child mortality rate in the world, with over half of all child deaths occurring in this region. In contrast, countries in Western Europe have achieved some of the lowest child mortality rates globally. These disparities emphasize the importance of addressing the underlying factors that contribute to child mortality and implementing targeted interventions to ensure that progress is equitable across all regions. Numerous studies have been conducted on child mortality to understand its causes and develop strategies for prevention [1,3,4]. These studies have highlighted factors such as poverty, lack of access to healthcare, malnutrition, and infectious diseases as major contributors to high child mortality rates.

The use of mathematical modelling has proven to be an invaluable tool for understanding complex phenomena. It can provide insights into a range of real-world problems, from climate change to economic

forecasting as well as to evaluate the effectiveness of proposed solutions. Using the Lives Saved Tool mathematical model [4] examined the impact of the service utilization interruptions on maternal and child mortality. In their study, [5] utilized an integrated model to assess outcomes of tuberculosis treatment in children under 5. Yerramsetti et al. [6] conducted a study using data sources to develop their mathematical model of paediatric tuberculosis where they incorporated risk factors like HIV, BCG non-vaccination and malnutrition. Abioye et al. [7] proposed a deterministic malaria transmission model by using the Adomian decomposition method to approximate the solution for the model. Also, James Peter [8] provided valuable insights into the transmission dynamics of measles and highlighted the effectiveness of combined control strategies by using a deterministic mathematical model that sheds light on the impact of vaccination rates and hospitalization on the spread of the disease.

Studies have shown that, the modelling of problems using caputo fractional derivatives has a lot of advantages over conventional methods [9]. For example, it can accurately capture the effects of non-linearity, provide better predictions of the behavior of complex systems, and allow for more efficient numerical solution. Caputo fractional derivatives has been used in numerous applications include finance, engineering, science, and medicine [10–12]. For instance, Rahul, & Prakash [13] examined the applicability of a fractional Susceptible Infected Recovered (SIR) model to understand childhood diseases through the use of Caputo, Caputo-Fabrizio, and Atangana-Baleanu, to analyze the proposed model. The proposed Constant Proportional-Caputo (CPC) operator by [14] offered a comprehensive approach to modeling childhood disease epidemics. Additionally, the incorporation of reproductive and strength numbers allows for a thorough assessment of the dynamics of the biological system. Based on these studies we propose the Child-Sick-Recovery model using Caputo fractional derivative. In contrast to other existing models in the field, our model takes into account the global and African parameters in our stimulation which provides a real and practical view of the current state of child mortality. This allows for a more accurate representation of the dynamics involved and can lead to improved predictions and treatment strategies. The rest of the paper is structured as follows: the next section deals with materials and methods followed by the numerical stimulations and the last section is the conclusion.

2. Materials and methods

We used fractional calculus in this section to model child mortality and study the dynamics associated with it. We begin with the model formulation, followed by stability and equilibrium point as well as the existence and uniqueness of the solution.

2.1. Preliminary results and definitions

We will recall some essential definitions of fractional calculus and explore some of its' features in this section. As a first step, we define Caputo derivative as express by [15] and then introduce the fractional derivative with nonsingular exponential kernel [16].

Definition 1. Given a differentiable function f, it's Caputo derivative of order$a \in (0,1)$is define as

$$^{C}D^{a}f(t) = \frac{1}{\Omega(n-a)}\int_0^t f'(s)\frac{1}{(t-s)}ds. \tag{1}$$

Definition 2. given that $T > 0, f \in \mathbb{Z}^1(0,T)$ with the order$a \in (0,1)$, then the ath order Caputo derivative of the differentiable function f is

$$^{CF}D^{a}f(t) = \frac{1}{2}\frac{\Omega(a)(2-a)}{1-a}f'(s)\,e^{-n(t-s)}ds. \tag{2}$$

where $n = \frac{a}{1-a}$ and $\Omega(a)$been a normalizing function which depends on a in that $\Omega(0) = \Omega(1) = 1$.

Using the formula provided by [17] for $\Omega(a)$as $\Omega(a) = \frac{2}{2-a}$. The Caputo–Fabrizio derivative can be reduced to

$$^{CF}D^{a}f(t) = \frac{1}{1-a}\int_0^t f'(s)\,e^{-n(t-s)}ds. \tag{3}$$

Table 1. State variable description

Compartments and parameters	Description	Value	Reference
C(t)	Population of children under 5 at time t	1000	[2]
S(t)	Population of sick children at time t	0	Assumed
R(t)	Population of children under 5 who are able to recover from their sickness at time t	0	Assumed
α	Rate at which under five children become sick and move to the sick children compartment	0.8	Assumed
β	Rate at which sick children recover and move to the recovery compartment	0.75	[18]
		0.58	[19]
γ	Rate at which children who recover from one sickness move to the children compartment again (as they are still susceptible to other childhood diseases)	0.1	Assumed
μ	Natural death rate	0.042	[3]
κ	Rate of death due to childhood diseases	0.037(Global)	[2]
		0.07(Africa)	[2]

Definition 3. Given a Caputo–Fabrizio derivative, it's Laplace transform is given by [17]:

$$L\left[{}^{CF}D^{a}f\left(t\right)\right]=\frac{1}{2}\frac{\Omega(a)\left(2-a\right)}{1-a}\frac{sL[f(t)]-f(0)}{s+\frac{a}{1-a}}. \tag{4}$$

After the concept of fractional derivative, fractional integral becomes a necessity. As such, using order a of the Riemann–Liouville integral we obtained:

$$D^{a}f\left(t\right)=\frac{1}{\Omega\left(a\right)}\int_{0}^{t}f\left(s\right)\frac{1}{\left(t-s\right)^{1-a}}ds, \tag{5}$$

where the integral exists. Then following [17], the ath-order Caputo–Fabrizio integral can be written as:

$${}^{CF}D^{a}f\left(t\right)=\frac{2(1-a)}{n(a)(2-a)}f(t)+\frac{2a}{n(a)(2-a)}\int_{0}^{t}f(s)ds. \tag{6}$$

2.2. Model formulation

We model the proposed fractional child mortality model in this section.

Let the population of children at time **t**, the population of sick children at time t and the population of recovered children at time t be represented by $C(t)$, $S(t)$, and $R(t)$respectively. As such, the model could be described as

$$\begin{cases}\frac{dC(t)}{dt} = \gamma R(t)-(\mu+\alpha)C(t),\\ \frac{dS(t)}{dt} = \alpha C(t)-(\kappa+\mu+\beta)S(t),\\ \frac{dR(t)}{dt} = \beta S(t)-(\mu+\gamma)R(t),\end{cases} \tag{7}$$

where the initial condition are$(C(0),S(0),R(0))=(I,0)$ and $(I>0)$.

Modifying Eq. (7) in fractional-order Caputo–Fabrizio sense we obtain

$$\begin{cases}{}^{CF}D^{a}C(t) = \gamma R(t)-(\mu+\alpha)C(t),\\ {}^{CF}D^{a}S(t) = \alpha C(t)-(\kappa+\mu+\beta)S(t),\\ {}^{CF}D^{a}R(t) = \beta S(t)-(\mu+\gamma)R(t).\end{cases} \tag{8}$$

2.3. Equilibrium and stability

At the origin, $E^* = (0,0,0)$, Eq. (8) has a unique equilibrium since it is a system of homogeneous fractional differential equations. Therefore, the matrix coefficient for Eq. (8) is

$$J = \begin{bmatrix} -(\mu+\alpha) & 0 & \gamma \\ \alpha & -(\kappa+\mu+\beta) & 0 \\ 0 & \beta & -(\mu+\gamma) \end{bmatrix}. \tag{9}$$

Having the characteristic polynomial

$$\begin{aligned} &\lambda^3 + (\alpha+\beta+k+\gamma+3\mu)\lambda^2 + (\alpha\beta+\alpha\kappa+\alpha\gamma+2\alpha\mu+\beta\gamma+2\beta\mu+\kappa\gamma+2\kappa\mu+2\gamma\mu+3\mu^2)\lambda \\ &+\alpha\beta\mu+\alpha\kappa\gamma+\alpha\kappa\mu+\alpha\gamma\mu+\alpha\mu^2+\beta\gamma\mu+\beta\mu^2+\kappa\gamma\mu+\kappa\mu^2+\gamma\mu^2+\mu^3 = 0. \end{aligned} \tag{10}$$

Since $(\alpha+\beta+k+\gamma+3\mu) > 0$, $(\alpha\beta+\alpha\kappa+\alpha\gamma+2\alpha\mu+\beta\gamma+2\beta\mu+\kappa\gamma+2\kappa\mu+2\gamma\mu+3\mu^2) > 0$ and $\alpha\beta\mu+\alpha\kappa\gamma+\alpha\kappa\mu+\alpha\gamma\mu+\alpha\mu^2+\beta\gamma\mu+\beta\mu^2+\kappa\gamma\mu+\kappa\mu^2+\gamma\mu^2+\mu^3 > 0$ it implies that J is having eigenvalues with negative real parts. Hence, Eq. (8) is asymptotically stable.

2.4. Existence and uniqueness

We now analyse the existence of a unique solution for the system (8) using the fixed point theory and Picard–Lindelöf technique. With the initial conditions$(C(0), S(0), R(0)) = (I, 0)$ we convert the system (8) into an integral equation by using the ath-order Caputo–Fabrizio integral define in (6). We now obtain

$$\begin{cases} C(t) - I = & \frac{2(1-a)}{\Omega(a)}\{\gamma R(t) - (\mu+\alpha)C(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\gamma R(t) - (\mu+\alpha)C(t)\}\,ds, \\ S(t) - 0 = & \frac{2(1-a)}{\Omega(a)}\{\alpha C(t) - (\kappa+\mu+\beta)S(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\alpha C(t) - (\kappa+\mu+\beta)S(t)\}\,ds, \\ R(t) - 0 = & \frac{2(1-a)}{\Omega(a)}\{\beta S(t) - (\mu+\gamma)R(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\beta S(t) - (\mu+\gamma)R(t)\}\,ds. \end{cases} \tag{11}$$

Let $C_0(t) = 1,\ S_0(t) = 0\ and\ R_0(t) = 0$ then the Picard iteration is defined as

$$\begin{cases} C_{i+1}(t) = & \frac{2(1-a)}{\Omega(a)}\{\gamma R_i(t) - (\mu+\alpha)C_i(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\gamma R_i(t) - (\mu+\alpha)C_i(t)\}\,ds, \\ S_{i+1}(t) = & \frac{2(1-a)}{\Omega(a)}\{\alpha C_i(t) - (\kappa+\mu+\beta)S_i(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\alpha C_i(t) - (\kappa+\mu+\beta)S_i(t)\}\,ds, \\ R_{i+1}(t) = & \frac{2(1-a)}{\Omega(a)}\{\beta S_i(t) - (\mu+\gamma)R_i(t)\} + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\beta S_i(t) - (\mu+\gamma)R_i(t)\}\,ds. \end{cases} \tag{12}$$

To show the existence of a unique solution we define

$$\begin{cases} f_1(t,C,S,R) = & \gamma R - (\mu+\alpha)C, \\ f_2(t,C,S,R) = & \alpha C - (\kappa+\mu+\beta)S, \\ f_3(t,C,S,R) = & \beta S - (\mu+\gamma)R, \end{cases} \tag{13}$$

where $f_1(t,C,S,R), f_2(t,C,S,R)\ and\ f_3(t,C,S,R)$ are contraction.

As such we define the Picard operator as

$$P(\zeta(t)) = \zeta_0 + \frac{2(1-a)}{\Omega(a)(2-a)}\Delta(t,\zeta(t)) + \frac{2a}{\Omega(a)(2-a)}\int_0^t \Delta(s,\zeta(t))ds, \tag{14}$$

where $\zeta(t) = (C(t), S(t), R(t))$, $\zeta_0 = (I, 0)$ and

$$\Delta(t,\zeta(t)) = (f_1(t,C(t),S(t),R(t)), f_2(t,C(t),S(t),R(t)), f_3(t,C(t),S(t),R(t))).$$

Hence the solution of the system (8) is bounded. Also, since f_1, $f_2$and $f_3$are contraction, we write

$$\|\Delta(t,\zeta_1(t)) - \Delta(t,\zeta_2(t)) - \Delta(t,\zeta_3(t))\| \leq \delta\,\|\zeta_1(t) - \zeta_2(t) - \zeta_3(t)\|, \tag{15}$$

where $\delta < 1$. Again using Eq. (11) we obtain

$$\begin{aligned}\|\zeta(t)-\zeta_0\| &= \left\| \frac{2\,(1-a)}{\Omega(a)(2-a)}\Delta(t,\zeta(t)) + \frac{2a}{\Omega(a)(2-a)}\int_0^t \Delta(s,\zeta(t))ds \right\| \\ &\leq \frac{2\,(1-a)}{\Omega(a)(2-a)}\|\Delta(t,\zeta(t))\| + \frac{2a}{\Omega(a)(2-a)}\int_0^t \|\Delta(s,\zeta(t))\|\,ds \\ &\leq \left(\frac{2\,(1-a)}{\Omega(a)(2-a)} + \frac{2at_0}{\Omega(a)(2-a)}\right)\delta \leq c\delta, \end{aligned} \tag{16}$$

where we need $c\delta < 1$, as such by using the definition of (14), we obtain

$$\begin{aligned}\|P(\zeta_1(t)) - P(\zeta_2(t)) - P(t,\zeta_3(t))\| &= \left\| \frac{2\,(1-a)}{\Omega(a)(2-a)}\{\Delta(t,\zeta_1(t)) - \Delta(t,\zeta_2(t)) - \Delta(t,\zeta_3(t))\} \right. \\ &\quad \left. + \frac{2a}{\Omega(a)(2-a)}\int_0^t \{\Delta(s,\zeta_1(s)) - \Delta(s,\zeta_2(s)) - \Delta(s,\zeta_3(s))\}\,ds \right\| \\ &\leq \frac{2\,(1-a)}{\Omega(a)(2-a)}\|\Delta(t,\zeta_1(t)) - \Delta(t,\zeta_2(t)) - \Delta(t,\zeta_3(t))\| \\ &\quad + \frac{2a}{\Omega(a)(2-a)}\int_0^t \|\Delta(s,\zeta_1(s)) - \Delta(s,\zeta_2(s)) - \Delta(s,\zeta_3(s))\|\,ds \\ &\leq \frac{2\,(1-a)}{\Omega(a)(2-a)}\delta\,\|\zeta_1(t) - \zeta_2(t) - \zeta_3(t)\| \\ &\quad + \frac{2a\delta}{\Omega(a)(2-a)}\int_0^t \|\zeta_1(s) - \zeta_2(s) - \zeta_3(s)\|\,ds \\ &\leq \left\{\frac{2\,(1-a)}{\Omega(a)(2-a)} + \frac{2at_0}{\Omega(a)(2-a)}\right\}\delta\,\|\zeta_1(t) - \zeta_2(t) - \zeta_3(t)\| \\ &\leq c\delta\,\|\zeta_1(t) - \zeta_2(t) - \zeta_3(t)\|, \end{aligned} \tag{17}$$

where $c\delta < 1$ from Eq. (16). As a result, the system (8) has a unique solution since the operator $P$is contraction.

2.5. Stability analysis

At this point to prove the stability of the Picard iteration established in the previous section, we will use the fixed point theory.

Theorem 1. [20] *If $Q$is self-map ofΦ such that it's Banach space is $(\Phi, \|.\|)$. Then for all$u, v \in \Phi$, $Q$is Picard$Q-$Stableif the inequality below holds*

$$\|Q_u - Q_v\| \leq K\,\|v - Q_v\| + k\,\|u - v\|, \tag{18}$$

where $K \geq 0$ and $0 \leq k \leq 1$.

We now consider the iterative formula for system (8) as:

$$\begin{cases} C_{i+1}(t) = & C_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\gamma R_i(t) - (\mu+\alpha)C_i(t)\right]\right], \\ S_{i+1}(t) = & S_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\alpha C_i(t) - (\kappa+\mu+\beta)S_i(t)\right]\right], \\ R_{i+1}(t) = & R_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\beta S_i(t) - (\mu+\gamma)R_i(t)\right]\right], \end{cases} \tag{19}$$

where the Lagrange multiplier is represented by$\lambda\,(m,a) = \frac{m+a(1-m)}{m}$.

Theorem 2. *let Q be a self-map then,*

$$\begin{cases} Q\,(C_i(t)) = & C_{i+1}(t) = C_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\gamma R_i(t) - (\mu+\alpha)C_i(t)\right]\right], \\ Q\,(S_i(t)) = & S_{i+1}(t) = S_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\alpha C_i(t) - (\kappa+\mu+\beta)S_i(t)\right]\right], \\ Q\,(R_i(t)) = & R_{i+1}(t) = R_i(t) + L^{-1}\left[\lambda\,(m,a)\,L\left[\beta S_i(t) - (\mu+\gamma)R_i(t)\right]\right]. \end{cases} \tag{20}$$

Thus Eq. (19) *is conditionally* $Q-$*Stable in* $B^2(0,T)$.

Proof. First we show that Q having a fixed point. Thus, $Q(C_i(t)) - Q(C_j(t)) \; \forall (i,j) \in N \times N$

$$\begin{aligned} Q(C_i(t)) - Q(C_j(t)) =& C_i(t) - C_j(t) + L^{-1}[\lambda(m,a) L[\gamma R_i(t) - (\mu+\alpha) C_i(t)]] \\ & - L^{-1}[\lambda(m,a) L[\gamma R_j(t) - (\mu+\alpha) C_j(t)]] \\ =& C_i(t) - C_j(t) + L^{-1}[\lambda(m,a) L[\gamma(R_i(t) - R_j(t)) - (\mu+\alpha)(C_i(t) - C_j(t))]]. \end{aligned} \tag{21}$$

Taking the supreme norm of Eq. (21), gives us

$$\|Q(C_i(t)) - Q(C_j(t))\| \leq \|C_i(t) - C_j(t)\| + \left\|L^{-1}[\lambda(m,a) L[\gamma(R_i(t) - R_j(t)) - (\mu+\alpha)(C_i(t) - C_j(t))]]\right\|. \tag{22}$$

Similarly, we have

$$\begin{aligned} \|Q(S_i(t)) - Q(S_j(t))\| \leq & \|S_i(t) - S_j(t)\| \\ & + \left\|L^{-1}[\lambda(m,a) L[\alpha(C_i(t) - C_j(t)) - (\kappa+\mu+\beta)(S_i(t) - S_j(t))]]\right\|, \\ \|Q(R_i(t)) - Q(R_j(t))\| \leq & \|R_i(t) - R_j(t)\| \\ & + \left\|L^{-1}[\lambda(m,a) L[\beta(S_i(t) - S_j(t)) - (\mu+\gamma)(R_i(t) - R_j(t))]]\right\|. \end{aligned} \tag{23}$$

Assuming the solutions perform the same roles:

$$\|Q(C_i(t)) - Q(C_j(t))\| \cong \|Q(S_i(t)) - Q(S_j(t))\| \cong \|Q(R_i(t)) - Q(R_j(t))\|. \tag{24}$$

By considering the assumption (24), from Eqs. (22) and (23) we get

$$\begin{cases} \|Q(C_i(t)) - Q(C_j(t))\| \leq & (1 + \gamma\bar{n}_1(t) - (\mu+\alpha)\bar{n}_2(t)) \|C_i(t) - C_j(t)\|, \\ \|Q(S_i(t)) - Q(S_j(t))\| \leq & (1 + \alpha\bar{n}_3(t) - (\kappa+\mu+\beta)\bar{n}_4(t)) \|S_i(t) - S_j(t)\|, \\ \|Q(R_i(t)) - Q(R_j(t))\| \leq & (1 + \beta\bar{n}_5(t) - (\mu+\gamma)\bar{n}_6(t)) \|R_i(t) - R_j(t)\|, \end{cases} \tag{25}$$

where $\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4, \bar{n}_5$ *and* $\bar{n}_6$are functions gotten through the use of $L^{-1}[\lambda(m,a) L[.]]$.

By using the condition

$$\begin{cases} 1 + \gamma\bar{n}_1(t) - (\mu+\alpha)\bar{n}_2(t) < 1 \\ 1 + \alpha\bar{n}_3(t) - (\kappa+\mu+\beta)\bar{n}_4(t) < 1 \\ 1 + \beta\bar{n}_5(t) - (\mu+\gamma)\bar{n}_6(t) < 1 \end{cases} \tag{26}$$

It implies the self-map $Q$is contraction, as such it is having a fixed point. Hence we show that Qsatisfies conditions of Theorem 1. we assume that

$$K = (0,0,0), \; k = \begin{cases} 1 + \gamma\bar{n}_1(t) - (\mu+\alpha)\bar{n}_2(t) \\ 1 + \alpha\bar{n}_3(t) - (\kappa+\mu+\beta)\bar{n}_4(t) \\ 1 + \beta\bar{n}_5(t) - (\mu+\gamma)\bar{n}_6(t) \end{cases} \tag{27}$$

Hence, Q is Picard $Q-$*Stable* since all conditions of Theorem 1 holds. □

2.6. Numerical stimulation

We used JULIA [21] as well as real world parameters to perform numerical stimulations of our model. The simulations were used to identify potential areas of improvement. It allowed us to test the model's assumptions and predictions which helped us to evaluate the performance of child mortality and make necessary improvements. We first perform stimulations with the global parameter followed by the Africa parameters.

From Figure 1 we could see that at the end of 5 years, every surviving child would have pass through at least one illness and has recovered. This is an indication of the effectiveness of the healthcare system and its

success in treating diseases. It is also a sign that children are resilient and can fight off illnesses. One of the key factors contributing to the effectiveness of the healthcare system in treating diseases and preventing illnesses in children is the implementation of vaccinations. By ensuring that children are immunized against various diseases, vaccinations play a crucial role in boosting their immune systems and reducing the likelihood of illness. Again, Figure 1 confirms the global trends of child mortality where surviving rates are increasing [2] which is a very positive sign that child mortality has decreased globally. This decrease in child mortality is largely due to the implementation of effective health policies, improved access to healthcare, and increased investments in public health [22].

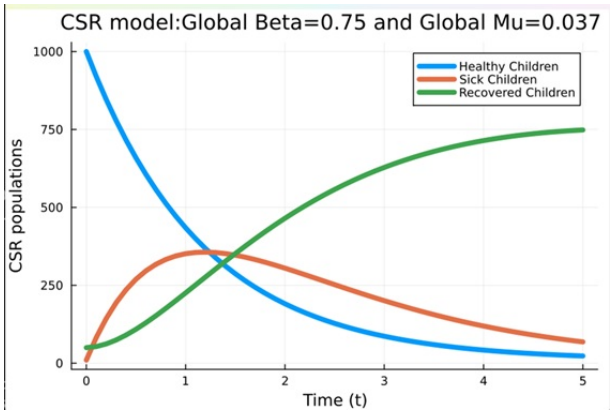

Figure 1. Global recovery rate of 75% and mortality rate of 37 per 1000 births

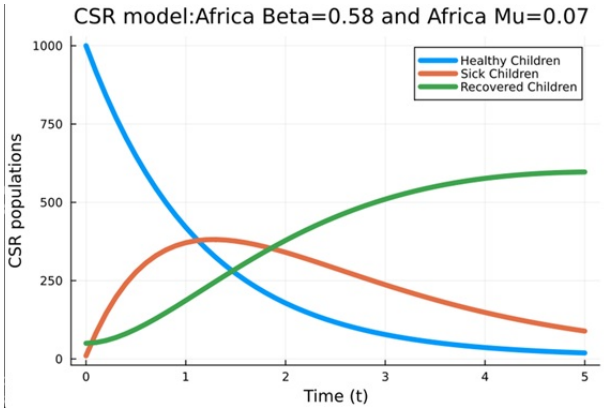

Figure 2. Africa recovery rate of 58% and mortality rate of 70 per 1000 births

From Figure 2 we can see that the diagram really depicts the African situation on child mortality. There are several factors contributing to high child mortality in Africa, including limited access to healthcare and immunization, poor nutrition and sanitation, inadequate prenatal and postnatal care, and the prevalence of infectious diseases such as malaria, pneumonia, and HIV/AIDS [5]. These challenges highlight the urgent need for targeted interventions and investment in healthcare infrastructure in order to improve child survival rates in the region. Although there has been much improvement in subsequent years, still much needs to be done so as to achieve the sustainable development goals.

From Figure 3 we can see that reducing the natural death rate and the rate at which recovered children return to the susceptible compartment does not have any influence on reducing child mortality. This indicates that reducing child mortality requires interventions that prevent children from contracting the disease in the first place. Such interventions might include providing access to vaccines, improving sanitation and nutrition, and providing better health care [23].

It can be seen from Figure 4 that reducing child mortality rate and the rate at which recovered children return to the susceptible compartment alone do not have much impact on reducing child mortality. However, it is much better than Figure 3.

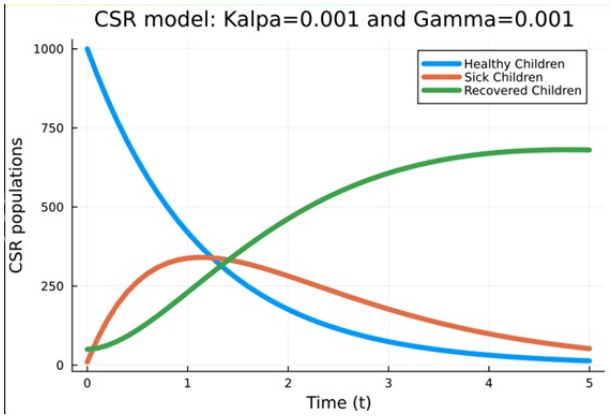

Figure 3. Reduced natural death rate and reduced Gamma rate

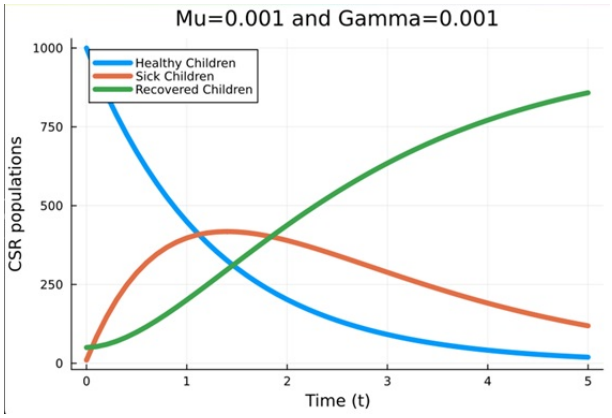

Figure 4. Reduced gamma and mortality rates

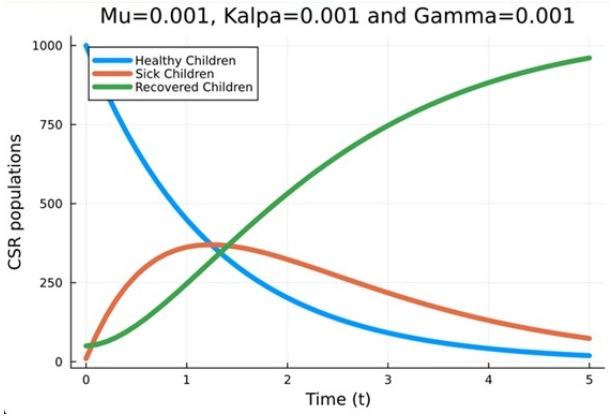

Figure 5. reduced natural death rate, reduced Gamma rate and reduced mortality rates

It can be seen from Figure 5 that reducing child mortality rates, natural death rates and the rate at which recovered children return to the susceptible compartment have much impact on the reduction of child mortality. This is because child mortality is reduced by preventing deaths due to infectious diseases, malnutrition, and vaccine-preventable illnesses. In addition, a decrease in natural death rates and the rate at which recovered children return to the susceptible compartment allows more resources to be directed towards improving the health of children. This, in turn, leads to improved health outcomes and better outcomes for the overall population. Additionally, improved health outcomes for children lead to greater economic productivity, as healthier children have better educational outcomes and are more likely to enter the workforce. It confirms other studies that reducing child mortality does not depend solely on lowering mortality rates [24,25]. Consequently, we must also try to reduce natural death rates as well as reduce

the likelihood that recovered children will become susceptible to other diseases. To reduce natural death rates in children, it is essential to focus on improving access to quality healthcare services, implementing effective nutrition programs, and promoting hygiene and sanitation practices. Additionally, investing in early childhood development programs and providing comprehensive immunization coverage can also play a significant role in reducing natural death rates among children.

3. Conclusion

The fractional model in the sense of Caputo derivative was used to model Child mortality. the equilibrium points, existence and uniqueness of a solution and its stability were obtained. The numerical results indicate that reducing just child mortality rates will not result in drastic survival rates for children under five. In addition to reducing natural death rates, it is also necessary to reduce the rate at which recovered children become susceptible to other diseases. One strategy for reducing the rate of susceptibility in recovered children is to improve their overall immune system through proper nutrition and access to healthcare. Vaccination programs can also play a crucial role in preventing the spread of infectious diseases and reducing the chances of recovered children falling ill again. Additionally, implementing hygiene practices and providing clean water sources can further protect children from contracting new infections. Regular health check-ups are also important to ensure that recovered children remain healthy and do not relapse. To further improve the survival rates of children under five, it is recommended to implement comprehensive public health campaigns that educate communities about the importance of proper nutrition, hygiene practices, and access to healthcare. Additionally, providing ongoing support for vaccination programs and ensuring regular health check-ups for recovered children can significantly reduce the risks of relapse and susceptibility to other diseases. These findings have important implications for future studies on child health and mortality. It highlights the need to take a holistic approach to improving the survival rates of children under five, focusing not only on reducing mortality but also on strengthening their immune systems and addressing factors such as nutrition, healthcare access, vaccination, and hygiene practices. By incorporating these elements into public health interventions, we can create a more comprehensive and effective strategy for reducing susceptibility and improving overall child health outcomes. However, it is important to acknowledge the limitations of this study. The study focused on a limited number of factors and did not consider other potential variables that could impact child health and mortality. Further research is needed to explore the complex interactions between various factors and develop targeted interventions that address the specific needs of different populations. Other studies can also include recruitment rates which may provide different results.

Conflicts of Interest: The authors declare no conflict of interest.

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