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Mellin transform of some trigonometric functions

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Abstract: In this paper, we focus on calculating the Mellin transform of three types of trigonometric functions, namely, $\sum_{k=0}^{n} c_k \sin(a_k x)$, $\sum_{k=0}^{n} c_k \cos(a_k x)$ and $\sum_{k=1}^{n} c_k(1-\cos(a_k x))$, where n is an integer, $c_k \in \mathbb{R}^*$ and $0 < a_0 < \cdots < a_n$. Our approach is based on the application of techniques from linear algebra, calculus, Laplace transform, and special functions. In particular, we give an evaluation of the integral $\int_0^\infty \frac{\sin^n x}{x^\alpha} dx$, $n \in \mathbb{N}^*$, $0 < \alpha < n+1$.

Keywords: Mellin transform, improper integrals, trigonometric functions, special functions

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1. Introduction

The Mellin transform is a powerful integral transform that serves as the multiplicative analogue of the Laplace transform. It finds extensive applications in pure and applied mathematics, including number theory [1], complex analysis [2], applied mathematics [3], and engineering [4]. Its utility extends to diverse domains such as electromagnetics, signal processing, quantum calculus, medical imaging, and mathematical physics [5–8]. A key strength of the Mellin transform is its scale invariance, which makes it particularly effective for solving partial differential equations and analyzing the local behavior of functions near singularities, especially when such functions admit series expansions.

Beyond its practical utility, the Mellin transform is valued for its algorithmic structure, enabling the systematic evaluation of complex integrals and the derivation of asymptotic expansions. It also plays a central role in the study of special functions, due to its close connection with the Gamma and Beta functions. Recent advances have introduced generalized versions of the Mellin transform, further enhancing its theoretical and computational scope [8].

Moreover, the Mellin transform facilitates the simplification of functions by mapping them into a more tractable domain, from which the original function can often be recovered via inversion.

In this paper, we are interested in the calculus of the Mellin transform for real variables of the following three types of trigonometric functions.

1. $S_n(x) = \sum_{k=0}^{n} c_k \sin(a_k x)$, where $n \in \mathbb{N}$, $c_k \in \mathbb{R}^*$, $0 \leq k \leq n$, and $0 < a_0 < \cdots < a_n$.
2. $T_n(x) = \sum_{k=0}^{n} c_k \cos(a_k x)$, where $n \in \mathbb{N}$, $c_k \in \mathbb{R}^*$, $0 \leq k \leq n$, and $0 < a_0 < \cdots < a_n$.
3. $J_n(x) = \sum_{k=1}^{n} c_k(1-\cos(a_k x))$, where $n \in \mathbb{N}^*$, $c_k \in \mathbb{R}^*$, $1 \leq k \leq n$, and $0 < a_1 < \cdots < a_n$.

Our results, stated below in Theorems 1, 2, 3, allow us to evaluate improper integrals of the following forms:

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^\alpha} dx, \text{ for } n \in \mathbb{N} \text{ and } 0 < \alpha < 2n+2,$$

$$\int_0^\infty \frac{\cos^{2n+1}(x)}{x^\sigma} dx, \text{ for } n \in \mathbb{N} \text{ and } 0 < \sigma < 1,$$

and

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^\alpha}dx, \text{ for } n \in \mathbb{N}^* \text{ and } 1 < \alpha < 2n+1.$$

As shown in Corollaries 1, 2, 3, our results extend and generalize those established in the literature. Namely, the authors in [9] considered only the special case of the first type of improper integrals:

$$\int_0^\infty \frac{\sin^n x}{x^n}dx, \text{ for } n \in \mathbb{N}^*.$$

In [10], the author examined improper integrals of the forms:

$$\int_0^\infty \frac{\sin^n x}{x^m}dx, \text{ for integers } n, m \text{ such that } n \geq m, \tag{1}$$

$$\int_0^\infty \frac{\sin^n x}{x^\mu}dx, \text{ for } n \in \mathbb{N},\ \mu \in \mathbb{R} \text{ such that } (0 < \mu < n+1) \text{ or } (\mu < 1 \text{ and } n \text{ is odd}), \tag{2}$$

and

$$\int_0^\infty \frac{\cos^n(x)}{x^\mu}dx, \text{ for } n \in \mathbb{N},\ \mu \in \mathbb{R} \text{ such that } 0 < \mu < 1 \text{ and } n \text{ is odd}. \tag{3}$$

In order to address the calculus of these integrals, Trainin [10] used various techniques, including Taylor series, trigonometric transformations, and special substitutions. A significant part of the article was dedicated to demonstrating the validity of these techniques through concrete examples and specific cases. Indeed, the author furnished in the appendix a synopsis of the evaluation of integrals in (1) depending on the parities of integers n and m, without a proof. For the evaluation of the integrals of forms (2) and (3), the author provided examples without giving general formulae. The principal novelty of our work is that we provide explicit closed-form expressions for every type of integral studied in [9,10]. In particular, we derive unified formulas for the Mellin transform of any linear combination of sines and cosines at distinct frequencies, valid throughout their maximal domains of convergence.

The outline of this article is as follows. In §2, we recall essential definitions and present preliminary results that will be crucial in the following analysis. §3, §4 and §5 are dedicated to evaluating the Mellin transform of S_n, T_n, and J_n, respectively.

2. Preliminaries

We begin this section by introducing key definitions related to the Mellin transform, as well as the Gamma and Beta functions.

Definition 1. Let f be a piecewise continuous function on $(0, \infty)$. The Mellin transform of f is defined by

$$M(f)(s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

where s is a real number for which the integral converges.

Definition 2. For $x > 0$, the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Definition 3. For $x > 0$, $y > 0$, the Beta function is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

We recall below some fundamental identities that will be used throughout the paper. For further details on the properties of the Gamma and Beta functions, we refer the interested reader to [11].

Proposition 1. *1. For $x > 0$, we have*

$$\Gamma(x+1) = x\Gamma(x).$$

2. For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

3. For $0 < x < 1$, we have the Euler's reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

4. For $\alpha > 0$, we have

$$\frac{1}{t^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tx} x^{\alpha-1} dx. \tag{4}$$

5. For $x, y > 0$, we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

6. For $0 < x < 1$, we have the classical identity

$$B(x, 1-x) = \frac{\pi}{\sin(\pi x)}. \tag{5}$$

In what follows, we give some examples of the Mellin transform.

Example 1. 1. Let a, β, λ be positive real numbers and define the function f on $(0,\infty)$ by

$$f(x) = \frac{1}{(x^\lambda + a)^\beta}.$$

Then, $M(f)(s)$ is defined for $0 < s < \lambda\beta$ and is given by

$$M(f)(s) = \frac{a^{\frac{s}{\lambda}-\beta}}{\lambda} B\left(\frac{s}{\lambda}, \beta - \frac{s}{\lambda}\right).$$

2. Let $f(x) = \dfrac{1}{e^x - 1}$. Then, $M(f)(s)$ is defined for $s > 1$ and is given by

$$M(f)(s) = \Gamma(s)\zeta(s),$$

where ζ is the Zeta function defined by $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$, for $s > 1$.

3. Let $\alpha \neq 0$, $\beta \geq 0$ and define the function f on $(0,\infty)$ by

$$f(x) = e^{-x^\alpha} x^{-\beta}.$$

Then, $M(f)(s)$ is defined for $\dfrac{s-\beta}{\alpha} > 0$ and is given by

$$M(f)(s) = \frac{1}{|\alpha|} \Gamma\left(\frac{s-\beta}{\alpha}\right).$$

4. Let $f(x) = \ln(1+x)$. Then, $M(f)(s)$ is defined for $-1 < s < 0$ and is given by

$$M(f)(s) = \frac{\pi}{s\sin(\pi s)}.$$

The following lemmas are essential to obtain our main results.

Lemma 1. *We consider $a > 0$.*

1. Let $-1 < \theta < 1$. Then, we have

$$\int_0^\infty \frac{t^\theta}{t^2 + a^2} dt = \frac{\pi a^{\theta-1}}{2\cos(\frac{\pi\theta}{2})} = \frac{\pi a^{\theta-1}}{2\sin(\frac{\pi(1-\theta)}{2})} = \frac{\pi a^{\theta-1}}{2\sin(\frac{\pi(1+\theta)}{2})}. \tag{6}$$

2. *Let* $0 < \alpha < 2$. *Then, we have*

$$\int_0^\infty \frac{\sin(at)}{t^\alpha}dt = \frac{\pi a^{\alpha-1}}{2\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}. \tag{7}$$

3. *Let* $0 < \alpha < 1$. *Then, we have*

$$\int_0^\infty \frac{\cos(at)}{t^\alpha}dt = \frac{\pi a^{\alpha-1}}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}.$$

Proof. 1. By making the change of variables $u = \frac{t^2}{a^2}$, we get that

$$\int_0^\infty \frac{t^\theta}{t^2+a^2}dt = \frac{a^{\theta-1}}{2}\int_0^\infty \frac{u^{\frac{\theta-1}{2}}}{1+u}du.$$

Then by using the transformation $z = \frac{u}{1+u}$, we obtain that

$$\begin{aligned}\int_0^\infty \frac{t^\theta}{t^2+a^2}dt &= \frac{a^{\theta-1}}{2}\int_0^1 z^{\frac{\theta-1}{2}}(1-z)^{-\frac{\theta+1}{2}}dz\\ &= \frac{a^{\theta-1}}{2}B\left(\frac{1+\theta}{2},\frac{1-\theta}{2}\right).\end{aligned}$$

This gives by (5) that

$$\int_0^\infty \frac{t^\theta}{t^2+a^2}dt = \frac{\pi a^{\theta-1}}{2\sin(\frac{\pi(1+\theta)}{2})}.$$

Noting that

$$\sin\left(\frac{\pi(1+\theta)}{2}\right) = \cos\left(\frac{\pi\theta}{2}\right), \tag{8}$$

and

$$\sin\left(\frac{\pi(1-\theta)}{2}\right) = \cos\left(\frac{\pi\theta}{2}\right),$$

we deduce (6).

2. By integration by parts, we have

$$\int_0^\infty \frac{\sin(at)}{t^\alpha}\,dt = \left[\frac{1-\cos(at)}{at^\alpha}\right]_0^\infty + \frac{\alpha}{a}\int_0^\infty \frac{1-\cos(at)}{t^{\alpha+1}}\,dt.$$

Using (4), this yields

$$\int_0^\infty \frac{\sin(at)}{t^\alpha}\,dt = \frac{\alpha}{a\,\Gamma(\alpha+1)}\int_0^\infty\int_0^\infty (1-\cos(at))\,x^\alpha e^{-tx}\,dt\,dx.$$

Since the integrand $(1-\cos(at))\,x^\alpha e^{-tx}$ is nonnegative on $(0,\infty)\times(0,\infty)$, the Fubini-Tonelli theorem applies, allowing us to interchange the order of integration, and we obtain

$$\begin{aligned}\int_0^\infty \frac{\sin(at)}{t^\alpha}dt &= \frac{\alpha}{a\Gamma(\alpha+1)}\int_0^\infty x^\alpha\left(\int_0^\infty e^{-tx}(1-\cos(at)dt\right)dx\\ &= \frac{1}{a\Gamma(\alpha)}\int_0^\infty x^\alpha\left(\frac{1}{x}-\frac{x}{x^2+a^2}\right)dx\\ &= \frac{a}{\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha-1}}{x^2+a^2}dx.\end{aligned}$$

This gives, using (6) for $\theta = \alpha - 1$, the intended result.

3. By an integration by parts, we get that

$$\int_0^\infty \frac{\cos(at)}{t^\alpha}dt = \left[\frac{\sin(at)}{at^\alpha}\right]_0^\infty + \frac{\alpha}{a}\int_0^\infty \frac{\sin(at)}{t^{\alpha+1}}dt$$

$$= \frac{\alpha}{a}\int_0^{\infty}\frac{\sin(at)}{t^{\alpha+1}}dt.$$

Since $1<\alpha+1<2$, we achieve, using (7) and (8), the desired result.
□

Lemma 2. *Let $a>0$, $t>0$, and $m\in\mathbb{N}$. Then we have*

$$\frac{t^m}{t+a} = \begin{cases} \sum_{j=0}^{m-1}(-1)^{m-j-1}a^{m-j-1}t^j + (-1)^m\frac{a^m}{t+a}, & \text{if } m\in\mathbb{N}^*,\\ \frac{1}{t+a}, & \text{if } m=0. \end{cases}$$

Remark 1 (Parity-Endpoint Mechanism)**.** The Mellin transforms of the trigonometric functions $S_n(x)$, $T_n(x)$, and $J_n(x)$ exhibit a unified behavior with respect to the exponent in the denominator of the integrand. Specifically, for each function, there exists an integer (denoted p, q, or r, as defined in Lemmas 3, 5 and 7 below) that determines the convergence interval of the Mellin transform. The evaluation of the integrals

$$\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx,\quad \int_0^{\infty}\frac{T_n(x)}{x^{\sigma}}dx,\quad \int_0^{\infty}\frac{J_n(x)}{x^{\alpha}}dx$$

depends critically on whether the exponent α or σ lies in a specific discrete set:

- For $S_n(x)$, the integral simplifies to a closed form involving trigonometric and Gamma functions when $\alpha\notin 2\mathbb{N}$, but yields logarithmic terms when $\alpha\in 2\mathbb{N}^*$ (Theorem 1).
- For $T_n(x)$, the closed form holds for $\sigma\notin 2\mathbb{N}+1$, while logarithmic terms appear for $\sigma\in 2\mathbb{N}+1$ (Theorem 2).
- For $J_n(x)$, the closed form applies for $\alpha\notin 2\mathbb{N}+3$, and logarithmic terms arise for $\alpha\in 2\mathbb{N}+3$ (Theorem 3).

This dichotomy stems from the pole structure of the Mellin transform and is a direct consequence of the identities established in Lemmas 1 and 2. The minimal indices p,q,r govern the convergence domains and the appearance of logarithmic singularities at certain values of the exponent.

3. Mellin transform of S_n, $n\in\mathbb{N}$

In this section, we consider $n\in\mathbb{N}$ and define $S_n(x)=\sum_{k=0}^{n}c_k\sin(a_kx)$, where for $0\le k\le n$, $c_k\in\mathbb{R}^*$, and $0<a_0<\cdots<a_n$.

Lemma 3. *Let $p=\min\{l\in\mathbb{N}:\sum_{k=0}^{n}c_ka_k^{2l+1}\neq 0\}$. Then, we have $0\le p\le n$.*

Proof. Suppose that $p\ge n+1$. This implies that for all $0\le l\le n$, we have $\sum_{k=0}^{n}c_ka_k^{2l+1}=0$. Specifically, this system can be written as:

$$(S)\begin{cases} a_0c_0+\cdots+a_nc_n=0\\ a_0^3c_0+\cdots+a_n^3c_n=0\\ \vdots\\ a_0^{2n+1}c_0+\cdots+a_n^{2n+1}c_n=0. \end{cases}$$

(S) is a system of $n+1$ linear equations with $n+1$ unknowns $c_0, \cdots, c_n$. Its determinant is given by

$$\det_S = \begin{vmatrix} a_0 & . & . & . & a_n \\ a_0^3 & . & . & . & a_n^3 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_0^{2n+1} & . & . & . & a_n^{2n+1} \end{vmatrix} = (\prod_{k=0}^{n} a_k) V(a_0^2, \dots, a_n^2),$$

where $V(a_0^2, \dots, a_n^2)$ is a Vandermonde determinant. This implies that

$$\det_S = \prod_{k=0}^{n} a_k \prod_{0 \le i < j \le n} (a_j^2 - a_i^2) > 0.$$

Thus, the system (S) is a Cramer system, which implies that its unique solution is the $(n+1)-$tuple $(0, \cdots, 0)$. This contradicts the assumption that for $0 \le k \le n$, $c_k \in \mathbb{R}^*$. Therefore $p \le n$. □

Lemma 4. *There exists $C > 0$ such that for all $x \ge 0$, we have*

$$|S_n(x)| \le C \, \min(x^{2p+1}, 1).$$

Proof. From the Taylor expansion of the sine function at 0 and the definition of p, we obtain, for $x > 0$,

$$S_n(x) = \frac{(-1)^p}{(2p+1)!} \left(\sum_{k=0}^{n} c_k a_k^{2p+1} \right) x^{2p+1} + o(x^{2p+2}),$$

so that $x \mapsto S_n(x)/x^{2p+1}$ is continuous on $[0,1]$. Moreover, since $|S_n(x)| \le \sum_{k=0}^{n} |c_k|$ for $x \ge 0$, it follows that there exists $C > 0$ such that

$$|S_n(x)| \le C \min\{x^{2p+1}, 1\}, \qquad x \ge 0.$$

□

Remark 2. By the definition of the Mellin transform

$$M(S_n)(s) = \int_0^\infty x^{s-1} S_n(x) \, dx,$$

we have

$$\int_0^\infty \frac{S_n(x)}{x^\alpha} \, dx = \int_0^\infty x^{(-\alpha+1)-1} S_n(x) \, dx = M(S_n)(1-\alpha).$$

In particular, this integral converges if and only if

$$0 < \alpha < 2p+2,$$

so that the strip of definition for $M(S_n)(s)$ is

$$-(2p+1) < s < 1.$$

Besides, we have that the Mellin transform $M(S_n)(s)$ is analytic on the vertical strip

$$-(2p+1) < s < 1.$$

Moreover, singularities occur at the points

$$s = 1 - 2m, \qquad m \in \mathbb{N}^*,$$

which correspond to even values of the parameter α.

Theorem 1. *Assume* $0 < \alpha < 2p+2$. *Then*

$$\int_0^\infty \frac{S_n(x)}{x^\alpha}dx = M(S_n)(1-\alpha) = \begin{cases} \frac{\pi}{2\,\Gamma(\alpha)\,\sin(\frac{\pi\alpha}{2})}\sum_{k=0}^{n} c_k\, a_k^{\alpha-1}, & \text{if } \alpha \notin 2\mathbb{N}, \\ \frac{(-1)^{\frac{\alpha}{2}}}{\Gamma(\alpha)}\sum_{k=0}^{n} c_k\, a_k^{\alpha-1}\,\ln a_k, & \text{if } \alpha \in 2\mathbb{N}^*. \end{cases}$$

Proof. Let $0 < \alpha < 2p+2$. We distinguish two cases.

Case 1. If $0 < \alpha < 2$, then by applying Lemma 1, we obtain that

$$\begin{aligned} \int_0^\infty \frac{S_n(x)}{x^\alpha}dx &= \sum_{k=0}^{n} c_k \int_0^\infty \frac{\sin(a_k x)}{x^\alpha}dx \\ &= \frac{\pi}{2\,\Gamma(\alpha)\,\sin(\frac{\pi\alpha}{2})}\sum_{k=0}^{n} c_k\, a_k^{\alpha-1}. \end{aligned}$$

Case 2. If $2 \le \alpha < 2p+2$, then we have from (4),

$$\begin{aligned} \int_0^\infty \frac{S_n(x)}{x^\alpha}dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(\int_0^\infty e^{-tx}t^{\alpha-1}dt\right)S_n(x)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty e^{-tx}t^{\alpha-1}S_n(x)dtdx. \end{aligned}$$

Using Lemma 4, we have on $(0,\infty)\times(0,\infty)$,

$$|e^{-tx}t^{\alpha-1}S_n(x)| \le C\,e^{-tx}t^{\alpha-1}\min(x^{2p+1},1).$$

Since the function $(x,t) \mapsto e^{-tx}t^{\alpha-1}\min(x^{2p+1},1)$ is integrable over $(0,\infty)\times(0,\infty)$, then Fubini-Tonelli theorem applies and we obtain that

$$\begin{aligned} \int_0^\infty \frac{S_n(x)}{x^\alpha}dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(\int_0^\infty e^{-tx}t^{\alpha-1}dt\right)S_n(x)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\left(\int_0^\infty e^{-tx}S_n(x)dx\right)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\left(\int_0^\infty e^{-tx}\sum_{k=0}^{n} c_k\sin(a_k x)dx\right)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \sum_{k=0}^{n} c_k t^{\alpha-1}\left(\int_0^\infty e^{-tx}\sin(a_k x)dx\right)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \sum_{k=0}^{n} c_k a_k \frac{t^{\alpha-1}}{t^2+a_k^2}dt. \end{aligned}$$

Due to the parity-endpoint mechanism (Remark 1), the interval $[2,2p+2)$ decomposes as:

$$[2,2p+2) = \left(\bigcup_{m=1}^{p}(2m,2m+2)\right)\cup\left(\bigcup_{m=0}^{p-1}\{2m+2\}\right).$$

We therefore distinguish two subcases:

Subcase 1. If $\alpha \notin 2\mathbb{N}$, then there exist $m \in \{1,\cdots,p\}$ and $-1 < \epsilon < 1$ such that $\alpha = 2m+1+\epsilon$. This is equivalent to $2m < \alpha < 2m+2$.

Then, we get by using Lemma 2, that

$$\sum_{k=0}^{n} c_k a_k \frac{t^{\alpha-1}}{t^2+a_k^2} = t^\epsilon \sum_{k=0}^{n} c_k a_k \frac{t^{2m}}{t^2+a_k^2}$$

$$
\begin{aligned}
&= t^{\epsilon}\sum_{k=0}^{n}c_ka_k\left(\sum_{j=0}^{m-1}(-1)^{m-j-1}a_k^{2(m-j-1)}t^{2j}+(-1)^m\frac{a_k^{2m}}{t^2+a_k^2}\right)\\
&= t^{\epsilon}\left[\sum_{j=0}^{m-1}(-1)^{m-j-1}\left(\sum_{k=0}^{n}c_ka_k^{2(m-j-1)+1}\right)t^{2j}+(-1)^m\sum_{k=0}^{n}\frac{c_ka_k^{2m+1}}{t^2+a_k^2}\right].
\end{aligned}
$$

Since for $0 \le j \le m-1$, we have $0 \le m-j-1 \le m-1 \le p-1$, it follows that for $0 \le j \le m-1$, we have $\sum_{k=0}^{n} c_k a_k^{2(m-j-1)+1} = 0$. Hence we reach that

$$
\begin{aligned}
\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty}(-1)^m t^{\epsilon}\sum_{k=0}^{n}\frac{c_ka_k^{2m+1}}{t^2+a_k^2}dt\\
&= \frac{(-1)^m}{\Gamma(\alpha)}\sum_{k=0}^{n}c_ka_k^{2m+1}\int_0^{\infty}\frac{t^{\epsilon}}{t^2+a_k^2}dt.
\end{aligned}
$$

Using (6), we obtain that

$$
\begin{aligned}
\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx &= \frac{(-1)^m}{\Gamma(\alpha)}\sum_{k=0}^{n}c_ka_k^{2m+1}\frac{\pi a_k^{\epsilon-1}}{2\sin(\frac{\pi(1+\epsilon)}{2})}\\
&= \frac{(-1)^m}{\Gamma(\alpha)}\sum_{k=0}^{n}c_k\frac{\pi a_k^{2m+\epsilon}}{2\sin(\frac{\pi(\alpha-2m)}{2})}\\
&= \frac{\pi}{2\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}\sum_{k=0}^{n}c_ka_k^{\alpha-1}.
\end{aligned}
$$

Subcase 2. If $\alpha \in 2\mathbb{N}^*$, then there exists $m \in \{0, \cdots, p-1\}$ such that $\alpha = 2m+2$. So, we have

$$
\sum_{k=0}^{n}c_ka_k\frac{t^{\alpha-1}}{t^2+a_k^2} = t\sum_{k=0}^{n}c_ka_k\frac{t^{2m}}{t^2+a_k^2}.
$$

As in subcase 1, we obtain that

$$
\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx = \frac{(-1)^m}{\Gamma(\alpha)}\lim_{t\to\infty}\sum_{k=0}^{n}c_ka_k^{2m+1}\int_0^{t}\frac{s}{s^2+a_k^2}ds.
$$

This implies that

$$
\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx = \frac{(-1)^m}{2\Gamma(\alpha)}\lim_{t\to\infty}\sum_{k=0}^{n}c_ka_k^{2m+1}\left[\ln(s^2+a_k^2)\right]_0^t.
$$

Using the fact that for $0 \le m \le p-1$, we have $\sum_{k=0}^{n} c_k a_k^{2m+1} = 0$, we get that

$$
\begin{aligned}
\int_0^{\infty}\frac{S_n(x)}{x^{\alpha}}dx &= \frac{(-1)^m}{2\Gamma(\alpha)}\left[\lim_{t\to\infty}\sum_{k=0}^{n}c_ka_k^{2m+1}\ln(1+\frac{a_k^2}{t^2})-\sum_{k=0}^{n}c_ka_k^{2m+1}\ln(a_k^2)\right]\\
&= \frac{(-1)^{m+1}}{\Gamma(\alpha)}\sum_{k=0}^{n}c_ka_k^{2m+1}\ln(a_k)\\
&= \frac{(-1)^{\frac{\alpha}{2}}}{\Gamma(\alpha)}\sum_{k=0}^{n}c_ka_k^{\alpha-1}\ln(a_k).
\end{aligned}
$$

This ends the proof. □

As an application of Theorem 1, we give the following examples.

Example 2. 1. Consider the improper integral $\int_0^\infty \frac{5\sin x - 4\sin 2x + \sin 3x}{x^\alpha}dx$. Here, we have $n = 2$, $c_0 = 5$, $c_1 = -4$, $c_2 = 1$, $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$. It can be easily verified that $p = 2$. Therefore, $\int_0^\infty \frac{5\sin x - 4\sin 2x + \sin 3x}{x^\alpha}dx$ converges if and only if $0 < \alpha < 6$. By Theorem 1, we have

$$\int_0^\infty \frac{5\sin x - 4\sin 2x + \sin 3x}{x^\alpha}dx = \begin{cases} \frac{\pi}{2\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}(5 - 2^{\alpha+1} + 3^{\alpha-1}), & \text{if} \quad \alpha \notin 2\mathbb{N}, \\ \frac{(-1)^{\frac{\alpha}{2}}}{\Gamma(\alpha)}(-2^{\alpha+1}\ln(2) + 3^{\alpha-1}\ln(3)), & \text{if} \quad \alpha \in 2\mathbb{N}^*. \end{cases}$$

To illustrate this result further, we consider the following specific cases.

a. For $\alpha = \frac{5}{2}$, we have

$$\int_0^\infty \frac{5\sin x - 4\sin 2x + \sin 3x}{x^{\frac{5}{2}}}dx = \frac{2\sqrt{2\pi}}{3}(8\sqrt{2} - 5 - 3\sqrt{3}).$$

b. For $\alpha = 4$, we have

$$\int_0^\infty \frac{5\sin x - 4\sin 2x + \sin 3x}{x^4}dx = \frac{9}{2}\ln(3) - \frac{16}{3}\ln(2).$$

2. For $a > 0$, $b > 0$, we consider the improper integral $\int_0^\infty \frac{\cos(ax)\sin(bx)}{x^\alpha}dx$. We have

$$\cos(ax)\sin(bx) = \frac{1}{2}sign(b-a)\sin(|b-a|x) + \frac{1}{2}\sin((b+a)x),$$

where

$$sign(b-a) = \begin{cases} -1, & \text{if} \quad a > b, \\ 0, & \text{if} \quad a = b, \\ 1, & \text{if} \quad a < b. \end{cases}$$

So,

$$\begin{cases} n = 1, c_0 = \frac{1}{2}sign(b-a), c_1 = \frac{1}{2}, a_0 = |b-a|, a_1 = b+a, & \text{if} \quad a \neq b, \\ n = 0, c_0 = \frac{1}{2}, a_0 = 2a, & \text{if} \quad a = b. \end{cases}$$

It is easy to see that $p = 0$. Therefore, $\int_0^\infty \frac{\cos(ax)\sin(bx)}{x^\alpha}dx$ converges if and only if $0 < \alpha < 2$. Hence by applying Theorem 1, we obtain that

$$\int_0^\infty \frac{\cos(ax)\sin(bx)}{x^\alpha}dx = \frac{\pi}{4\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}(sign(b-a)|b-a|^{\alpha-1} + (b+a)^{\alpha-1}).$$

Corollary 1. *Let $n \in \mathbb{N}$ and $0 < \alpha < 2n+2$. Then, we have*

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^\alpha}dx = \begin{cases} \frac{\pi}{2^{2n+1}\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}\sum_{k=0}^{n}(-1)^k C_{2n+1}^{n-k}(2k+1)^{\alpha-1}, & \textit{if} \quad \alpha \notin 2\mathbb{N}, \\ \frac{(-1)^{\frac{\alpha}{2}}}{2^{2n}\Gamma(\alpha)}\sum_{k=0}^{n}(-1)^k C_{2n+1}^{n-k}(2k+1)^{\alpha-1}\ln(2k+1), & \textit{if} \quad \alpha \in 2\mathbb{N}^*. \end{cases}$$

Proof. Let $n \in \mathbb{N}$. Through linearization, which relies on Euler's formula and the binomial theorem, we have that for $x \in \mathbb{R}$,

$$\sin^{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k C_{2n+1}^{n-k}}{2^{2n}}\sin((2k+1)x). \tag{9}$$

To apply Theorem 1, we observe that for $0 \le k \le n$,

$$c_k = \frac{(-1)^k C_{2n+1}^{n-k}}{2^{2n}},$$

and

$$a_k = 2k + 1.$$

On the other hand, we have

$$\sin^{2n+1}(x) = x^{2n+1} + o(x^{2n+2}).$$

This implies that $p = n$ and for $0 \le m \le n-1$, $n \ge 1$,

$$\sum_{k=0}^{n} c_k (2k+1)^{2m+1} = 0.$$

Hence, applying Remark 2, we deduce that the integral $\int_0^\infty \frac{\sin^{2n+1}(x)}{x^\alpha} dx$ converges if and only if $0 < \alpha < 2n+2$. Moreover, Theorem 1 gives that

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^\alpha} dx = \begin{cases} \frac{\pi}{2^{2n+1}\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})} \sum_{k=0}^{n} (-1)^k C_{2n+1}^{n-k} (2k+1)^{\alpha-1}, & \text{if} \quad \alpha \notin 2\mathbb{N}, \\ \text{and} & \\ \frac{(-1)^{\frac{\alpha}{2}}}{2^{2n}\Gamma(\alpha)} \sum_{k=0}^{n} (-1)^k C_{2n+1}^{n-k} (2k+1)^{\alpha-1} \ln(2k+1), & \text{if} \quad \alpha \in 2\mathbb{N}^*. \end{cases}$$

□

In the following example, we present specific cases of Corollary 1.

Example 3.
1. By taking $n = 1$ and $\alpha = \frac{1}{2}$, we obtain that

$$\int_0^\infty \frac{\sin^3(x)}{\sqrt{x}} dx = \frac{\sqrt{2\pi}}{8} \left(3 - \frac{\sqrt{3}}{3}\right).$$

2. For $n \in \mathbb{N}^*$, we have:

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^{2n}} dx = \frac{(-1)^n}{2^{2n}(2n-1)!} \sum_{k=0}^{n} (-1)^k C_{2n+1}^{n-k} (2k+1)^{2n-1} \ln(2k+1).$$

3. For $n \in \mathbb{N}$, we have:

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^{2n+1}} dx = \frac{(-1)^n \pi}{2^{2n+1}(2n)!} \sum_{k=0}^{n} (-1)^k C_{2n+1}^{n-k} (2k+1)^{2n}.$$

4. Mellin transform of T_n, $n \in \mathbb{N}$

In this section, we consider $n \in \mathbb{N}$ and define $T_n(x) = \sum_{k=0}^{n} c_k \cos(a_k x)$, where for $0 \le k \le n$, $c_k \in \mathbb{R}^*$, and $0 < a_0 < \cdots < a_n$.

Using similar arguments as in the proofs of Lemmas 3 and 4, we obtain the following results.

Lemma 5. *Let* $q = \min\{l \in \mathbb{N} : \sum_{k=0}^{n} c_k a_k^{2l} \neq 0\}$. *Then, we have* $0 \le q \le n$.

Lemma 6. *There exists* $C > 0$ *such that for* $x \ge 0$, *we have*

$$|T_n(x)| \le C \min(x^{2q}, 1).$$

Remark 3. By the definition of the Mellin transform

$$M(T_n)(s) = \int_0^\infty x^{s-1} T_n(x)\, dx,$$

we have

$$\int_0^\infty \frac{T_n(x)}{x^\sigma}\, dx = \int_0^\infty x^{(-\sigma+1)-1} T_n(x)\, dx = M(T_n)(1-\sigma).$$

Hence this integral converges precisely when

$$0 < \sigma < 2q+1,$$

so that the Mellin transform $M(T_n)(s)$ is defined on the vertical strip

$$-2q < s < 1.$$

Furthermore, the Mellin transform $M(T_n)(s)$ is analytic on the strip $-2q < s < 1$. Its poles are located at $s = -2m$, with $m \in \mathbb{N}^*$, and correspond to odd values of the parameter σ.

Theorem 2. *Assume* $0 < \sigma < 2q+1$. *Then*

$$\int_0^\infty \frac{T_n(x)}{x^\sigma} dx = M(T_n)(1-\sigma) = \begin{cases} \frac{\pi}{2\,\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})} \sum_{k=0}^{n} c_k\, a_k^{\sigma-1}, & \text{if } \sigma \notin 2\mathbb{N}+1, \\ \frac{(-1)^{\frac{1+\sigma}{2}}}{\Gamma(\sigma)} \sum_{k=0}^{n} c_k\, a_k^{\sigma-1} \ln a_k, & \text{if } \sigma \in 2\mathbb{N}+1. \end{cases}$$

Proof. Let $0 < \sigma < 2q+1$. We distinguish two cases.

Case 1. If $0 < \sigma < 1$, then by applying Lemma 1, we obtain that

$$\begin{aligned} \int_0^\infty \frac{T_n(x)}{x^\sigma} dx &= \sum_{k=0}^{n} c_k \int_0^\infty \frac{\cos(a_k x)}{x^\sigma} dx \\ &= \frac{\pi}{2\,\Gamma(\sigma)\,\cos(\frac{\pi\sigma}{2})} \sum_{k=0}^{n} c_k\, a_k^{\sigma-1}. \end{aligned}$$

Case 2. If $1 \le \sigma < 2q+1$, then we have from (4),

$$\begin{aligned} \int_0^\infty \frac{T_n(x)}{x^\sigma} dx &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left(\int_0^\infty e^{-tx} t^{\sigma-1} dt \right) T_n(x) dx \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-tx} t^{\sigma-1} T_n(x) dt dx. \end{aligned}$$

Using Lemma 6, we have on $(0,\infty) \times (0,\infty)$,

$$|e^{-tx} t^{\sigma-1} T_n(x)| \le C\, e^{-tx} t^{\sigma-1} \min(x^{2q}, 1).$$

Since the function $(x,t) \mapsto e^{-tx} t^{\sigma-1} \min(x^{2q}, 1)$ is integrable over $(0,\infty) \times (0,\infty)$, then Fubini-Tonelli theorem applies and we obtain that

$$\begin{aligned} \int_0^\infty \frac{T_n(x)}{x^\sigma} dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_0^\infty e^{-tx} t^{\sigma-1} dt \right) T_n(x) dx \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} \left(\int_0^\infty e^{-tx} T_n(x) dx \right) dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} \left(\int_0^\infty e^{-tx} \sum_{k=0}^{n} c_k \cos(a_k x) dx \right) dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{k=0}^n c_k t^{\sigma-1} \left(\int_0^\infty e^{-tx} \cos(a_k x) dx \right) dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{k=0}^n c_k \frac{t^\sigma}{t^2 + a_k^2} dt. \end{aligned}$$

Due to the parity-endpoint mechanism (Remark 1), the interval $[1, 2q+1)$ decomposes as:

$$[1, 2q+1) = \bigcup_{m=1}^{q} \left((2m-1, 2m+1) \cup \{2m-1\} \right).$$

We therefore distinguish two subcases:

Subcase 1. If $\sigma \notin 2\mathbb{N}+1$, then there exist $m \in \{1, \cdots, q\}$ and $-1 < \epsilon < 1$ such that $\sigma = 2m + \epsilon$. This is equivalent to $2m - 1 < \sigma < 2m + 1$.

Then, we get by using Lemma 2, that

$$\begin{aligned} \sum_{k=0}^n c_k \frac{t^\sigma}{t^2 + a_k^2} &= t^\epsilon \sum_{k=0}^n c_k \frac{t^{2m}}{t^2 + a_k^2} \\ &= t^\epsilon \sum_{k=0}^n c_k \left(\sum_{j=0}^{m-1} (-1)^{m-j-1} a_k^{2(m-j-1)} t^{2j} + (-1)^m \frac{a_k^{2m}}{t^2 + a_k^2} \right) \\ &= t^\epsilon \left[\sum_{j=0}^{m-1} (-1)^{m-j-1} \left(\sum_{k=0}^n c_k a_k^{2(m-j-1)} \right) t^{2j} + (-1)^m \sum_{k=0}^n \frac{c_k a_k^{2m}}{t^2 + a_k^2} \right]. \end{aligned}$$

Given that $0 \le j \le m-1$ implies $0 \le m-j-1 \le m-1 \le q-1$, it follows that for $0 \le j \le m-1$, we have $\sum_{k=0}^n c_k a_k^{2(m-j-1)} = 0$. Thus, applying (6), we conclude that

$$\begin{aligned} \int_0^\infty \frac{T_n(x)}{x^\sigma} dx &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{k=0}^n c_k \frac{t^\sigma}{t^2 + a_k^2} dt \\ &= \frac{(-1)^m}{\Gamma(\sigma)} \sum_{k=0}^n c_k a_k^{2m} \int_0^\infty \frac{t^\epsilon}{t^2 + a_k^2} dt \\ &= \frac{(-1)^m}{\Gamma(\sigma)} \sum_{k=0}^n c_k a_k^{2m} \frac{\pi a_k^{\epsilon-1}}{2\cos(\frac{\pi\epsilon}{2})} \\ &= \frac{(-1)^m}{\Gamma(\sigma)} \sum_{k=0}^n c_k \frac{\pi a_k^{2m+\epsilon-1}}{2\cos(\frac{\pi(\sigma-2m)}{2})} \\ &= \frac{\pi}{2\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})} \sum_{k=0}^n c_k a_k^{\sigma-1}. \end{aligned}$$

Subcase 2. If $\alpha \in 2\mathbb{N}+1$, then there exists $m \in \{0, \cdots, q-1\}$ such that $\sigma = 2m+1$. So, we have by similar arguments as above that

$$\begin{aligned} \int_0^\infty \frac{T_n(x)}{x^\sigma} dx &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \sum_{k=0}^n c_k \frac{t^{2m+1}}{t^2 + a_k^2} dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty t \sum_{k=0}^n c_k \frac{t^{2m}}{t^2 + a_k^2} dt \\ &= \frac{(-1)^m}{2\Gamma(\sigma)} \lim_{t\to\infty} \sum_{k=0}^n c_k a_k^{2m} \int_0^t \frac{2s}{s^2 + a_k^2} ds. \end{aligned}$$

Taking into account that for $0 \leq m \leq q-1$, we have $\sum_{k=0}^{n} c_k a_k^{2m} = 0$, we get that

$$\begin{aligned}
\int_0^\infty \frac{T_n(x)}{x^\sigma}dx &= \frac{(-1)^m}{2\Gamma(\sigma)} \lim_{t\to\infty} \sum_{k=0}^{n} c_k a_k^{2m} \left[\ln(t^2+a_k^2)\right]_0^t \\
&= \frac{(-1)^{m+1}}{\Gamma(\sigma)} \sum_{k=0}^{n} c_k a_k^{2m} \ln(a_k) \\
&= \frac{(-1)^{\frac{1+\sigma}{2}}}{\Gamma(\sigma)} \sum_{k=0}^{n} c_k a_k^{\sigma-1} \ln(a_k).
\end{aligned}$$

This ends the proof. □

To illustrate Theorem 2, we provide the following examples.

Example 4. 1. Consider the improper integral $\int_0^\infty \frac{5\cos x - 8\cos 2x + 3\cos 3x}{x^\sigma}dx$. Here, we have $n = 2$, $c_0 = 5$, $c_1 = -8$, $c_2 = 3$, $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$. It can be easily verified that $q = 2$. Therefore, $\int_0^\infty \frac{5\cos x - 8\cos 2x + 3\cos 3x}{x^\sigma}dx$ converges if and only if $0 < \sigma < 5$. By Theorem 2, we have

$$\int_0^\infty \frac{5\cos x - 8\cos 2x + 3\cos 3x}{x^\sigma}dx = \begin{cases} \frac{\pi}{2\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})}(5 - 2^{\sigma+2} + 3^\sigma), & \text{if} \quad \sigma \notin 2\mathbb{N}+1, \\ \frac{(-1)^{\frac{1+\sigma}{2}}}{\Gamma(\sigma)}(-2^{\sigma+2}\ln(2) + 3^\sigma \ln(3)), & \text{if} \quad \sigma \in 2\mathbb{N}+1. \end{cases}$$

To further describe this result, we examine the following specific cases.

a. For $\sigma = \frac{7}{2}$, we have

$$\int_0^\infty \frac{5\cos x - 8\cos 2x + 3\cos 3x}{x^{\frac{7}{2}}}dx = \frac{16\sqrt{2\pi}}{15}(5 - 32\sqrt{2} + 27\sqrt{3}).$$

b. For $\sigma = 3$, we have

$$\int_0^\infty \frac{5\cos x - 8\cos 2x + 3\cos 3x}{x^3}dx = \frac{27}{2}\ln(3) - 16\ln(2).$$

2. For $0 < a < b$, we consider the improper integral $\int_0^\infty \frac{\cos(ax)\cos(bx)}{x^\sigma}dx$.
Using the trigonometric identity

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((b-a)x) + \frac{1}{2}\cos((b+a)x),$$

we get that

$$n = 1, c_0 = c_1 = \frac{1}{2}, a_0 = b-a, a_1 = b+a.$$

It is straightforward to see that $q = 0$. Therefore, $\int_0^\infty \frac{\cos(ax)\cos(bx)}{x^\sigma}dx$ converges if and only if $0 < \sigma < 1$. Hence by applying Theorem 2, we obtain that

$$\int_0^\infty \frac{\cos(ax)\cos(bx)}{x^\sigma}dx = \frac{\pi}{4\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})}((b-a)^{\sigma-1} + (b+a)^{\sigma-1}).$$

3. For $0 < a < b$, we consider the improper integral $\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^\sigma}dx$.
Using the trigonometric identity

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((b-a)x) - \frac{1}{2}\cos((b+a)x),$$

we get that

$$n = 1, c_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, a_0 = b-a, a_1 = b+a.$$

It is easy to see that $q = 1$. Thus, the integral $\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^\sigma}dx$ converges if and only if $0 < \sigma < 3$. By applying Theorem 2, we find

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^\sigma}dx = \begin{cases} \frac{\pi}{4\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})}((b-a)^{\sigma-1} - (b+a)^{\sigma-1}), & \text{if} \quad \sigma \in (0,3)\backslash\{1\}, \\ -\frac{1}{2}((b-a)^{\sigma-1}\ln(b-a) - (b+a)^{\sigma-1}\ln(b+a)), & \text{if} \quad \sigma = 1. \end{cases}$$

Corollary 2. *Let $n \in \mathbb{N}$ and $0 < \sigma < 1$. Then, we have*

$$\int_0^\infty \frac{\cos^{2n+1}(x)}{x^\sigma}dx = \frac{\pi}{2^{2n+1}\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})}\sum_{k=0}^{n} C_{2n+1}^{n-k}(2k+1)^{\sigma-1}.$$

Proof. Let $n \in \mathbb{N}$. Using linearization, we obtain that for $x \in \mathbb{R}$,

$$\cos^{2n+1}(x) = \sum_{k=0}^{n} \frac{C_{2n+1}^{n-k}}{4^n}\cos((2k+1)x).$$

To apply Theorem 2, observe that for $0 \le k \le n$, we have

$$c_k = \frac{C_{2n+1}^{n-k}}{4^n},$$

and

$$a_k = 2k+1.$$

Since $\sum_{k=0}^{n} \frac{C_{2n+1}^{n-k}}{4^n} \neq 0$, it follows that $q = 0$. Therefore, applying Remark 3, we deduce that $\int_0^\infty \frac{\cos^{2n+1}(x)}{x^\sigma}dx$ converges if and only if $0 < \sigma < 1$. Furthermore, Theorem 2 gives that

$$\int_0^\infty \frac{\cos^{2n+1}(x)}{x^\sigma}dx = \frac{\pi}{2^{2n+1}\Gamma(\sigma)\cos(\frac{\pi\sigma}{2})}\sum_{k=0}^{n} C_{2n+1}^{n-k}a_k^{\sigma-1}.$$

□

To illustrate Corollary 2, we provide the following example.

Example 5. $\int_0^\infty \frac{\cos^3(x)}{\sqrt{x}}dx = \frac{\sqrt{6\pi}}{24}\left(1+3\sqrt{3}\right).$

5. Mellin transform of $\mathbf{J_n}$, $\mathbf{n} \in \mathbb{N}^*$

In this section, we consider $n \in \mathbb{N}^*$ and define $J_n(x) = \sum_{k=1}^{n} c_k(1-\cos(a_k x))$, where for $1 \le k \le n$, $c_k \in \mathbb{R}^*$, and $0 < a_1 < \cdots < a_n$.

Proceeding as in the proofs of Lemmas 3 and 4, we obtain the following results.

Lemma 7. *Let* $r = \min\{l \in \mathbb{N}^* : \sum_{k=1}^{n} c_k a_k^{2l} \neq 0\}$*. Then, we have* $1 \leq r \leq n$*.*

Lemma 8. *There exists* $C > 0$ *such that for* $x \geq 0$*, we have*

$$|J_n(x)| \leq C \min(x^{2r}, 1).$$

Remark 4. By definition of the Mellin transform

$$M(J_n)(s) = \int_0^\infty x^{s-1} J_n(x)\, dx,$$

we observe

$$\int_0^\infty \frac{J_n(x)}{x^\alpha}\, dx = \int_0^\infty x^{(-\alpha+1)-1} J_n(x)\, dx = M(J_n)(1-\alpha).$$

Therefore, this integral converges precisely when

$$1 < \alpha < 2r+1,$$

which corresponds to the definition strip

$$-2r < s < 0$$

for $M(J_n)(s)$. The Mellin transform $M(J_n)(s)$ is analytic on the strip $-2r < s < 0$, with singularities at $\alpha \in 2\mathbb{N}+3$, corresponding to $s = 1-\alpha$, that is, odd integers shifted.

Theorem 3. *Assume* $1 < \alpha < 2r+1$*. Then*

$$\int_0^\infty \frac{J_n(x)}{x^\alpha} dx = M\{J_n\}(1-\alpha) = \begin{cases} -\frac{\pi}{2\,\Gamma(\alpha)\,\cos(\frac{\pi\alpha}{2})} \sum_{k=1}^{n} c_k\, a_k^{\alpha-1}, & \text{if } \alpha \notin 2\mathbb{N}+3, \\ \frac{(-1)^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)} \sum_{k=1}^{n} c_k\, a_k^{\alpha-1} \ln a_k, & \text{if } \alpha \in 2\mathbb{N}+3. \end{cases}$$

Proof. Let $1 < \alpha < 2r+1$. In view of (4), we have

$$\begin{aligned} \int_0^\infty \frac{J_n(x)}{x^\alpha} dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_0^\infty e^{-tx} t^{\alpha-1} dt \right) J_n(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^\infty e^{-tx} t^{\alpha-1} J_n(x) dt dx. \end{aligned}$$

Using Lemma 8, we have on $(0,\infty) \times (0,\infty)$,

$$|e^{-tx} t^{\alpha-1} J_n(x)| \leq C\, e^{-tx} t^{\alpha-1} \min(x^{2r}, 1).$$

Since the function $(x,t) \mapsto e^{-tx} t^{\alpha-1} \min(x^{2r}, 1)$ is integrable over $(0,\infty) \times (0,\infty)$, then Fubini-Tonelli theorem applies and we obtain that

$$\begin{aligned} \int_0^\infty \frac{J_n(x)}{x^\alpha} dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left(\int_0^\infty e^{-tx} J_n(x) dx \right) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=1}^{n} c_k t^{\alpha-1} \left(\int_0^\infty e^{-tx} (1 - \cos(a_k x)) dx \right) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=1}^{n} c_k t^{\alpha-1} \left(\frac{1}{t} - \frac{t}{t^2 + a_k^2} \right) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=1}^{n} c_k a_k^2 \frac{t^{\alpha-2}}{t^2 + a_k^2} dt. \end{aligned}$$

Due to the parity-endpoint mechanism (Remark 1), the interval $(1, 2r+1)$ decomposes as:

$$(1,2r+1)=\left(\bigcup_{m=0}^{r-1}(2m+1,2m+3)\right)\cup\left(\bigcup_{m=0}^{r-2}\{2m+3\}\right).$$

We therefore distinguish two cases.

Case 1. If $\alpha\notin 2\mathbb{N}+3$, then there exist $m\in\{0,\cdots,r-1\}$ and $-1<\epsilon<1$ such that $\alpha=2m+2+\epsilon$. This is equivalent to $2m+1<\alpha<2m+3$.

Then, by applying Lemmas 1 and 2, and using the fact that for $0\le j\le m-1$, we have $\sum_{k=1}^{n}c_k a_k^{2(m-j)+1}=0$, we obtain:

$$\begin{aligned}\int_0^\infty \frac{J_n(x)}{x^\alpha}dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty (-1)^m t^\epsilon \sum_{k=1}^{n}\frac{c_k a_k^{2(m+1)}}{t^2+a_k^2}dt\\ &= \frac{(-1)^m}{\Gamma(\alpha)}\sum_{k=1}^{n}c_k a_k^{2(m+1)}\frac{\pi a_k^{\epsilon-1}}{2\sin(\frac{\pi(1-\epsilon)}{2})}\\ &= \frac{(-1)^m}{\Gamma(\alpha)}\sum_{k=1}^{n}c_k\frac{\pi a_k^{2m+\epsilon+1}}{2\sin(\frac{\pi(2m-\alpha+3)}{2})}\\ &= -\frac{\pi}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}\sum_{k=0}^{n}c_k a_k^{\alpha-1}.\end{aligned}$$

Case 2. If $\alpha\in 2\mathbb{N}+3$, then there exists $m\in\{0,\cdots,r-2\}$ such that $\alpha=2m+3$. So,

$$\int_0^\infty \frac{J_n(x)}{x^\alpha}dx=\frac{1}{\Gamma(\alpha)}\int_0^\infty\sum_{k=1}^{n}c_k a_k^2\frac{t^{2m+1}}{t^2+a_k^2}dt.$$

Using similar arguments as in case 1, we get that

$$\begin{aligned}\int_0^\infty \frac{J_n(x)}{x^\alpha}dx &= \frac{(-1)^m}{2\Gamma(\alpha)}\lim_{t\to\infty}\sum_{k=1}^{n}c_k a_k^{2(m+1)}\int_0^t\frac{2s}{s^2+a_k^2}ds\\ &= \frac{(-1)^m}{2\Gamma(\alpha)}\lim_{t\to\infty}\sum_{k=1}^{n}c_k a_k^{2(m+1)}\left[\ln(s^2+a_k^2)\right]_0^t.\end{aligned}$$

Since for $0\le m\le r-2$, we have $\sum_{k=1}^{n}c_k a_k^{2(m+1)}=0$, it follows that

$$\begin{aligned}\int_0^\infty \frac{J_n(x)}{x^\alpha}dx &= \frac{(-1)^{m+1}}{\Gamma(\alpha)}\sum_{k=1}^{n}c_k a_k^{2(m+1)}\ln(a_k)\\ &= \frac{(-1)^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)}\sum_{k=1}^{n}c_k a_k^{\alpha-1}\ln(a_k).\end{aligned}$$

This achieves the proof. □

As an application of Theorem 3, we present the following examples.

Example 6. 1. Consider the improper integral $\int_0^\infty \frac{15(1-\cos(x))-6(1-\cos(2x))+(1-\cos(3x))}{x^\alpha}dx$. Here, we have $n=3$, $c_1=15$, $c_2=-6$, $c_3=1$, $a_1=1$, $a_2=2$, and $a_3=3$. It can be easily verified that $r=3$.

Therefore, $\int_0^\infty \frac{15(1-\cos(x)) - 6(1-\cos(2x)) + (1-\cos(3x))}{x^\alpha} dx$ converges if and only if $1 < \alpha < 7$. By Theorem 3, we have

$$\int_0^\infty \frac{15(1-\cos x) \; - \; 6(1-\cos 2x) \; + \; (1-\cos 3x)}{x^\alpha} \, dx = \begin{cases} \frac{\pi\left(3 \cdot 2^\alpha - 3^{\alpha-1} - 15\right)}{2\,\Gamma(\alpha)\,\cos\left(\frac{\pi\alpha}{2}\right)}, & \text{if } \alpha \notin 2\mathbb{N}+3, \\ \frac{(-1)^{\frac{\alpha-1}{2}}\left(-3\cdot 2^\alpha \ln 2 + 3^{\alpha-1}\ln 3\right)}{\Gamma(\alpha)}, & \text{if } \alpha \in 2\mathbb{N}+3. \end{cases}$$

To elaborate on this result, we consider the following specific cases.

a. For $\alpha = \frac{13}{2}$, we have

$$\int_0^\infty \frac{15(1-\cos(x)) - 6(1-\cos(2x)) + (1-\cos(3x))}{x^{\frac{13}{2}}} dx = \frac{32\sqrt{2\pi}}{10395}(243\sqrt{3} + 15 - 192\sqrt{2}).$$

b. For $\alpha = 5$, we have

$$\int_0^\infty \frac{15(1-\cos(x)) - 6(1-\cos(2x)) + (1-\cos(3x))}{x^5} dx = -4\ln(2) + \frac{27}{8}\ln(3).$$

2. Consider the improper integral $\int_0^\infty \frac{\cos^4 x \sin^4 x}{x^\alpha} dx$. We have

$$\cos^4 x \sin^4 x = \frac{1}{32}(1-\cos(4x)) - \frac{1}{128}(1-\cos(8x)).$$

Here,

$$n = 2,\ c_1 = \frac{1}{32},\ c_2 = -\frac{1}{128},\ a_1 = 4,\ a_2 = 8.$$

We can easily see that $r = 2$. Therefore, $\int_0^\infty \frac{\cos^4 x \sin^4 x}{x^\alpha}$ converges if and only if $1 < \alpha < 5$. Hence by applying Theorem 3, we obtain that

$$\int_0^\infty \frac{\cos^4 x \sin^4 x}{x^\alpha} dx = \begin{cases} -\frac{4^{\alpha-4}\pi}{\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}(1-2^{\alpha-3}), & \text{if} \quad \alpha \in (1,5)\backslash\{3\}, \\ \frac{\ln(2)}{4}, & \text{if} \quad \alpha = 3. \end{cases}$$

3. Let $n \in \mathbb{N}^*$ and consider the integral $\int_0^\infty \frac{1-\cos^{2n} x}{x^\alpha} dx$. By linearization, we have

$$1 - \cos^{2n} x = \sum_{k=1}^{n} \frac{2}{4^n} C_{2n}^{n-k}(1-\cos(2kx)).$$

We note that for $1 \le k \le n$,

$$c_k = \frac{2}{4^n} C_{2n}^{n-k},$$

and

$$a_k = 2k.$$

It is obvious that $r = 1$. Thus, by Remark 4, the integral $\int_0^\infty \frac{1-\cos^{2n} x}{x^\alpha} dx$ converges if and only if $1 < \alpha < 3$. Moreover, Theorem 3 gives that

$$\int_0^\infty \frac{1-\cos^{2n} x}{x^\alpha} dx = -\frac{\pi}{4^n\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \sum_{k=1}^{n} C_{2n}^{n-k}(2k)^{\alpha-1}.$$

To further describe this result, we consider two specific cases where $n = 2$.

a. For $\alpha = 2$, we have
$$\int_0^\infty \frac{1-\cos^4 x}{x^2}dx = \frac{3\pi}{4}.$$

b. For $\alpha = \frac{5}{2}$, we have
$$\int_0^\infty \frac{1-\cos^4 x}{x^{\frac{5}{2}}}dx = \frac{2\sqrt{2\pi}}{3}(1+\sqrt{2}).$$

4. Let $n \in \mathbb{N}$ and consider the integral $\int_0^\infty \frac{1-\cos^{2n+1} x}{x^\alpha}dx$. By linearization, we have

$$\begin{aligned} 1-\cos^{2n+1} x &= \sum_{k=0}^{n} \frac{C_{2n+1}^{n-k}}{4^n}(1-\cos((2k+1)x)) \\ &= \sum_{k=1}^{n+1} \frac{C_{2n+1}^{n-k+1}}{4^n}(1-\cos((2k-1)x)). \end{aligned}$$

We note that for $1 \le k \le n+1$,
$$c_k = \frac{C_{2n+1}^{n-k+1}}{4^n},$$
and
$$a_k = 2k-1.$$

It is evident that $r = 1$. Thus, by Remark 4, the integral $\int_0^\infty \frac{1-\cos^{2n+1} x}{x^\alpha}dx$ converges if and only if $1 < \alpha < 3$. Furthermore, the integral evaluates to

$$\int_0^\infty \frac{1-\cos^{2n+1} x}{x^\alpha}dx = -\frac{1}{2.4^n\Gamma(\alpha)}\frac{\pi}{\cos(\frac{\pi\alpha}{2})}\sum_{k=0}^{n} C_{2n+1}^{n-k}(2k+1)^{\alpha-1}.$$

To further illustrate this result, we provide the two following specific cases.

a. For $n = 1$ and $\alpha = 2$, we have
$$\int_0^\infty \frac{1-\cos^3 x}{x^2}dx = \frac{3\pi}{4}.$$

b. For $n = 2$ and $\alpha = \frac{3}{2}$, we have
$$\int_0^\infty \frac{1-\cos^5 x}{x^{\frac{3}{2}}}dx = \frac{\sqrt{2\pi}}{16}(10+5\sqrt{3}+\sqrt{5}).$$

Corollary 3. *Let $n \in \mathbb{N}^*$ and $1 < \alpha < 2n+1$. Then, we have*

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^\alpha}dx = \begin{cases} \dfrac{\pi}{2^{2n}\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}\displaystyle\sum_{k=1}^{n}(-1)^k C_{2n}^{n-k}(2k)^{\alpha-1}, & \text{if} \quad \alpha \notin 2\mathbb{N}+3, \\ \dfrac{(-1)^{\frac{\alpha-1}{2}}}{2^{2n-1}\Gamma(\alpha)}\displaystyle\sum_{k=1}^{n}(-1)^{k+1}C_{2n}^{n-k}(2k)^{\alpha-1}\ln(k), & \text{if} \quad \alpha \in 2\mathbb{N}+3. \end{cases}$$

Proof. Let $n \in \mathbb{N}^*$. Using linearization, we find that for $x \in \mathbb{R}$,

$$\sin^{2n}(x) = \frac{1}{2^{2n-1}}\sum_{k=1}^{n}(-1)^{k+1}C_{2n}^{n-k}(1-\cos(2kx)). \tag{10}$$

To apply Theorem 3, we note that for $1 \le k \le n$,
$$c_k = \frac{(-1)^{k+1}C_{2n}^{n-k}}{2^{2n-1}},$$

and
$$a_k = 2k.$$

On the other hand, we have
$$\sin^{2n}(x) = x^{2n} + o(x^{2n+1}).$$
It follows that $r = n$ and for $1 \leq m \leq n-2$, $n \geq 2$,
$$\sum_{k=1}^{n} \frac{(-1)^{k+1} C_{2n}^{n-k}}{4^n} (2k)^{2m} = 0.$$

Thus, applying Remark 4, we conclude that the integral $\int_0^\infty \frac{\sin^{2n}(x)}{x^\alpha} dx$ converges if and only if $1 < \alpha < 2n+1$. Additionally, Theorem 3 gives that

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^\alpha} dx = \begin{cases} \frac{\pi}{2^{2n}\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \sum_{k=1}^{n} (-1)^k C_{2n}^{n-k} (2k)^{\alpha-1}, & \text{if} \quad \alpha \notin 2\mathbb{N}+3, \\ \frac{(-1)^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)} \sum_{k=1}^{n} 2 \frac{(-1)^{k+1} C_{2n}^{n-k}}{4^n} (2k)^{\alpha-1} \ln(2k), & \text{if} \quad \alpha \in 2\mathbb{N}+3. \end{cases}$$

Now, for $\alpha \in 2\mathbb{N}+3$, we note that there exists $m \in \{0, \cdots, r-2\}$ such that $\alpha = 2m+3$. This implies by using (5), that

$$\begin{aligned} \int_0^\infty \frac{\sin^{2n}(x)}{x^\alpha} dx &= \frac{(-1)^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)} \sum_{k=1}^{n} \frac{(-1)^{k+1} C_{2n}^{n-k}}{2^{2n-1}} (2k)^{2m+2} \ln(2k) \\ &= \frac{(-1)^{\frac{\alpha-1}{2}}}{2^{2n-1}\Gamma(\alpha)} \sum_{k=1}^{n} (-1)^{k+1} C_{2n}^{n-k} (2k)^{2m+2} \ln(k) + \frac{2(-1)^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)} \underbrace{\sum_{k=1}^{n} \frac{(-1)^{k+1} C_{2n}^{n-k}}{4^n} (2k)^{2(m+1)}}_{0} \ln(2) \\ &= \frac{(-1)^{\frac{\alpha-1}{2}}}{2^{2n-1}\Gamma(\alpha)} \sum_{k=1}^{n} (-1)^{k+1} C_{2n}^{n-k} (2k)^{2m+2} \ln(k) \\ &= \frac{(-1)^{\frac{\alpha-1}{2}}}{2^{2n-1}\Gamma(\alpha)} \sum_{k=1}^{n} (-1)^{k+1} C_{2n}^{n-k} (2k)^{\alpha-1} \ln(k). \end{aligned}$$

This ends the proof. □

In the following example, we exhibit particular cases of Corollary 3.

Example 7.
1. By taking $n = 1$ and $\alpha = \frac{5}{2}$, we obtain that
$$\int_0^\infty \frac{\sin^2 x}{x^{\frac{5}{2}}} dx = \frac{4\sqrt{\pi}}{3}.$$
2. By taking $n = 2$ and $\alpha = 2$, we obtain that
$$\int_0^\infty \frac{\sin^4 x}{x^2} dx = \frac{\pi}{4}.$$
3. For $n \geq 1$, we have:
$$\int_0^\infty \frac{\sin^{2n}(x)}{x^{2n}} dx = \left(-\frac{1}{4}\right)^n \frac{\pi}{(2n-1)!} \sum_{k=1}^{n} (-1)^k C_{2n}^{n-k} (2k)^{2n-1}.$$

4. For $n \geq 2$, we have:

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^{2n-1}}dx = \frac{(-1)^{n-1}}{2(2n-2)!}\sum_{k=1}^{n}(-1)^{k+1}C_{2n}^{n-k}k^{2n-2}\ln(k).$$

In particular, for $n = 2$, we have $\int_0^\infty \frac{\sin^4(x)}{x^3}dx = \ln(2)$.

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