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On the moment problem and related problems

Cristian Octav Olteanu

Independent researcher, 060042 Bucharest, Romania; octav.olteanu50@gmail.com

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Abstract: Necessary and sufficient conditions for the existence of the solutions of a class of scalar and mainly for operator-valued moment problems are reviewed. This was the first motivation for proving our constrained extension results for linear operators. Polynomial approximations on bounded and on unbounded closed subsets are very useful in proving the uniqueness of the solution. We also reviewed earlier results on the extension of positive linear functional and operators. Such results are applied to ensure the extension of our linear solution from the subspace of polynomials to a larger function space. In most of the cases from below, this is made using polynomial approximation in one and several variables. Besides positivity, our solution is bounded from above by a dominating linear, sublinear or only convex continuous operator, on the entire domain space or only on its positive cone. This allows estimating the norm of the linear solution.

Keywords: Hahn-Banach type theorems, approximation, unbounded subsets, moment problem, existence, uniqueness

MSC: 41A10, 44A60, 46A22.

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1. Introduction and preliminaries

Being given a sequence $s = (s_j)_{j\in\mathbb{N}^d}$ of real numbers, a closed subset $S \subseteq \mathbb{R}^d$, the moment problem consists of finding necessary and sufficient conditions on the sequence s for the existence and uniqueness of a linear positive functional T defined on a function space X containing polynomials $\varphi_j(t) = t^j,\ t \in S,\ j \in \mathbb{N}^d$, such that the following conditions are satisfied:

$$T(\varphi_j) = s_j, \quad j \in \mathbb{N}^d. \tag{1}$$

Here we have denoted $\mathbb{N} := \{0, 1, 2, \dots\}$. These conditions are usually expressed in terms of the given numbers s_j, that are called the moments of order j for the linear form T. Next, we recall the motivation of this terminology. Usually, the function space X is topologically complete and contains the subspace $C_c(S)$ of all real valued compactly supported functions which are continuous on S, having their support contained in S. Obviously, the function space might contain some other function subspaces. Such usual spaces are topologically complete (Banach spaces, Fréchet spaces or other locally convex complete spaces obtained by those mentioned above). In all these cases, the linear positive functional T is represented by a positive regular Borel measure ν on S. Thus, returning to the above notations, Eq. (1) is written as

$$\int_S t^j \, d\nu = s_j, \quad j \in \mathbb{N}^d. \tag{2}$$

This is the full (the entire) scalar moment problem, when the numerical moments of all orders are required to satisfy the moment conditions (2). When $d = 1$ and (2) must be satisfied with all indices $j \in \{0, 1, \dots, n\}$, the corresponding moment problem is called reduced (or truncated). Since a polynomial function $q \in \mathbb{R}[t]$ is nonnegative on the entire real axes if and only if q is a sum of squares of some other polynomials with real coefficients, the one-dimensional moment problem for $S = \mathbb{R}$, called the one-dimensional Hamburger moment problem, is the simplest one to be studied in terms of quadratic forms.

Indeed, to ensure the positivity of the linear form T on $C_c(\mathbb{R})$, we first need to know that $T(q) \geq 0$ for all $q \in \mathbb{R}[t]$, with $q(t) \geq 0$ for all $t \in \mathbb{R}$. Since any such polynomial q can be written as a sum of squares of polynomials r_m^2, for some $r_m \in \mathbb{R}[t]$, $m = 1, \ldots, N_m$,

$$r_m^2(t) = \left(\sum_{j \in J_{0,m}} \alpha_{j,m} t^j \right)^2, \quad m = 0, 1, \ldots, N_m,$$

the positivity $T(q) \geq 0$ is expressed by $\sum_{i,j \in J_{0,m}} \alpha_{i,m}\alpha_{j,m} T(\varphi_{i+j}) \geq 0,\ m = 1, \ldots, N_m$, that is

$$\sum_{i,j \in J_{0,m}} \alpha_{i,m}\alpha_{j,m} s_{i+j} \geq 0, \quad m = 0, 1, \ldots, N_m. \tag{3}$$

The conditions (3) say that all the symmetric Hankel matrices $H_m := (s_{i+j})_{i,j=0}^{N_m}$ are positive semidefinite. In other words, the sequence s is positive semidefinite. Since for each multi-index j, $\int_S t^j d\nu$ is, by definition, the moment defined by $j \in \mathbb{N}^d$ with respect to a positive regular Borel measure ν, the sequence (2) is called a moment sequence. Inequalities (3) say that any moment sequence is positive semidefinite. As we can see from [1–9], there is a strong relationship between positive polynomial and approximating positive functions from important function spaces by positive polynomials on $\mathbb{R}$ and on $\mathbb{R}_+$. One of the aims of this review paper is to extend such type results from the one-dimensional case $d = 1$ to the multidimensional case $d \geq 2$. Since the explicit form of nonnegative polynomials on $\mathbb{R}$ and on $\mathbb{R}_+ := [0, +\infty)$ in terms of sums of squares is quite simple, these cases were the first moment problems under attention. In several dimensions, the relationship between positive polynomials on $\mathbb{R}^d$ or $\mathbb{R}_+^d$, $d \geq 2$ and sums of squares is studied. Namely, there exist positive polynomials that are not sums of squares and there exist positive definite sequences which are not moment sequences. In [10], a criterion for the representation of a linear functional by a positive Borel measure, in terms of its positivity on the subspace of polynomials, is provided (Haviland theorem). The next problem consists of studying the case when $S = [0, +\infty)$. This is the one-dimensional Stieltjes moment problem. As is well known, a polynomial $q \in \mathbb{R}[t]$ is nonnegative on the entire semiaxes $[0, +\infty)$ if and only if q is a sum of polynomials of the form $r^2(t) + tw^2(t)\ \forall t \in [0, +\infty)$, for some $r, w \in \mathbb{R}[t]$. Consequently, the positivity condition $T(q) \geq 0$ for all $q \in \mathbb{R}[t]$ with $q(t) \geq 0\ \forall t \in [0, +\infty)$ must be written as follows:

$$\sum_{i,j \in J_{0,m}} \alpha_{i,m}\alpha_{j,m} s_{i+j+r_i} \geq 0, \quad m = 0, 1, \ldots, N_m, \quad r_i \in \{0, 1\}. \tag{4}$$

Once (3) (respectively (4)) is satisfied, from Haviland theorem, the existence of a representing positive Borel measure for T on $\mathbb{R}$ (respectively on $[0, +\infty)$), follows. Since for $d \geq 2$ the positive polynomials on $\mathbb{R}^d$ are not necessarily sums of squares, characterization for the existence of a positive linear solution T satisfying (1) in terms of quadratic forms is not so simple as in the one-dimensional case. In [11], results referring mainly to convex functions are under attention. The book [12] represents a very useful source on real and complex analysis, including measure theory. References [13–15] contain general and specific results in functional analysis or respectively in potential theory [16] and approximation theory [17]. Main results on the multidimensional moment problem have been published in references [18–22]. Namely, in the article [18], the multidimensional moment problem on a compact with nonempty interior in $\mathbb{R}^d$ is solved. The form of positive polynomials on such compact subsets is also determined. In [19], the authors provide an example showing that there exist moment determinate measures μ on $\mathbb{R}^d$, $d \geq 2$, such that the subspace of polynomials is not dense in $L^2_\mu(\mathbb{R}^d)$. As we shall see in the present study, we recall that for any determinate measure μ on a closed unbounded subset F of $\mathbb{R}^d$, the subspace of polynomials is dense in $L^1_\mu(F)$. The articles [20–22] refer to the moment problem on semi-algebraic compact subsets $K \cap \mathbb{R}^d$. In this case, the positive polynomials on K can be expressed as sums of squares multiplied by the polynomials f_i defining the compact subset K (for details, see Theorem 3 below). The article [23] deals with Hamburger as Stieltjes moment problems in several variables. In [24], an operator valued moment problem is under attention. In the area of the multidimensional moment problem, other results have been published in [25–27]. The book [28] refers to polynomial and semi-algebraic optimization. In [29], checkable criterions for determinacy of the one-dimensional Hamburger and Stieltjes moment problem have been published. Basic results, some of them involving determinacy, probability theory, geometric interpretations, approximation and optimization, have been published in [30] and recently in [31] and [32]. In the articles [33] and [34], results in operator theory are applied to the moment problem. The article [35] is not directly related to the moment problem. However, it contains proof of the fact that any positive linear operator acting on ordered Banach spaces is continuous, as well as other results. The articles [36–40] are devoted to recent results in approximation theory, applied to expansions and to a class of generalized Bernstein operators. The refences [41–45] concern characterization of the existence and uniqueness of the positive linear solution of moment problems. The case when the linear operator-solution is dominated by a convex operator is under attention. In this case, control on the norm of the solution is possible to be made.

The first aim of this work is to review results from [18] and [20] to the moment problem on special compact subsets in $\mathbb{R}^d$. As a consequence, the decomposition of positive polynomials on such compacts as sums of important positive polynomials is deduced. Generally, constrained extension of linear operators' results are applied to infer the existence of a solution. On the other hand, in case of moment problems on unbounded closed subsets, approximation by positive polynomials of functions from the positive cone of certain function spaces is reviewed. On the other hand, the relationship between positivity and continuity of classes of linear operators is reviewed. We proposed a method of solving the multidimensional Hamburger and Stieltjes moment problems in terms of quadratic forms. If $d \geq 2$ and $s = (s_j)_{j \in \mathbb{N}^d}$ is a sequence, the multidimensional Stieltjes moment problem consists of characterizing those sequences that are moment sequences on $[0, +\infty)^d$. The existence and uniqueness of the solution positive Borel representing measure ν are under attention. In a few cases, the construction of the solution is also possible to be made, although this is not an aim of the present work. §3.1 contains solutions for a few moment problems on compact subsets. In §3.2, similar results on closed unbounded subsets are under attention. In what follows, all the function spaces containing polynomials are functions whose elements are real valued functions, or classes of such functions. The rest of the paper is organized as follows. In §2, the basic methods applied in this review paper are summarized. §3 is devoted to the results. §4 concludes the paper.

2. Methods

Here are the basic methods applied in the Results section.

1. Applying a result on constrained extension of a positive linear operator, that preserves positivity and the property of being dominated by a given convex operator, on the positive cone of the domain space. The codomain is an order complete ordered vector space. The extension is made from the subspace of polynomials to a larger function space (see [14,15,41,44] and the references there).
2. Using continuity of any positive linear operator acting between ordered Banach spaces [35].
3. Evaluating the norm of a positive linear operator on Banach lattices with order units [4,5].
4. Recalling and applying results from [18,41] regarding solution of the moment problem on compact subsets with nonempty interior in $\mathbb{R}^d$, $d \geq 2$.
5. Applying separation theorem to infer the decomposition of positive polynomials on special compact subsets, in terms of special positive polynomials (see [18–20]). Using Hahn-Banach type theorems in solving the existence of the vector valued (and operator valued) linear solution for the moment problem.
6. Applying polynomial approximation on unbounded subsets, to elements of the positive cone of certain function spaces, by nonnegative polynomials. Consequently, we obtain solution for the multidimensional moment problem on $[0, +\infty)^2$, using approximation of any nonnegative function from $C(S_1 \times S_2)$, by tensor products of polynomials $(r_{1,m} \otimes r_{2,m})(t_1, t_2) : r_{1,m}(t_1) r_{2,m}(t_2)$, $r_{1,i}(t_i) \geq 0 \ \forall t_i \in \mathbb{R}_+$. This leads to the results of [45], including Theorem 13 recalled below.
7. Using functional calculus for self-adjoint operators.
8. Recalling the methods applied in [18–20], regarding solving moment problem on special compact subsets of $\mathbb{R}^d$. The case of semi-algebraic compact subsets and Schmudgen's Positivstellensatz are recalled.
9. Results from [42–45] on polynomial approximation on unbounded closed subsets are reviewed and applied. In this case, the density of polynomials and uniqueness of the solution in $L^1_\mu(\mathbb{R}^d)$ spaces for a moment determinate measure μ on $\mathbb{R}^d$, $d \geq 2$ is pointed out. As has been discussed and has been proved in [19], such results are no longer valid when we replace $L^1_\mu(\mathbb{R}^d)$ with $L^2_\mu\left(\mathbb{R}^d\right)$, $d \geq 2$, even for moment determinate measures μ.

10. Expressing our necessary and sufficient conditions in terms of quadratic forms, even in the case of some multidimensional moment problems on $\mathbb{R}^d$ and on $\mathbb{R}^d_+$. To do this, we use approximation of any element from the positive cone of the domain function space by sums of squares of polynomials as $(p_1(t_1)\cdots p_d(t_d))^2$, $p_i \in \mathbb{R}[t_i]$, or respectively by sums of products $\prod_{i=1}^d (p_i^2(t_i) + t_i q_i^2(t_i))$, where p_i, $q_i \in \mathbb{R}[t_i]$. Since our solutions are continuous linear functionals (or operators), passing to the limit, the positivity on the positive cone of the domain space is maintained. We also point out the connection with the notion of positive sequence on an interval.

3. Results

3.1. Existence of the solutions for moment problems on compact subsets

One of the main applications of convexity consists of proving Hahn-Banach type results for extension of linear functionals and operators. For example, separation theorem for convex disjoint sets has been applied in references [7,18], and extension of linear operators was applied in [41,42] and reviewed in [44]. As we have already seen in the Introduction, when solving the existence problem, we are interested only in the linear solutions T satisfying (1), which are positive on the positive cone of the domain function space X. In other words, if we are looking for such a linear solution Tmapping Xinto the ordered vector space Y, then Tmust verify $Tx \geq 0$ for all $x \in X_+$. As we have recalled in §2, if X, Y are ordered Banach spaces, then the positivity of a linear operator from X to Y implies its continuity. The implication in the reverse sense stays valid under natural assumptions on density of positive polynomials in the positive cone X_+. We start this paper by recalling solutions of moment problems on compact subsets of $\mathbb{R}^d$. Next, the statement of a general Hahn-Banach type theorem, formulated in the framework of positive linear operators on ordered vector spaces is reviewed. This general result can be applied to the moment problem on a compact subset K with nonempty interior [18,41]. The review will continue with the problem of moments on semi-algebraic compact subsets K (see [7,20]). The form of positive polynomials q, $q(t) > 0\ \forall t \in K$, namely their decomposition as sums of special positive polynomials involved in the definition of K, plays an important role in both types of compacts mentioned above. The form of positive polynomials q, $q(t) > 0\ \forall t \in K$, namely their decomposition as sums of special positive polynomials involved in the definition of K, plays an important role in both types of compacts mentioned above. For example, in Theorems 1 and 2 stated below the special nonnegative polynomials are those from ΔK, while in case of a semi-algebraic compact defined by equality (5) below, these special polynomials are $f_1, \ldots, f_m$, (multiplied by a sum of squares $\sum_i r_i^2$, $r_i \in \mathbb{R}[t_1, \ldots, t_d]$). In the last section, moment problems on unbounded subsets are discussed. Here the point is the uniqueness of the solution. In what follows, we recall notions and results from the article [18]. Let K be a compact subset with a nonempty interior in $\mathbb{R}^d$, $d \geq 2$. Since K is a closed subset in $\mathbb{R}^d$, there exists a family $(p_i)_{i \in I}$ of polynomials of degrees smaller or equal to two, such that $K = \bigcap_{i \in I} p_i^{-1}([0, +\infty))$. One denotes by $E(K)$ the vector space generated by the polynomials of degree at most one and the polynomials p_i, $i \in I$. So, $E(K)$ is contained in the vector space $\mathbb{R}_2[t_1, \ldots, t_d]$ of all polynomials having the degree at most two. One defines the convex cone $E_+(K)$ that consists of elements in $E(K)$ which take nonnegative values at all points of K. Let $G(K)$ be the subset of those $T \in E_+(K)$ that generate an extremal ray of $E_+(K)$. One also defines the important subset

$$G_1(K) : \{T \in G(K) \,;\, \|T\|_K = 1\}.$$

In the case when K is convex, one can take as $E(K)$ the subspace of all polynomials of degree at most one. For example, when $d=1$ and $K=[0,1]$, we see that

$$G_1(K) = \{t, 1-t\}.$$

In the case $d \geq 2$, the family of polynomials $t^m(1-t)^n$, $m, n \in \mathbb{N}$ appearing in the Hausdorff moment problem, can be replaced by the family $\Delta(K)$ of polynomials that are finite products of elements from $G_1(K)$, the constant 1 corresponding to an "empty" product. Since $\Delta(K)$ has only a multiplicative structure with order unit, one considers the convex cone $\Gamma = \Gamma(K)$ formed by convex combinations with nonnegative coefficients of elements from $\Gamma(K)$. Finaly, one introduces the set $\pi(K)$ of all linear forms L on $\mathbb{R}[t_1, \ldots, t_d]$ having the properties $L(1)=1$ and $L(T)\geq 0$ for all $T\in\Delta(K)$. In [11], the set $\pi(K)$ is studied in key Lemma 3, preceded by proving that $\pi(K)$ is a compact convex, if we endow it by the weak topology $w(\pi(K),\Delta(K))$. An interesting geometric property is that any extreme point T of $\Delta(K)$ is multiplicative on $E{:}\mathbb{R}[t]$ (see [16], Lemma 3, pp. 256-257). However, we do not apply this last remark in our statements, that concern directly only the way of solving moment problems. The order of stating the following results published in research articles or specialized books follows the chronological order of their publication date.

Theorem 1 (see [18]). *(Solution for the moment problem). Let K be compact in $\mathbb{R}^d$, $d\geq 2$ with nonempty interior in $\mathbb{R}^d$. A sequence $(s_j)_{j\in\mathbb{N}^d}$ is a moment sequence on K if and only if the linear form L_s defined on $\mathbb{R}[t]$ by $L_s(t^j):=s_j, j\in\mathbb{N}^d$, verifies the condition $L(T)\geq 0$ for all polynomials $T\in\Delta(K)$.*

The next result provides the expression of positive polynomials on a compact having nonempty interior.

Theorem 2 (see [18]). *Any positive polynomial on a compact having nonempty interior in $\mathbb{R}^d$ is a linear combination with nonnegative coefficients of polynomials from $\Delta(K)$.*

Next, we continue with the Schmüdgen's Positivstellensatz, which is related to the solution of the moment problem on semi-algebraic compact subsets. There exists Putinar's Positivstellensatz. If $f_1,\ldots,f_m\in\mathbb{R}[t]$ are such that the set

$$K:=\{t\in\mathbb{R}^d\,;\,f_1(t)\geq 0,\ldots,f_m(t)\geq 0\},\tag{5}$$

is compact, then K is called a semi-algebraic compact. We denote by Σ^2 the convex cone generated in $\mathbb{R}[t]$ by all squares of polynomials:

$$\Sigma^2:=\mathrm{co}\{q^2\,;\,q\in\mathbb{R}[t]\},$$

and by

$$\Sigma^2\langle f_1,\ldots,f_m\rangle:=\Sigma^2+f_1\Sigma^2+\cdots+f_m\Sigma^2+f_1f_2\Sigma^2+\cdots+f_1\cdots f_m\Sigma^2,$$

the multiplicative convex cone generated by Σ^2 and $f_1,\ldots,f_m$.

Theorem 3 (see [7], [20]). *(Decomposition of positive polynomials on semi-algebraic compact subsets). With the above notations, the following assertion holds true. Let K be a semi-algebraic compact and $q\in\mathbb{R}[t], q(t)>0$ for all $t\in K$. Then $q\in\Sigma^2\langle f_1,\ldots,f_m\rangle$.*

In other words, Theorem 3 says that in several dimensions, positive polynomials on semi-algebraic compact subsets can be decomposed as sums whose terms are squares of polynomials multiplied by products of polynomials f_i appearing in the definition of the involved semi-algebraic compact K.

Here is a possible relationship between Theorems 2 and 3 recalled in this review paper. For example, Theorem 3 involves sums of squares multiplied with products $f_1\cdots f_l$, $l\in\{0,1,\ldots,m\}$, while Theorem 2 involves linear combinations with nonnegative coefficients of elements from $\Delta(K)$. The common point seems to be that the polynomials appearing in these linear combinations are products of polynomials that define the compact under attention (see the definitions of $\Delta(K)$ in [16] and respectively of $\Sigma^2\langle f_1,\ldots,f_m\rangle$ in [8] and [20]). Hence, in Theorem 3 positivity conditions on positive polynomials can be written in terms of quadratic forms. Consequences of Theorem 2 are derived below in Theorems 6–8. For the next three theorems, see [18], [41]. Theorems 4–8 stated below represent the author's contribution to this and related subjects. For example, Theorem 4 provides characterization of the existence of a positive linear extension T of T_1, such that T is dominate by P on the positive cone X_+. A quite similar type result, written in the framework of the moment problem, is stated in Theorem 5. Theorems 6 and 7 follow as applications to a concrete domain space, using the notion of positive sequence on an interval. Generally, the terms of this sequence are elements of an order complete Banach lattice. Theorem 8 represents a constrained extension result for linear operators applied to a multidimensional moment problem. Our extension satisfies not only the positivity constraint on X_+, but also the condition of being dominated by a given convex operator. For example, in Theorem 8 this convex constraint is defined by a vector valued norm. Usually, the convex operator defining the upper constraint on the solution may be defined on the entire domain space X, or on the positive cone X_+.

Theorem 4. *Let X be an ordered vector space, Y an order complete vector space, $M \subset X$ a vector subspace, $T_1 \colon M \to Y$ a linear operator, $P \colon X_+ \to Y$ a convex operator. The following two statements are equivalent.*

(a) *There exists a positive linear extension $T \colon X \to Y$ of T_1 such that $T|_{X_+} \leq P$;*

(b) *We have $T_1(h) \leq P(x)$ for all $(h, x) \in M \times X_+$ such that $h \leq x$.*

Theorem 5. *Let X be as in Theorem 4, Y an order complete vector lattice, $\{x_j\}_{j \in J}$, respectively $\{y_j\}_{j \in J}$ finite or infinite given families in X, respectively in Y, $P \colon X \to Y$ a convex operator. The following assertions are mutually equivalent.*

(a) *There exists a positive linear operator $T \colon X \to Y$ such that*

$$Tx_j = y_j \quad \text{for all } j \in J, \quad \text{and} \quad Tx \leq Px \quad \text{for all } x \in X.$$

(b) *For any finite subset $J_0 \subset J$, any family $\{\alpha_j \colon j \in J_0\} \subset \mathbb{R}$, the relation*

$$\sum_{j \in J_0} \alpha_j x_j \leq x \in X \implies \sum_{j \in J_0} \alpha_j y_j \leq Px.$$

In the next result, we use the notion of positive sequence on an interval $I \subset \mathbb{R}$, pointing out the relationship between such a sequence and the moment problem. Theorem 6 stated below follows quite rapidly with the aid of Theorem 5. Namely, if $\mathbf{Y}$ is an ordered vector lattice, and $\mathbf{y} = \{y_0, y_1, \ldots, y_n, \ldots\}$ a sequence in $\mathbf{Y}$, we say that $\mathbf{y}$ is of positive type on the interval I if and only if for any polynomials $\sum_{j=0}^{n} \alpha_j t^j$ that take nonnegative values at all points t of $[0, b]$, we have $\sum_{j=0}^{n} \alpha_j y_j \geq 0$.

In the following theorem, the domain space is $L^1([0, b])$, $0 < b < +\infty$, with respect to Lebesgue measure dt. The motivation of choosing this function space (containing all the polynomials), consists of the fact that among all $L^p([0, b])$ spaces with $p \in [1, +\infty]$, $L^1([0, b])$ is the largest one. This is a consequence of Hölder's inequality. Thus, if we are looking for an extension of the linear positive functional or operator, verifying the moment conditions (1), from the subspace of polynomials to $L^p([0, b])$, the case when $p = 1$ provides a maximal extension. With the above-mentioned notations, the following theorem holds.

Theorem 6. *Let $X := L^1_{\mathrm{d}t}([0, b])$, x_j the class of polynomial t^j, $j \in \mathbb{N}$, $\tilde{y} \in Y_+ \setminus \{0\}$. If the sequence $\{\tilde{y}, y_0, 2y_1, \ldots, ny_{n-1}, \ldots\}$ is positive on $[0, b]$, then there exists a positive linear operator T from X into Y such that $Tx_j = y_j \quad \forall j \in J$, and*

$$T(x) \leq P(x) := \left(\int_0^b |x(t)| \, dt \right) \tilde{y}, \quad \forall x \in L^1_{\mathrm{d}t}([0, b]).$$

Proof. To apply Theorem 5, (b) implies (a), we must verify the implication stated at point (b). More specifically, with the notation of the present theorem, the following implication should hold true. If we define the linear operator on the subspace of polynomials by $T(x_j) := y_j$, $j \in \mathbb{N}$, for $\alpha_0 := \int_0^b |x(t)| dt$, we must prove that $\sum_{j=0}^{n} \alpha_j t^j \leq x(t)$ almost everywhere in $[0, b]$ implies:

$$\sum_{j=1}^{n} \alpha_j x_j \leq \left(\int_0^b |x(t)| dt \right) x_0,$$

that is,

$$\left(\int_0^b |x(t)| dt \right) x_0 - \sum_{j=1}^{n} \alpha_j x_j \geq 0,$$

implies

$$\left(\int_0^b |x(t)| dt \right) \tilde{y} - \sum_{j=1}^{n} \alpha_j y_j \geq 0.$$

In other words, using the notations that define the y_j, we must prove that

$$\sum_{j=0}^{n} \alpha_j t^j \leq x(t) \text{ almost everywhere in } [0,b], \tag{6}$$

implies:

$$\sum_{j=1}^{n} \alpha_j y_j \leq \left(\int_0^b |x(t)| dt \right) \tilde{y}. \tag{7}$$

On the other hand, by integrating in (6), we have:

$$\sum_{j=0}^{n} \alpha_j \frac{u^{j+1}}{j+1} \leq \int_0^u x(t) dt \leq \int_0^u |x(t)| dt, \quad u \in [0,b].$$

This means that

$$\sum_{j=0}^{n} \alpha_j \frac{u^{j+1}}{j+1} \leq \int_0^u |x(t)| dt \leq \int_0^b |x(t)| dt, \quad u \in [0,b].$$

Considering the above notations $T(x_j) := y_j$, that define T on the subspace of polynomials, we infer:

$$\alpha_0 y_0 + \sum_{j=1}^{n} \alpha_j (j+1) \frac{y_j}{j+1} = \sum_{j=0}^{n} \alpha_j y_j \leq \left(\int_0^b |x(t)| dt \right) \tilde{y} = P(x).$$

Thus, condition (b) of Theorem 5 is satisfied. Hence the assertions (a) of the same theorem hold true. The desired conclusion follows. □

Theorem 7. *Let $X, x_j, Y, \tilde{y}$ be as in Theorem 6 and $\{y_0, y_1, \ldots, y_n, \ldots\} \subset Y$. We consider the following statements:*

(a) *There exists a positive linear operator T from X into Y such that $Tx_j = y_j \quad \forall j \in J$, and*

$$T(x) \leq P(x) := \left(\int_0^b |x(t)| dt \right) \cdot \tilde{y}, \quad \forall x \in L^1_{dt}([0,b]).$$

(b) *The sequence $\{y_1, (1/2)y_2, \ldots, (1/(n+1))y_{n+1}, \ldots\}$ is positive on the interval $[0,b]$ and*

$$y_j \leq \left(\frac{b^{j+1}}{j+1} \right) \cdot \tilde{y}, \quad j \in \mathbb{N}.$$

(c) *The sequence $\{y_0, y_1, \ldots, y_n, \ldots\}$ is positive on the interval $[0,b]$ and*

$$y_j \leq \left(\frac{b^{j+1}}{j+1} \right) \cdot \tilde{y}, \quad j \in \mathbb{N}.$$

Then $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$. Assume that (a) holds. If $\sum_{j=0}^{n} \alpha_j x_j(t) = \sum_{j=0}^{n} \alpha_j t^j \geq 0 \quad t \in [0,b]$, by integrating on $[0,u] \subseteq [0,b]$, one obtains

$$\sum_{j=0}^{n} \alpha_j \left(\frac{u^{j+1}}{j+1} \right) \geq 0, \quad u \in [0,b].$$

This can be written as

$$\sum_{j=0}^{n} \alpha_j \left(\frac{x_{j+1}}{j+1} \right) = \alpha_0 x_1 + \alpha_1 \left(\frac{x_2}{2} \right) + \cdots + \alpha_n \left(\frac{x_{n+1}}{n+1} \right) \geq 0,$$

in $L^1_{\mathrm{d}t}([0,b])$. Since $T\colon L^1_{\mathrm{d}t}([0,b]) \to Y$ is positive and linear, with $T(x_j) = y_j, j \in \mathbb{N}$, we infer that

$$\sum_{j=0}^{n} \alpha_j \left(\frac{y_{j+1}}{j+1}\right) \geq 0 \text{ in } Y.$$

Hence, the sequence $\{y_1, (1/2)y_2, \ldots, (1/(n+1))y_{n+1}, \ldots\}$ is positive on $[0,b]$. On the other hand, from (a), we know that

$$y_j = T(x_j) \leq \left(\int_0^b t^j dt\right)\tilde{y} = \left(\frac{b^{j+1}}{j+1}\right)\tilde{y}, \quad j \in \mathbb{N}.$$

Thus, the implication (a) implies (b) is proved. The assertion (a) implies (c) is almost obvious and can be proved in a similar way. □

Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a compact subset with nonempty interior, $\mathbf{Y}$ an order complete vector lattice endowed with a linear topology such that the positive cone Y_+ is normal (see [4]), μ a positive regular Borel measure on K, and $X := L^1_\mu(K)$. We denote by x_j the class of the polynomial $t^j := t_1^{j_1} \cdots t_d^{j_d}$, $t = (t_1, \ldots, t_d) \in K$, $j = (j_1, \ldots, j_d) \in \mathbb{N}^d$, $(y_j)_{j \in \mathbb{N}^d}$ a sequence in Y, $d \geq 2$. Let $\tilde{y}$ be a fixed not null element from Y_+. With the above notations and using results recalled above, we have the following result (see [41]).

Theorem 8. *There exists a unique positive linear operator $T \in L_+(X,Y)$ such that*

$$Tx_j = y_j, \quad \forall j \in \mathbb{N}^d, \quad Tx \leq \left(\int_K |x(t)| d\mu\right) \cdot \tilde{y}, \quad \forall x \in X,$$

if and only if for any polynomial $p = \sum_{j \in J_0} \alpha_j x_j \in \Delta(K)$ the following inequalities hold:

$$0 \leq \sum_{j \in J_0} \alpha_j y_j \leq \left(\int_K \left|\sum_{j \in J_0} \alpha_j t^j\right| d\mu\right) \cdot \tilde{y}.$$

3.2. Approximation by nonnegative polynomials on unbounded subsets

Theorems 9, 10, 13 represent the **author's** contribution to the type of approximation mentioned in the title of this subsection. Then such approximations are applied to prove the relationship between positivity and continuity for classes of linear operators (see Theorem 12).

Theorem 9. *Let $K \subset [0, +\infty)$ be a compact subset, and $f\colon K \to [0, +\infty)$ a continuous function. Then there exists a sequence $(p_m)_{m \in \mathbb{N}}$ of polynomial functions from $\mathbb{R}[t]$ such that $p_m(t) \geq f(t)$ for all $t \in [0, +\infty)$ and all m, with $\lim_m p_m|_K = f$. The convergence holds uniformly on K.*

For proof of Theorem 9 see [43], Lemma 2.

Next, we recall the notion of a moment determinate measure on a closed unbounded subset. If $F \subseteq \mathbb{R}^d$ is an unbounded closed subset, a positive Borel measure μ on F is called M-determinate (moment determinate, or simply determinate), if and only if the polynomials $p_n(t) := t^n := t_1^{n_1} \cdots t_d^{n_d}$, $n \in \mathbb{N}^d$, are elements of the space $L^1_\mu(F)$ and μ is uniquely determined by its moments

$$\int_F t^n d\mu, \quad n \in \mathbb{N}^d.$$

This means that for any other such positive Borel measure λ with

$$\int_F t^n d\lambda = \int_F t^n d\mu, \quad \forall n \in \mathbb{N}^d,$$

we have $\lambda = \mu$ as measures $\left(\int_F \varphi d\lambda = \int_F \varphi d\mu \text{ for all } \varphi \in C_c(F)\right)$.

Theorem 10 (see [42], [44]). *Let F be a closed unbounded subset in $\mathbb{R}^d$, $d \geq 1$, and μ a positive M-determinate Borel regular measure on F. Then for any $\varphi \in (C_c(F))_+$, there exists a sequence $(q_m)_{m\in\mathbb{N}}$ of polynomials, $q_m(t) \geq \varphi(t) \geq 0\ \forall t \in F, \forall m \in \mathbb{N}$, $\lim_m q_m = \varphi$ in $L^1_\mu(F)$. In particular,*

$$\lim_m \int_F q_m\, d\mu = \int_F \varphi\, d\mu,$$

the positive cone of nonnegative polynomials on F is dense in $\left(L^1_\mu(F)\right)_+$, and the subspace $\mathbb{R}[t]$ of all polynomials is dense in $L^1_\mu(F)$.

Before continuing with the moment problem, we recall the following useful general-type result on continuity of any positive linear operator acting on ordered Banach spaces (see [35]).

Theorem 11. *Let X, Y be ordered Banach spaces and $T\colon X \to Y$ a positive linear operator. Then T is continuous.*

The implication in the reverse sense also holds under the conditions mentioned below.

Theorem 12. *Let X be an ordered Banach space of functions on $F \subseteq \mathbb{R}^d$, containing a convex sub-cone $\mathcal{C} \subseteq (\mathbb{R}[t])_+ := \{q \in \mathbb{R}[t], q(t) \geq 0\ \forall t \in F\}$, with $\bar{\mathcal{C}} = X_+$, Y an ordered Banach space, and $T \in B(X,Y)$ a bounded linear operator mapping X into Y, with $Tq \geq 0$ for all $q \in \mathcal{C}$. Then $Tx \geq 0$ for all $x \in X_+$ (hence T is positive).*

The proof of Theorem 12 is obvious. In the context of this study, the difficulties might arrive in verifying the assumptions from the statement of Theorem 12. The point is to prove the density of the sub-cone $\mathcal{C}$ in the positive cone X_+ of the domain space X. For the one-dimensional case $d = 1$, this follows from Theorem 9.

Namely, if $X = \mathcal{C}(K)$ for some compact subset $K \subset [0, +\infty)$, the dense sub-cone $\mathcal{C} \subset X_+$ which is under attention is the sub-cone of polynomials p with $p(t) \geq 0$ for all $t \in [0, +\infty)$. For arbitrary $d \geq 2$, it can be deduced from Theorems 9 and 10, via Theorems 1 and 2 of [45].

Namely, if $d = 2$ and $S := S_1 \times S_2$, with S_i compact subsets of $[0, +\infty)$, $i = 1, 2$, and $X := \mathcal{C}(S)$, the interesting dense sub-cone in X_+ is the convex cone $\mathcal{C}$ generated by polynomials q with $q(t_1, t_2) = q_1(t_1)q_2(t_2)$, $q_i(t_i) \geq 0\ \forall t_i \geq 0$, $i = 1, 2$. Since any nonnegative polynomial $q_i(t_i)$ on $\mathbb{R}_+$ is a sum of special polynomials $t_1^{s_1 r_1(t_1)} t_2^{s_2 r_2(t_2)}$ for some $r_i \in \mathbb{R}[t_i]$ and $s_i \in \{0, 1\}$, the condition $Tx \geq 0$ for all $x \in \mathcal{C}(S)$ is expressible using signature of quadratic expressions, as described in [45].

In a way, Theorem 12 claims the converse implication of that from Theorem 11 also holds true. In other words, for a large class of linear operators T and ordered Banach spaces X, continuity of T on X and its positivity on $\mathcal{C} \subset (\mathbb{R}[t])_+$, where $\mathcal{C}$ is a dense sub-cone in X_+, implies positivity on the entire positive cone X_+ of the domain space X.

Thus, via linearity, continuity of T implies positivity (hence T is monotone increasing). In this context, we have denoted by $\bar{\mathcal{C}}$ the topological closure of $\mathcal{C}$ in X_+. For example, if $K \subset [0, +\infty)$ is compact and $X = \mathcal{C}(K)$, $\bar{\mathcal{C}}$ is the topological closure of $\mathcal{C}$ in X_+, which is equal to the closure of $\mathcal{C}$ in X, with respect to the topology of uniform convergence on K.

Example 1. Let $F = \mathbb{R}$, μ a moment determinate measure on $\mathbb{R}$, $X = L^1_\mu(\mathbb{R})$, Y an ordered Banach space, $T \in B(X,Y)$ a bounded linear operator such that if $J_0 \subset \mathbb{N}$ is a finite subset and $\{\alpha_j\colon j \in J_0\} \subset \mathbb{R}$, then the following inequality holds:

$$\sum_{i,j \in J_0} \alpha_i \alpha_j\, T(x_{i+j}) \geq 0.$$

Then $Tx \geq 0$ for all $x \in X_+$, hence T is positive.

Example 1 stated above refers to Theorems 10 and 12. Since T is supposed to be a continuous linear operator which takes nonnegative values at any positive polynomial which is nonnegative on the entire real axes, according to Theorem 12 and using the notations there, it follows that

$$T(x) = T\left(\lim_m q_m\right) = \lim_m T(q_m) \geq 0, \quad \forall x \in X_+.$$

Thus, T is positive, although we have supposed only its positivity on the subcone $\mathcal{C}$ of all polynomials that are nonnegative on $\mathbb{R}$. The equality $\bar{\mathcal{C}} = X_+$ appearing in the hypothesis of Theorem 12 holds true due to Theorem 10.

The next example also refers to Theorem 12. It completes the discussion regarding the relationship between continuity and positivity for usual classes of linear operators.

Remark 1. The uniqueness of the solution is obvious for the scalar moment problem. It is a direct consequence of the assumption on determinacy of μ. For the general case, it seems easier to prove first Theorem 10, then deducing from this Theorem 12, as described above.

Example 2. Let H be a Hilbert space, $A\colon H \to H$ be a positive self-adjoint operator, $\sigma(A)$ the spectrum of A, $X := \mathcal{C}(\sigma(A))$, $\mathcal{A}$ the ordered Banach space (see [4,5]) of all self-adjoint mappings H into H, $Y := Y(A)$ the commutative Banach algebra and order complete Banach lattice of self-adjoint operators studied in [15], pp. 303–305.

Let $(S_j)_{j\in\mathbb{N}}$ be a sequence in Y, such that for any finite subset $J_0 \subset \mathbb{N}$, any $\{\alpha_j\colon j \in J_0\} \subset \mathbb{R}$, and $k \in \{0,1\}$, the following inequalities hold:

$$0 \le \sum_{i,j\in J_0} \alpha_i\alpha_j S_{i+j+k} \le \sum_{i,j\in J_0} \alpha_i\alpha_j A^{i+j+k}.$$

Then the linear operator T_S defined on $\mathbb{R}[t]$ by

$$T_S\left(\sum_{j\in J_0} \alpha_j t^j\right) := \sum_{j\in J_0} \alpha_j S_j,$$

admits a positive linear extension T which is dominated by the extension Φ_A of the linear operator defined by $\Phi\left(\sum_{j\in J_0} \alpha_j t^j\right) := \sum_{j\in J_0} \alpha_j A^j$ on the dense sub-cone $\mathcal{C}$ of restrictions to $\sigma(A)$ of positive polynomials on the entire interval $[0,+\infty)$. We recall that Φ_A is continuous on $\mathcal{C}(\sigma(A))$, due to functional calculus for self-adjoint operator A. On the other hand, Theorem 9 ensures the density of the sub-cone $\mathcal{C}$ mentioned above in $(\mathcal{C}(\sigma(A)))_+$.

The conclusion of Example 2 is that T_S admits a unique linear positive extension

$$T\colon \mathcal{C}(\sigma(A)) \to Y(A),$$

with

$$0 \le T(x) \le \Phi(x) = x(A), \quad \forall x \in (\mathcal{C}(\sigma(A)))_+ .$$

This follows via Theorems 9 and 12.

Sufficient conditions for determinacy and respectively indeterminacy of measures on $\mathbb{R}$ and on $[0,+\infty)$ have been proved in [29]. Next, we focus on determinacy of product measures $\mu_1 \times \mu_2$ on $\mathbb{R}^2$, respectively on $[0,+\infty)^2$, where $\mu_i, i = 1,2$ are moment determinate measures on $\mathbb{R}$, respectively on $[0,+\infty)$. The reason for doing this is to apply results like Theorem 12 stated above to the moment problem.

Theorem 13 (see [45]). *Let H be a Hilbert space, A_1 a positive self-adjoint operator mapping H into H, and let $Y := Y(A_1)$ be the commutative Banach algebra and order complete Banach lattice of self-adjoint operators studied in [5], pp. 303–305. Let $A_2 \in Y(A_1)$ be a positive operator. If we denote by S_i the spectrum of the positive operator A_i, $i = 1,2$, and $S := S_1 \times S_2$, let us consider the positive linear operator T_2 mapping $\mathcal{C}(S_1 \times S_2)$ into $Y(A_1)$ that verifies the equalities*

$$T_2(t_1^{j_1} t_2^{j_2}) = A_1^{j_1} A_2^{j_2} \quad \text{for all } (j_1, j_2) \in \mathbb{N}^2.$$

Being given a sequence $(U_{(j_1,j_2)})_{(j_1,j_2)\in\mathbb{N}^2}$ of elements from $Y(A_1)$, the following statements are equivalent:

(a) There exists a unique linear operator T mapping $\mathcal{C}(S)$ into Y, such that

$$T(t_1^{j_1} t_2^{j_2}) = U_{(j_1,j_2)}, \quad (j_1,j_2) \in \mathbb{N}^2, \quad \text{and} \quad 0 \le T(\varphi) \le T_2(\varphi), \quad \varphi \in (\mathcal{C}(S))_+ .$$

(b) *For $s_i \in \{0,1\}$, $i = 1,2$, any finite subsets $J_i \subset \mathbb{N}$, $i = 1,2$, and any subsets $\{\alpha_i, j_1 \in J_1\}$, and respectively $\{\beta_i, j_2 \in J_2\}$ of real numbers, the following inequalities hold true:*

$$0 \leq \sum_{i_1,j_1 \in J_1} \alpha_{i_1}\alpha_{j_1} \left(\sum_{i_2,j_2 \in J_2} \beta_{i_2}\beta_{j_2}\, U_{i_1+j_1+s_1, i_2+j_2+s_2} \right) \leq \sum_{i_1,j_1 \in J_1} \alpha_{i_1}\alpha_{j_1} \left(\sum_{i_2,j_2 \in J_2} \beta_{i_2}\beta_{j_2}\, A_1^{i_1+j_1+s_1} A_2^{i_2+j_2+s_2} \right).$$

For the proof of Theorem 13 stated above, see [45], pp. 6–12, Theorems 3, 4, 5.

Corollary 1 stated below provides estimates for $\|T\|$, $U_{(j_1,j_2)}$, and $\|U_{(j_1,j_2)}\|$, although the relationship between the involved operators T, $U_{(j_1,j_2)}$, and A_1, A_2 is not known from the beginning.

Corollary 2 below represents the matrix version of Theorem 13, *in the case where the Hilbert space H is $\mathbb{R}^n$*, $n \geq 2$.

Corollary 1. *With the notations and under the hypothesis of Theorem 13, if the sequence $\left(U_{(j_1,j_2)}\right)_{(j_1,j_2)\in\mathbb{N}^2}$ of self-adjoint operators satisfies conditions (b), and T is the solution of the constrained interpolation problem claimed at point (a), then the following estimates hold:*

$$\|T\| \leq 1, \quad 0 \leq U_{(j_1,j_2)} \leq A_1^{j_1} A_2^{j_2}, \quad \|U_{(j_1,j_2)}\| \leq \|A_1\|^{j_1} \cdot \|A_2\|^{j_2}.$$

Corollary 2. *The statement of Theorem 13 stays valid for $n \times n$ symmetric commuting matrices $A_1, A_2, U_{(j_1,j_2)}$, $(j_1, j_2) \in \mathbb{N}^2$, with real entries, that define linear operators satisfying the hypothesis of Theorem 13.*

4. Conclusions

We recall an earlier Hahn–Banach type result on characterizing the existence of a positive linear extension for a linear positive operator, preserving positivity and the condition of being dominated by a given convex continuous operator. In most cases, this condition involves a given linear positive operator that should dominate our extension on the positive cone of the domain space. We use the approximation results proved in [38] of any nonnegative function from the domain space by sums of squares or by sums of products of polynomials like

$$\prod_{i=1}^{d} \left(q_i^2(t_i) + t_i r_i^2(t_i)\right),$$

multiplied by products of functions f_i appearing in the definition of the compact semi-algebraic subset K (see Theorem 3 stated above).

For interested readers, future research directions might focus on solving moment problems on closed subsets F that are not Cartesian products of intervals. The polynomial approximation result claimed in Theorem 9 stated above of functions from $\left(L^1_\mu(F)\right)_+$ by nonnegative polynomials at each point of F is still working.

Here the problem is that the expression of such nonnegative polynomials in terms of special more simple nonnegative polynomials seems to be not known, unlike the case of semi-algebraic compact subsets K. In characterizing the existence of a positive linear solution, the idea is to find conditions that should be verified for classes of special nonnegative polynomials, for example polynomials expressible in terms of sums of squares. If this is not possible, we prove the density of such polynomials in the positive cone of our function spaces, as described above in the present work.

Our approximation results imply the uniqueness of the solution of the full moment problem on the corresponding function spaces. Almost all results are formulated in terms of quadratic expressions, even when we obtain operator valued solutions. The relationship between positivity and continuity for classes of linear operators is also discussed.

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