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On the sequence of even order moments of the function $\exp(-t^2)$ on compact intervals

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Abstract: First, this paper provides some approximation and estimation type results for some moments of the Gauss function, motivated by the fact that the moments of even orders $n = 2l$, $l \in \mathbb{N} = \{0, 1, \dots\}$ of the function $\exp(-t^2)$ on bounded intervals. Second, the problem of asymptotic behavior of the sequence of all orders for the same function on any interval $[0, b] \subseteq [0, 1/\sqrt{2}]$ is studied and solved. Here the point is using Jensen inequality. Third, the problem of asymptotic behavior of the sequence of all orders for the same function on any interval $[0, b] \subset [0, +\infty)$ is deduced, via elements of complex analysis (Vitali's theorem). The convergence holds uniformly on compact subsets of the complex plane. Fourth, the asymptotic behavior of the sequence of all moments on $[0, 1]$, as $n \rightarrow \infty$, for an arbitrary function $f \in C([0, 1])$ is determined precisely, by means of Korovkin's approximation theorem. Consequently, a similar result for complex analytic functions is deduced, using Vitali's theorem. This is the fifth aim of the paper.

Keywords: Jensen inequality, function e^{-t^2} , $t \in \mathbb{R}$, even order moments, asymptotic behavior, Korovkin's theorem, Vitali's theorem, uniform convergence, compact subsets

MSC: 26D15, 30B10, 30B40.

1. Introduction

The present work uses basic results in real and complex analysis and functional analysis (see [1–6], including integral form of Jensen's inequality), elements of approximation theory contained in some chapters of [5,6], as well as the classical Korovkin's theorem [7,8], quite recently recalled in [11,12]. For further information and generalizations of this main result see the reference [13]. For other main results in complex function theory and related special functions see [9,10]. In [13], the authors extend their framework from linear operators to special sublinear ones, and from the compact interval $[0, 1]$ appearing in the classical Korovkin's theorem, to arbitrary locally compact subsets of $\mathbb{R}^1$, $l \in \mathbb{N}$, $l \geq 1$. Also, the set $\{1, t, t^2\}$ of the test-functions from the classical Korovkin's theorem [11], Theorem 1, is replaced by the set of test-functions $\{1, \pm pr_1, \dots, \pm pr_l, \sum_{j=1}^l pr_j^2\}$ formed by the $2l + 2$ polynomials of degrees smaller or equal to two, mentioned above (see [13], Theorem 2). In [14], among other results, the author applies the classical Korovkin's theorem to approximate a Lipschitz-type function $v \in C_B(\setminus cite{0, \infty})$ by the rational functions $K_n^G(v)$, and determining the rates of convergence (see [14], Theorem 4). In [15], elements of functional calculus are reviewed. In the sequel, we review the motivations of the present paper. As is well-known, the antiderivative of the function $f(t) = e^{-t^2}$ is not expressible in terms of elementary functions. Even order moments of the same function on compact intervals are not known either. However, the limit

$$\lim_{n \rightarrow \infty} \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)},$$

can be determined exactly. In the present article, this is done from the beginning (see Theorem 1 below). Lower and upper bounds for each term of this sequence are also determined. This result is generalized in Theorem 2 (see (5) below), for the corresponding sequence of complex numbers. Namely, we discuss the limit

$$\lim_n \rightarrow \infty \frac{\int_0^z}{u^n e^{-u^2} du} z^{n+1} / (n+1) = e^{-z^2},$$

and prove that the convergence holds uniformly on compact subsets of the complex field. These results are generalized in Theorem 8, the last one in this work. In the sequel, for an entire complex function h of one complex variable, by $\int_0^z h(u) du$ we mean the curvilinear integral on any path of ends 0 and z . The rest of the paper is organized as follows. In Section 2, some methods used along with this work are reviewed. Section 3 contains the results and Section 4 (Conclusion) concludes the paper.

2. Methods

The methods used along with the proofs of the results are:

- 1) Elements of real analysis (Calculus, Jensen inequality (integral form) and appropriate other inequalities). For these results see [1–4].
- 2) Elements of complex analysis (zeros of holomorphic mappings [2] and Vitali's theorem [5]).
- 3) Using Korovkin's theorem [7,8], recalled in [11] and [12] and generalized in [13].
- 4) Applying approximations by sequences of functions.
- 5) Deducing equalities from inequalities, via a passing to the limit process.
- 6) Using functional calculus for continuous real valued functions defined on the spectrum of a self-adjoint operator. This way, we obtain operator inequalities from the corresponding scalar ones, valid at any point of the spectrum.

3. Results

Lemma 1. *The following iterative formulae for the moments of the function $f(t) = e^{-t^2}$, on any interval $[0, b]$, $0 < b < \infty$, hold true:*

$$\begin{aligned} m_1 &:= \int_0^b t e^{-t^2} dt = \frac{1}{2} (1 - e^{-b^2}), \\ m_{2k-1} &:= \int_0^b t^{2k-1} e^{-t^2} dt = \frac{1}{2k} (b^{2k} e^{-b^2} + 2m_{2k+1}), \quad k \in \mathbb{N}, k \geq 1, \\ m_{2k-2} &= \frac{1}{2k-1} (b^{2k-1} e^{-b^2} + 2m_{2k}), \quad k \in \mathbb{N}, k \geq 1. \end{aligned}$$

Proof. The first equality in the statement is obvious, since $m_1 = -\frac{1}{2} e^{-t^2} \Big|_0^b = \frac{1}{2} (1 - e^{-b^2})$. For the next one, integration by parts yields:

$$m_{2k-1} = \frac{1}{2k} t^{2k} e^{-t^2} \Big|_0^b + \frac{1}{k} \int_0^b t^{2k+1} e^{-t^2} dt = \frac{1}{2k} b^{2k} e^{-b^2} + \frac{1}{k} m_{2k+1}, \quad k \in \mathbb{N}, k \geq 1.$$

Thus, knowing m_1 written above, we derive m_3 , and then application of the last equality leads to inductively determining m_{2k+1} for all $k \in \mathbb{N}, k \geq 1$. Unlike the case of the moments of odd orders, which can be determined by induction, the exact values of the moments of even orders cannot be determined, since we cannot find the exact value for the first one, $m_0 = \int_0^b e^{-t^2} dt$. However, the induction works for these moments too, as follows:

$$\begin{aligned} m_{2k} &= \int_0^b t^{2k} e^{-t^2} dt \\ &= -\frac{1}{2} \int_0^b t^{2k-1} (-2te^{-t^2}) dt \\ &= -\frac{1}{2} t^{2k-1} e^{-t^2} \Big|_0^b + \frac{2k-1}{2} \int_0^b t^{2k-2} e^{-t^2} dt \end{aligned}$$

$$= -\frac{1}{2}b^{2k-1}e^{-b^2} + \frac{2k-1}{2}m_{2k-2}, \quad k \in \mathbb{N}, \quad k \geq 1.$$

From this last recurrence equality, it follows that for finding all the moments of even orders it is sufficient (and necessary) to determine one of them. This ends the proof. $\square$

The next theorem is motivated by the fact that the exact values of moments of even orders cannot be determined. Therefore, we apply related inequalities to establish estimations and asymptotic behavior as $n \rightarrow \infty$.

Theorem 1. For any $b \in (0, 1/\sqrt{2}]$, the following estimations hold:

$$e^{-b^2} \leq \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)} < e^{-((n+1)/(n+2))^2 b^2}, \quad n \in \mathbb{N}.$$

Consequently, the asymptotic behavior of the sequence of all moments for f is also valid:

$$\lim_{n \rightarrow \infty} \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)} = e^{-b^2}.$$

Proof. Let $f(t) := e^{-t^2}$, $0 \leq t \leq b \leq 1/\sqrt{2}$. Obviously, we can write $e^{-t^2} \geq e^{-b^2}$ for all $t \in [0, b]$. This implies that

$$\int_0^b t^n e^{-t^2} dt \geq e^{-b^2} b^{n+1}/(n+1), \quad n \in \mathbb{N}. \quad (1)$$

On the other hand, for a converse inequality observe that f is strictly concave on the interval $[0, 1/\sqrt{2}]$. Indeed, we have

$$f'(t) = -2te^{-t^2}, \quad f''(t) = -2e^{-t^2}(1 - 2t^2) < 0,$$

where the last inequality holds if and only if $t < 1/\sqrt{2}$. For any $b \in [0, 1/\sqrt{2}]$, we consider the probability measure $d\mu = \frac{t^n dt}{\int_0^b x^n dx}$. According to the integral form of Jensen's inequality (see [1] or [2], or [3] or [4]), applied to the continuous concave function f , we have:

$$\int_0^b \exp(-t^2) d\mu \leq \exp\left(-\left(\int_0^b t d\mu\right)^2\right) \frac{\int_0^b t^n e^{-t^2} dt}{\int_0^b x^n dx} \leq \exp\left(-\left(\frac{\int_0^b t^{n+1}}{\int_0^b x^{n+1} dx}\right)^2\right) = \exp\left(-\left(\frac{n+1}{n+2}b\right)^2\right). \quad (2)$$

Returning to the inequality (1), we write it under the equivalent form

$$\frac{\int_0^b t^n e^{-t^2} dt}{\int_0^b x^n dx} = \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)} \geq e^{-b^2}, \quad n \in \mathbb{N}.$$

Hence from (1) and (2) the following first conclusion (3) holds

$$e^{-b^2} \leq \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)} \leq e^{-((n+1)/(n+2))^2 b^2}, \quad n \in \mathbb{N}. \quad (3)$$

Consequently, these yields

$$\lim_{n \rightarrow \infty} \frac{\int_0^b t^n e^{-t^2} dt}{b^{n+1}/(n+1)} = e^{-b^2}. \quad (4)$$

Thus, the asymptotic behavior of the sequence of all moments of f is determined, and lower, as well as upper bound for each term of this sequence is pointed out. This ends the proof. $\square$

Corollary 1. Let A be a self-adjoint operator, $b \in (0, 1/\sqrt{2}]$. Assume that the spectrum $\sigma(A)$ is contained in the interval $(0, 1/\sqrt{2}]$. Then the following inequalities hold true:

$$e^{-A^2} \leq \sin(A) \leq e^{-((n+1)/(n+2))^2 A^2}, \quad \forall n \in \mathbb{N}.$$

Proof. According to (3), for any element $b \in \sigma(A) \subseteq (0, 1/\sqrt{2}]$ and any $n \in \mathbb{N}$, the following inequalities hold:

$$e^{-b^2} \leq (n+1) \cdot \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \cdot \frac{\int_0^b t^n \cdot t^{2l} dt}{b^{n+1}} \right) \leq e^{-((n+1)/(n+2))^2 b^2}.$$

The series appearing above in the center can be written as:

$$\frac{n+1}{b^{n+1}} \cdot \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \cdot \frac{b^{n+1+2l+1}}{(n+1)(2l+1)} \right) = \sum_{l=0}^{\infty} \frac{(-1)^l b^{2l+1}}{(2l+1)!} = \sin(b).$$

Inserting this into (3), we conclude that:

$$e^{-b^2} \leq \sin(b) \leq e^{-((n+1)/(n+2))^2 b^2}, \quad \forall b \in \sigma(A) \subseteq (0, 1/\sqrt{2}], \quad \forall n \in \mathbb{N}.$$

From functional calculus for continuous real valued functions on the spectrum $\sigma(A)$, we derive:

$$e^{-A^2} \leq \sin(A) \leq e^{-((n+1)/(n+2))^2 A^2}, \quad \forall n \in \mathbb{N}.$$

This ends the proof. $\square$

Corollary 2. The statement of Corollary 1 holds true for any symmetric $n \times n$ matrix with real entries, $n \geq 2$, whose all eigenvalues are in the interval $(0, 1/\sqrt{2}]$.

Remark 1. If A is self-adjoint and $f \in C(\sigma(A))$, then $\|f(A)\|_{B(H)} = \|f\|_{C(\sigma(A))}$. Hence, the norm of the self-adjoint operator $f(A)$ in the space $B(H)$ of all bounded linear operators acting on H , equals the norm of f in the Banach algebra $C(\sigma(A))$.

Our next purpose is to prove that the asymptotic behavior (4) holds uniformly on any compact interval. Moreover, the corresponding approximation result holds true for the function $f(z) = e^{-z^2}$, uniformly on compact subsets of $\mathbb{C}$. With the notations from the next theorem, this means that for any $\varepsilon > 0$ and for any compact $K \subset \mathbb{C}$, there exists $N = N(\varepsilon, K)$ such that

$$\left| \frac{\int_0^z u^n e^{-u^2} du}{z^{n+1}/(n+1)} - e^{-z^2} \right| \leq \varepsilon, \quad \forall z \in K, \quad \forall n \in \mathbb{N}, n \geq N(\varepsilon, K).$$

Theorem 2. The following asymptotic convergence result holds uniformly on compact subsets of $\mathbb{C}$:

$$\lim_{n \rightarrow \infty} \frac{\int_0^z u^n e^{-u^2} du}{z^{n+1}/(n+1)} = e^{-z^2}. \quad (5)$$

In particular, (4) is valid for all $b \in (0, \infty)$.

Proof. We consider the sequence of complex functions $(g_n)_{n \geq 0}$ in one complex variable, defined by:

$$g_n(\exp(-z^2)) := \frac{\int_0^z u^n e^{-u^2} du}{z^{n+1}/(n+1)}, \quad z \neq 0, \quad g_n(0) = 1, \quad n \in \mathbb{N}.$$

This sequence is suggested by the convergence (4) proved in Theorem 1. We start by observing that zero is a removable singularity for all terms of this sequence. Moreover, the assumptions of the statement of Vitali's theorem are satisfied. Namely, for $z \neq 0$, we have:

$$g_n(z) = (n+1) \cdot \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \cdot \frac{z^{n+2l+1}}{(n+2l+1)z^{n+1}} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \cdot \frac{n+1}{2l+n+1} z^{2l} \rightarrow 1, \quad z \rightarrow 0.$$

Hence g_n is holomorphic in $\mathbb{C}$ (is an entire complex function) for each $n \in \mathbb{N}$, and for any $R \in (0, \infty)$, the following estimation of a common upper bound for $|g_n(z)|$ holds:

$$|z| \leq R \implies |g_n(z)| \leq \sum_{l=0}^{\infty} \frac{1}{l!} \cdot \frac{n+1}{2l+n+1} R^{2l} \leq \sum_{l=0}^{\infty} \frac{R^{2l}}{l!} = e^{R^2} =: M(R), \quad \forall n \in \mathbb{N}.$$

Thus, there exists an upper bound e^{R^2} such that $|g_n(z)| \leq e^{R^2}$ for all natural numbers n and for all z in the closed disk $\{z; |z| \leq R\}$. Since any compact is contained in such a disk of sufficiently large radius, the sequence of holomorphic functions $(g_n)_n$ is uniformly bounded on any compact subset. The upper bound e^{R^2} does not depend on n or on z . It depends only on the compact disk containing the compact subset K . Now we recall that for such sequences of holomorphic functions on open connected subset $\tilde{U}$ of the complex plane, from the convergence of the sequence on a set having accumulation points in $\tilde{U}$, we conclude that the convergence holds uniformly on any compact $K \subset \Omega$, and the limit say h , is holomorphic in $\tilde{U}$. This is Vitali's theorem [5]. In our case, $\tilde{U}$ is the entire complex plane. Thus, $h := \lim_{n \rightarrow \infty} g_n$, is holomorphic. The conclusion is that the difference $h(z) - e^{-z^2} = 0$ for all $z \in \mathbb{C}$. Indeed, from Theorem 1, we already know that $\lim_{n \rightarrow \infty} g_n(z) = f(z) = e^{-z^2}$ for all $z \in [0, b] \subseteq [0, 1/\sqrt{2}]$. It results $f = h$ in $[0, 1/\sqrt{2}]$, and, since this interval has (infinitely many) accumulation points, via a well-known property of zeros of holomorphic functions [2], we conclude that

$$f(z) = h(z) := \lim_{n \rightarrow \infty} g_n(z) = e^{-z^2}, \quad z \in \mathbb{C}.$$

Indeed, in [2] or [5] it was proved that the set of zeros of a nonnull holomorphic function in a region $\tilde{U}$ has no accumulation point in $\tilde{U}$. According to Vitali's theorem, the convergence (5) claimed in the statement holds uniformly on compact subsets of $\mathbb{C}$. This ends the proof. $\square$

Corollary 3. For any $b \in (0, \infty)$, the following asymptotic behavior holds

$$\int_0^b u^{2n} e^{-u^2} du \sim \frac{b^{2n+1}}{2n+1} e^{-b^2}.$$

For the results appearing in the sequel, we need the following classical Korovkin's uniform approximation theorem.

Theorem 3. [See [7,8] and/or [11]]. Let $(L_n)_{n \geq 0}$ be a sequence of positive linear operators that map $C([0, 1])$ into itself. Suppose that the sequence $(L_n(f))_{n \geq 0}$ converges to f uniformly on $[0, 1]$, for each $f \in \{1, t, t^2\}$. Then this sequence converges to f uniformly on $[0, 1]$ for every $f \in C([0, 1])$.

Theorem 4. Let $f \in C([0, 1])$. Then

$$\lim_{n \rightarrow \infty} g_n(f)(x) := \lim_{n \rightarrow \infty} \frac{\int_0^x t^n f(t) dt}{x^{n+1}/(n+1)} = f(x), \quad x \in (0, 1], \quad \lim_{n \rightarrow \infty} g_n(f)(0) = f(0), \quad (6)$$

where $g_n = g_n(f)$, $n \in \mathbb{N}$ are defined below (see (7), (8)), the convergence being uniform on $[0, 1]$.

Proof. We start with the following remark. In (8), it is thereby understood that for any $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ the function

$$g_n(x) := g_n(f)(x) := \frac{\int_0^x t^n f(t) dt}{x^{n+1}/(n+1)}, \quad x \in (0, 1], \quad (7)$$

can be extended by continuity to the entire compact interval $[0, 1]$ by defining

$$g_n(0) := g_n(f)(0) := \lim_{x \downarrow 0} \frac{\int_0^x t^n f(t) dt}{x^{n+1}/(n+1)} = \lim_{x \downarrow 0} \frac{x^n f(x)}{x^n} = f(0). \quad (8)$$

Here L-Hôpital's rule has been applied for a limit $0/0$ involving two C^1 functions on $[0, 1]$. By abuse of notation, the extension of g_n defined by (7) to the entire interval $[0, 1]$ written in (8) has been denoted by g_n as well. Consider the sequence $(T_n)_{n \in \mathbb{N}}$ of linear positive operators $T_n : C([0, 1]) \rightarrow C([0, 1])$, defined by:

$$(T_n f)(x) := \frac{\int_0^x t^n f(t) dt}{x^{n+1}/(n+1)}, \quad x \in (0, 1], (T_n f)(0) := f(0). \quad (9)$$

The conclusion (8) of the theorem can be written as: $\lim_{n \rightarrow \infty} T_n f = f$ for all $f \in C([0, 1])$. According to Korovkin's theorem, it is sufficient to prove this convergence for the first three polynomial functions $p_0(t) = t^0 = 1, p_1(t) = t, p_2(t) = t^2$. That is we only have to prove that

$$\lim_{n \rightarrow \infty} T_n \mathbf{1} = \mathbf{1}, \quad \lim_{n \rightarrow \infty} T_n p_1 = p_1, \quad \lim_{n \rightarrow \infty} T_n p_2 = p_2,$$

the convergence being uniform on $[0, 1]$. The following easy computations hold true:

$$\begin{aligned} T_n(p_0)(x) &= \frac{\int_0^x t^n dt}{x^{n+1}/(n+1)} = 1, \quad x \in [0, 1], \\ T_n(p_1)(x) &= \frac{\int_0^x t^{n+1} dt}{x^{n+1}/(n+1)} = \frac{x^{n+2}/(n+2)}{x^{n+1}/(n+1)} = x \frac{n+1}{n+2} \rightarrow x = p_1(x), \quad n \rightarrow \infty, \quad x \in [0, 1], \\ T_n(p_2)(x) &= \frac{\int_0^x t^{n+2} dt}{x^{n+1}/(n+1)} = \frac{x^{n+3}/(n+3)}{x^{n+1}/(n+1)} = x^2 \frac{n+1}{n+3} \rightarrow x^2 = p_2(x), \quad n \rightarrow \infty, \quad x \in [0, 1]. \end{aligned}$$

It is easy to see that the convergences hold uniform with respect to $x \in [0, 1]$. Indeed, for $p_0 = \mathbf{1}$, $\max_{x \in [0, 1]} |T_n(p_0)(x) - \mathbf{1}| = 0$, for all $n \in \mathbb{N}$. If $p_1(x) = x$, then

$$\max_{x \in [0, 1]} |T_n(p_1)(x) - x| = \max_{0 \leq x \leq 1} |T_n(p_1)(x) - x| = \max_{0 \leq x \leq 1} \left| x \left(\frac{n+1}{n+2} - 1 \right) \right| = \frac{1}{n+2} \rightarrow 0, \quad n \rightarrow \infty.$$

If $p_2(x) = x^2$, then

$$\begin{aligned} \max_{x \in [0, 1]} |T_n(p_2)(x) - x^2| &= \max_{0 \leq x \leq 1} |T_n(p_2)(x) - x^2| \\ &= \max_{0 \leq x \leq 1} \left| x^2 \frac{n+1}{n+3} - x^2 \right| \\ &= \max_{0 \leq x \leq 1} \left| x^2 \left(\frac{n+1}{n+3} - 1 \right) \right| \\ &= \frac{2}{n+3} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Since all the three upper bounds converge to zero as $n \rightarrow \infty$, and they do not depend on $x \in [0, 1]$, the corresponding uniform convergences follow. Application of Korovkin's theorem ends the proof of the present theorem. $\square$

Corollary 4. With the notations from Theorem 4, we have

$$\int_0^x t^n f(t) dt \sim \frac{x^{n+1}}{n+1} f(x), \quad x \in [0, 1], \quad n \rightarrow \infty. \quad f \in C([0, 1])$$

Precisely, this means that (6) holds for all $x \in [0, 1]$.

Finally, we consider the case when f from Theorem 3 is the restriction of an entire function to the interval $[0, 1]$, and conditions like those emphasized in Theorems 1, 2 are satisfied.

Theorem 5. Let f be an entire function whose moment of zero-order on an arbitrary interval $[0, b]$ cannot be calculated exactly. Then the functions g_n , $n \in \mathbb{N}$ defined by

$$g_n(f)(z) := \frac{\int_0^z u^n f(u) du}{z^{n+1}/(n+1)}, \quad z \neq 0, g_n(0) = f(0), \quad (10)$$

have removable singularity at zero, are uniformly bounded on each closed disk $\{z; |z| \leq R\}$, and the asymptotic behavior (11) holds true:

$$f(z) = \lim_{n \rightarrow \infty} g_n(f)(z), \quad z \in \mathbb{C}. \quad (11)$$

In (11), the convergence holds uniformly on compact subsets of the complex plane.

Proof. The asymptotic behavior (6) follows from Theorem 3 since the restriction of f to $[0, 1]$ is continuous. The assumptions on uniform boundedness of the sequence of functions g_n , $n \in \mathbb{N}$ on compact subsets ensure the uniform convergence of this sequence on compact subsets, to a conformal mapping $h: \mathbb{C} \rightarrow \mathbb{C}$, via Vitali's theorem (each point in $[0, 1]$ is an accumulation point). Application of the theorem on zeros of holomorphic functions shows that $f = h$ on the entire complex plane. To conclude, we have:

$$f(z) = h(z) = \lim_{n \rightarrow \infty} g_n(z), \quad z \in \mathbb{C},$$

the convergence being uniform on compact subsets. This ends the proof. $\square$

Theorem 6. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z) = \sin(z)/z$, $z \neq 0$, $f(0) = 1$ and g_n , $n \in \mathbb{N}$ be defined by (10). Then f has removable singularity at zero and $f(z) = \lim_{n \rightarrow \infty} g_n(z)$, $z \in \mathbb{C}$ holds uniformly on compact subsets of the complex plane.

Proof. According to Theorem 5, to obtain the desired conclusion of Theorem 6, we only must check whether the requirements on the sequence $(g_n)_{n \geq 0}$ are satisfied. For $z \neq 0$, we have:

$$g_n(z) = \frac{\int_0^z u^n \left(\sum_{l=0}^{\infty} (-1)^l u^{2l} / (2l+1)! \right) du}{z^{n+1}/(n+1)} = 1 + \sum_{l=1}^{\infty} (-1)^l \frac{n+1}{2l+n+1} \cdot \frac{z^{2l}}{(2l+1)!} \rightarrow 1, \quad z \rightarrow 0.$$

For any $R > 0$ and all z with $|z| \leq R$, we obtain

$$|g_n(z)| \leq 1 + \sum_{l=1}^{\infty} \frac{R^{2l}}{(2l+1)!} \leq 1 + \sum_{l=1}^{\infty} \frac{R^{2l}}{(2l)!} = \cosh(R) =: M(R), \quad \forall n \in \mathbb{N}. \quad (12)$$

We observe that the right-hand side member of (12) does not depend on n . Application of Theorem 5 leads to the conclusion of the present theorem. This ends the proof. $\square$

Theorem 7. Let $f(z) := \sin(z)/z$, $z \neq 0$, $f(0) := 1$. Then f is the unique entire function whose moment of zero-order on an arbitrary interval $[0, b]$ cannot be calculated exactly, the functions g_n , $n \in \mathbb{N}$ defined by

$$g_n(z) := g_n(f)(z) := \frac{\int_0^z u^n f(u) du}{z^{n+1}/(n+1)}, \quad z \neq 0, g_n(0) = 1,$$

have removable singularity at zero, are uniformly bounded on each closed disc $\{z; |z| \leq R\}$, and the convergence

$$f(z) = \lim_{n \rightarrow \infty} g_n(f)(z), \quad (13)$$

holds uniformly on compact subsets of the complex plane.

Proof. According to Theorem 6, we already know that for f has all the properties claimed in the statement. It remains to prove that f is the unique mapping with these properties. Assume that there exists an entire mapping ψ with the property (14) written below. For such a function ψ , we should have

$$\psi(z) = \lim_{n \rightarrow \infty} g_n(f)(z), \quad z \in \mathbb{C}. \quad (14)$$

The convergence (14) holds uniformly on compact subsets. Then the function $d(z) := \psi(z) - f(z)$ is entire, $d(0) = \psi(0) - f(0) = 1 - 1 = 0$, and application of Theorem 6 and equality (14) lead to

$$\psi(z) - f(z) = \lim_{n \rightarrow \infty} g_n(f)(z) - \lim_{n \rightarrow \infty} (g_n(f))(z) = 0, \quad z \in \mathbb{C}.$$

□

Proceeding as in the proof of Theorem 7, the following general result follows as well.

Theorem 8. (Uniqueness of the function f). Let f be an arbitrary entire function satisfying the requirements stated in Theorem 7. Then (13) holds for f , and, if ψ is another entire function verifying the hypothesis of Theorem 7 and (14) holds, then $\psi \equiv f$.

4. Conclusion

We have pointed out sufficient conditions for determining exact asymptotic behavior of the sequence of their moments, for some entire functions. Two main examples are the functions emphasized in Theorems 1 and 2 and 6. For these mappings, the first moment (of order zero) and any other moment of even order, on a compact interval, cannot be determined or are very difficult to find. However, the limits of the sequences of moments (see (4), (5), (6), (11)), can be computed exactly. Of note, in Theorem 2, (5) is proved applying Jensen's inequality, then passing to the limit, without using Korovkin's theorem or other approximation result. Also, in Theorem 1, (3) gives estimates for each term of the even orders sequence of moments. As we have seen, we do not know the exact values of the even order moments of $f(t) = e^{-t^2}$ on an interval $[0, b]$, where $b \in (0, \infty)$. If $[0, b]$ is replaced by $(0, \infty)$, the moments of any order can be determined exactly by means of the properties of the Gamma function. Unlike the moments of even orders, those of odd orders can be easily determined on any interval $[0, b]$, since the antiderivative of $tf(t) = te^{-t^2}$ is clearly equal to $-(1/2)e^{-t^2}$ (see also Lemma 1 proved above). A first idea in solving this difficulty in the moments of even orders, is to appropriately estimate them. This can be done by means of integral form of Jensen's inequality [1,4] (see Theorem 1). Here we use the concavity of f on the interval $[0, 1/\sqrt{2}]$. According to this theorem, the asymptotic behavior of the sequence of all moments is given by (4). In turn, (4) is obtained from passing to the limit in (3). Consequently, we can write

$$\lim_{m \rightarrow \infty} \frac{\int_0^b t^{2m} e^{-t^2} dt}{b^{2m+1}/(2m+1)} = e^{-b^2}, \quad b \in (0, 1/\sqrt{2}].$$

The next step is to obtain the uniform convergence on any compact interval $[0, b]$. This can be done by means of Vitali's theorem [5], which leads to the result proved in Theorem 2 proved below. To obtain uniform convergence on any interval $[0, b]$ for any continuous real function on this interval, we apply Korovkin's theorem (Theorem 3 stated above), which leads to the proof of Theorem 4. Theorem 5 extends the result of Theorem 4 to the uniform convergence on compact subsets of $\mathbb{C}$, in the case where the involved function is the restriction to $[0, b]$ of an entire complex function of one complex variable. Here both Vitali's and Korovkin's theorems are applied. Theorem 6 is derived from Theorem 5 applied to $f(z) = \sin(z)/z$, $z \neq 0$, $f(0) = 1$, and g_n defined by (10). The last results, Theorem 7, provide the uniqueness of the solutions claimed in the statement of this theorem.

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