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Positivity preservation, elliptic boundary value problems, and green's operators

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Abstract: It is well known that positive Green's operators are not necessarily positivity preserving. This result is important, because many physical problems require positivity in their solutions in order to make sense. In this paper we investigate the matter of just how far from being positivity preserving a positive Green's operator can be. In particular, we will see that there exists positive Green's operators that takes some positive functions to functions with negative mean values. We will also identify a broad class of Green's operators that are not necessarily positivity preserving but have properties related to positivity preservation that one expects from positivity preserving Green's operators. Finally, we will compare the results contained in this paper with those that already exist in the literature on the subject.

Keywords: green's operators, green's functions, positivity preservation, elliptic boundary value problems, rods, plates

MSC: 35J25.

Let n be a positive integer. Let U be a bounded domain of $\mathbb{R}^n$ whose boundary S is smooth. Consider the equation

$$\begin{cases} A(x, \partial/\partial x)u(x) = f(x) & x \in U, \\ B_j(x, \partial/\partial x)u(x) = 0, & x \in S, \quad j = 1, 2, ... \frac{m}{2}, \end{cases} \tag{1}$$

where A is an elliptic operator of order m and the boundary operators $\{B_j\}$ satisfy: (i) At every point x of S, the normal derivative is not characteristic for any B_j; and (ii) the order m_j of B_j is less than m, and $m_j \neq m_k$. The domain $\mathcal{D}(A)$ of A is defined by

$$\mathcal{D}(A) = \left\{ u(x) | u \in H^m(U), \quad \text{and} \quad B_j(x, \partial/\partial x)u(x) = 0 \quad \text{for} \quad x \in S, \ j = 1, 2, ..., \frac{m}{2} \right\}. \tag{2}$$

Here $H^m(U)$ is a Sobolev space that consists of locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq m$, $D^\alpha u$ exists in the weak sense and belongs to $L^2(U)$. We will call the ordered collection

$$(U, A, B_1, ..., B_{m/2}),$$

admissable if and only if (i) U, A and $\{B_j\}$ are defined as above, (ii) A is a one-to-one mapping from $\mathcal{D}(A)$ onto $L^2(U)$; and (iii) A has the property that A^{-1} is self-adjoint in $L^2(U)$. The positive integer m will always be the order of A, and n will always be the positive integer such that U is a smooth, bounded domain of $\mathbb{R}^n$. We will call the inverse $G = A^{-1}$ a Green's operator.

Green's operators received a considerable amount of attention in the twentieth century. This is due in large part to the fact that elliptic boundary value problems serve as a model for many physical problems, such as the deflection of a clamped plate under the influence of gravity. It has been shown that $G[u] \in H^m(U)$ if $u(x) \in L^2(U)$ and that $G[u]$ depends continuously on u, [1]. In particular, if $m > n/2$, then by Sobolev's theorem G is a continuous mapping from $L^2(U)$ onto $C^0(\bar{U})$. It has also been shown that G is represented by an integral operator of Hilbert-Schmidt type, [2]. Namely, for any $f(x) \in L^2(U)$,

$$\begin{cases} (Gf)(x) = \int_U G(x,\xi) f(\xi) \, \mathrm{d}\xi, \\ \int_{U \times U} |G(x,\xi)|^2 \, \mathrm{d}(x,\xi) < +\infty. \end{cases} \tag{3}$$

In general, the function $G(x,\xi)$, obtained by the kernel representation of the Green's operator G, is called a Green's function.

1. higher order boundary value problems

For second-order elliptic differential operators there is a nice co-incedence: if A is a positive operator, and if A^{-1} is self-adjoint in $L^2(U)$, then A^{-1} is positivity preserving if homogeneous Dirichlet boundary conditions are used. Given this, it is natural to ask whether or not positive Green's operators [1] associated with higher-order elliptic differential operators and homogeneous Dirichlet boundary conditions are positivity preserving. There are many counter examples to this conjecture, although some Green's operators associated with higher-order elliptic differential operators and homogeneous Dirichlet boundary conditions are positivity preserving, [3]. One can still ask if the Green's operator has properties that are related to the positivity preserving property. Examples include whether or not the principal eigenvalue of A is simple and whether or not the corresponding eigenfunction is of one sign. These questions have received quite a bit of attention. Again, some admisssable ordered collections $(U, A, B_1, ..., B_{m/2})$ have these properties, some don't. (Again, F. Gazzola, H-C. Grunau, and G. Sweers have provided us with a comprehensive history of this topic, [3].) Also, surprising is the fact that there exists admisable ordered collections where A^{-1} is positive and such that the solution of

$$\begin{cases} A(x,\partial/\partial x)u(x) = 1 & x \in U, \\ B_j(x,\partial/\partial x)u(x) = 0, & x \in S, \quad j = 1,2,...,\frac{m}{2}, \end{cases} \tag{4}$$

is not non-negative, [4], [5]. Later in this paper we will see that this implies that it is possible that the mean value of a solution of

$$\begin{aligned} &A(x,\partial/\partial x)u(x) = f(x) \quad x \in U, \\ &B_j(x,\partial/\partial x)u(x) = 0, \quad x \in S, \quad j = 1,2,...,\frac{m}{2}, \end{aligned} \tag{5}$$

where f is a positive function, isn't necessarily non-negative. We will also see that the existence of a negative-somewhere solution to (3) implies that there are situations where the Green's operator, G, exists and is a positive operator, but we do not have $\int_U G(x,\xi)\, d\xi \geq 0$ for all x, where $G(x,\xi)$ is the Green's functions associated with G.

One statement involving positivity that is true for all Green's operators associated with positive elliptic operators is as follows:

Theorem 1. *Suppose that the ordered collection*

$$(U, A, B_1, ..., B_{m/2}),$$

is admissable, A is positive, and $m > \frac{n}{2}$. Then we have

$$\int_{U\times U} (z(x)G(x,\xi)z(\xi))\, d(x,\xi) \geq 0, \tag{6}$$

for all $z \in L^2(U)$, where $G(x,\xi)$ is the Green's function associated with the admissable ordered collection

$$(U, A, B_1, ..., B_{m/2}).$$

Proof. First note that we have $\int_U w(x)A\,(x,\partial/\partial x)\, w(x)\, dx \geq 0$, for all w in the domain of A because we assume that $A > 0$. The positivity of A also lets us conclude that a Green's operator G exists, so let us write $w = G[z]$. Then we have it that $\int_U z(x)G[z](x)\, dx \geq 0$ for all $z \in L^2(U)$. Since $G(x,\xi)$ is obtained by the kernel representation of the operator G, we can write $\int_U z(x) \left(\int_U G(x,\xi)z(\xi)\, d\xi\right) dx \geq 0$ for all $z \in L^2(U)$. Applying Fubini's Theorem to the above inequality then gives us the theorem. □

Some consequences of the above theorem are as follows:

[1] Here, and for the remainder of the paper, we will say that a Green's operator is positive if and only if all of its eigenvalues are real and positive.

Corollary 2. *Suppose that the ordered collection*

$$(U, A, B_1, ..., B_{m/2}),$$

is admissable, A is positive, and $m > \frac{n}{2}$. Then we have that

$$\int_{U\times U} G(x,\xi)\, d(x,\xi) \geq 0.$$

Proof. Recall (6) and set $z = 1$. □

Remark 1. Let $(U, A, B_1, ..., B_{m/2})$ be an admissable ordered collection. Note that the above corollary implies that the mean value of the solution of (3) is positive if A is a positive operator.

Corollary 3. *Suppose that the ordered collection*

$$(U, A, B_1, ..., B_{m/2}),$$

is admissable, A is positive, and $m > \frac{n}{2}$. then the solution of (1) is positive somewhere on U if f is positive.

Proof. Combining Theorem 1 with our hypotheses and Fubini's theorem we see that

$$0 < \int_U \left(\int_U f(x)G(x,\xi)f(\xi)\, \mathrm{d}\xi \right) \mathrm{d}x = \int_U f(x) \left(\int_U G(x,\xi)f(\xi)\, \mathrm{d}\xi \right) \mathrm{d}x.$$

The corollary follows. □

2. Equivalence of three properties

It is difficult to say much more about positivity preservation for arbitrary Green's operators, without adding additional hypotheses. Given that we want $G(x,y) \geq 0$, but only have $\int_{U\times U} G(x,y)\,\mathrm{d}(x,y) \geq 0$ in general, it seems natural to ask questions about the case where $\int_U G(x,y)\,\mathrm{d}y \geq 0$ for all $x \in U$. We reserve the remainder of this section to investigating this property. First we will prove two lemmas relating the positivity of $\int_U G(x,y)\,\mathrm{d}y$ to two different but important properties.

Lemma 1. *Let $(U, A, B_1, ..., B_{m/2})$ be an admissable ordered collection. Suppose $m > n/2$, and let $G(x,y)$ be the Green's function associated with this admissable ordered collection. Let $f : U \to \mathbb{R}$ be a smooth, non-negative function and let the solution of (1) be called u_f. Then we have $\int_U u_f(x)\,dx \geq 0$ iff $\int_U G(x,y)\,dy \geq 0$ for all $x \in U$.*

Proof. First note that we have

$$\int_U u_f(x)\,\mathrm{d}x = \int_U \left(\int_U f(y)G(x,y)\,\mathrm{d}y \right) \mathrm{d}x. \tag{7}$$

Invoking the symmetry of $G(x,y)$, we can write

$$\int_U u_f(x)\,\mathrm{d}x = \int_U \left(\int_U f(y)G(y,x)\,\mathrm{d}y \right) \mathrm{d}x. \tag{8}$$

We can now use Fubini's Theorem to obtain

$$\int_U u_f(x)\,\mathrm{d}x = \int_U \left(\int_U f(y)G(y,x)\,\mathrm{d}x \right) \mathrm{d}y. \tag{9}$$

This, in turn, allows us to write

$$\int_U u_f(x)\,\mathrm{d}x = \int_U f(y) \left(\int_U G(y,x)\,\mathrm{d}x \right) \mathrm{d}y. \tag{10}$$

Now, suppose that $\int_U G(x,y)\,\mathrm{d}y \geq 0$ for all $x \in U$. This, in turn, allows us to conclude that $\int_U G(y,x)\,\mathrm{d}x \geq 0$ for all $y \in U$. Since f is non-negative, we can now conclude that the right hand side of (10) is non-negative. This, in turn, allows us to write $\int_U u_f(x)\,\mathrm{d}x \geq 0$.

We now have $\int_U u_f(x)\,dx \geq 0$ if $\int_U G(x,y)\,dy \geq 0$ for all $x \in U$. It remains to show that there exists a smooth, non-negative function $f : U \to \mathbb{R}$ such that $\int_U u_f(x)\,dx < 0$ if

$$\int_U G(x,y)\,dy < 0 \quad \text{for some} \quad x \in U. \tag{11}$$

Assume (11) is the case. Combining this assumption with the continuity of

$$\int_U G(x,y)dy$$

we see that there would exist a smooth non-negative function $f : U \to \mathbb{R}$ such that

$$\int_U f(x)(\int_U G(x,y)\mathrm{d}y)\mathrm{d}x < 0.$$

This, in turn would allow us to write

$$\int_U (\int_U G(x,y)f(x)\mathrm{d}y)\mathrm{d}x < 0.$$

Applying Fubini's Theorem to the above inequality would then gives us

$$\int_U (\int_U G(x,y)f(x)\mathrm{d}x)\mathrm{d}y. < 0$$

Invoking the assumed symmetry of $G(x,y)$ we would obtain

$$\int_U (\int_U G(y,x)f(x)\mathrm{d}x)\mathrm{d}y < 0.$$

We now have that (11) implies the existence of a smooth, non-negative function f such that $\int_U u_f(x)dx < 0$, provided that the collection $\{U, B_1, ..., B_{m/2}, A\}$ is admissable and the Green's function associated with it is symmetric. The lemma follows. □

Next we prove:

Lemma 2. *Let $(U, A, B_1, ..., B_{m/2})$ be an admissable ordered collection. Suppose $m > n/2$, and let $G(x,y)$ be the Green's function associated with this admissable ordered collection. Then $\int_U G(x,y)\,dy \geq 0$ for all $x \in U$ iff the unique solution of the boundary value problem (3) is non-negative.*

Proof. Suppose that the solution, u, of the boundary value problem (3) is non-negative on U. Note that our hypotheses allow us to assume that a Green's operator G exists and that $u = G[1]$. Our hypotheses also guarantee that a Green's function $G(x,y)$ associated with G exists, and we can write $u(x) = \int_U G(x,y)\,\mathrm{d}y$. Recall that one of our hypotheses is that the solution of (3) is non-negative, so we have it that $\int_U G(x,y)\,\mathrm{d}y \geq 0$ for all $x \in U$, if the unique solution of the boundary value problem (3) is non-negative.

Now, let us assume that $\int_U G(x,y)\,\mathrm{d}y \geq 0$ for all $x \in U$, and let us write

$$w(x) = \int_U G(x,y)\,\mathrm{d}y.$$

Notice that $w \in \mathcal{D}(A)$. Applying A to both sides of the above equation gives us

$$A(x, \partial/\partial x)w(x) = \int_U A(x, \partial/\partial x)G(x,y)\,\mathrm{d}y = \int_U \delta(x-y)\,\mathrm{d}y = 1.$$

Since $w \in \mathcal{D}(A)$, we have it that w is the solution of (3). The lemma follows. □

With the above two lemmas in place, we can conclude that

Theorem 4. *Let m and n be positive integers, and suppose that $m > n/2$. Let $(U, A, B_1, ..., B_{m/2})$ be an admissable ordered collection. Then the following statements are equivalent to each other.*

(i) The solution of (3) is non-negative;

(ii) $\int_U G(x,y)\,dy \geq 0$ *for all* $x \in U$*, where* $G(x,y)$ *is the Green's function associated with the admissable ordered collection* $(U, A, B_1, ..., B_{m/2})$*; and*

(iii) the mean value of the solution of (1) *is non-negative if* f *is non-negative.*

Proof. Combine Lemma 1 and Lemma 2. □

3. Conclusion

In [4] and [5] we see that there exists admissable collections $(U, A, B_1, ..., B_{m/2})$ such that A^{-1} is a positive operator and such that the boundary value problem

$$A(x, \partial/\partial x)u(x) = 1 \quad x \in U,$$
$$B_j(x, \partial/\partial x)u(x) = 0, \quad x \in \partial U, \quad j = 1, 2, ..., \frac{m}{2}$$

has a unique solution that is negative somewhere on U. Let G be the Green's operator associated with $(U, A, B_1, ..., B_{m/2})$, and let $G(x,y)$ be the Green's function associated with G. Combining the above existence result with Theorem 4 we see that $\int_U G(x,y)\,dy$ is negative for some $x \in U$, and G takes some positive functions to functions with negative mean values. In short, positive Green's operators can be very far from positivity preserving. There is, though, a bottom limit on how far from positivity preserving they can be. Recalling work done in Section 1, we have $\int_{U \times U} G(x,y)\,d(x,y) \geq 0$, and u_f will be positive somewhere if f is a positive function.

At this point, we should note that there exists a whole separate theory of positivity preservation and Green's operators, [3], [6], [7], [8], [9]. It produces bounds on the negative part and the positive part of the Green's function being investigated and then compares the two. Typically, one has it that the negative part of the Green's function has much better bounds than the positive part. This, in turn, should assist in producing limits on how far positive Green's operators can be from being positivity preserving.

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