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Some refinements of Giaccardi and Petrović's inequalities

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Abstract: In this research article, the authors introduce the refinements of some special inequalities, like Lah-Ribarič type, Giaccardi, and Petrović's inequalities. Also, the authors define Fejér, Giaccardi, and Petrović's types of inequalities for different classes of convex functions.

Keywords: Refined convex functions, (α, m)-convex fuctions, (s, m)-convex fuctions, Lah-Ribarič inequality, Giaccardi inequality, Petrović's inequality

MSC: 26D10, 26D15, 26A51.

1. Introduction and preliminaries

Convex functions are one of the key elements that interconnect various areas of pure and applied mathematics, such as geometry, analysis, and optimization. In recent years, the study of convexity is evolving quickly by utilizing innovative and new techniques. In [1], Toader introduced the concept of m-convex functions as an interesting extension of convex functions. In a similar way, S. Varošanec introduced h-convex functions in [2]. Building upon these ideas, Özdemir et al. [3] presented the notion of (h, m)-convex functions and established various Hermite-Hadamard-type inequalities. Additionally, Dragomir and Fitzpatrick (see [4] and [5]) put forth the notion of s-convex functions, both in the first and second sense.

For the various choices of the auxiliary function h, we have various other classes of convex functions such as: s-Godunova-Levin type [6] and P-functions [7]. Since the appearance of this definition many researchers shown their special interest in studying this class of convex functions. Sarikaya et al. [8] has improved the Hermite-Hadamard's inequality for this class of convex functions. Recent research in this field has shown that theory of convexity and theory of inequalities have a close relationship. Many inequalities can be obtained with the help of convex functions and naturally they can be expanded for generalizations of convex functions.

Definition 1. A function $\mathcal{K} : [a, b] \to \mathbb{R}$, $a < b$, is convex, if

$$\mathcal{K}(\tau\theta + (1-\tau)w) \le \tau\mathcal{K}(\theta) + (1-\tau)\mathcal{K}(w), \quad \forall \theta, w \in [a, b], \quad \tau \in [0, 1].$$

Definition 2. [9] A function $\mathcal{K} : [0, \infty) \to \mathbb{R}$, $b > 0$, is (α, m)-convex, where $\alpha, m \in (0, 1]$, if

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \le \tau^{\alpha}\mathcal{K}(\theta) + m(1-\tau^{\alpha})\mathcal{K}(w), \ \forall \theta, w \in [0, \infty), \ \tau \in [0, 1]. \tag{1}$$

Setting $m = 1$ in (1), one gets the $(\alpha, 1)$-convex function given in [9].

Definition 3. [10] A function $\mathcal{K} : [0, \infty) \to \mathbb{R}$, $b > 0$, is (s, m)-convex, where $s, m \in (0, 1]$, if

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \le \tau^{s}\mathcal{K}(\theta) + m(1-\tau)^{s}\mathcal{K}(w), \quad \forall \theta, w \in [0, \infty), \quad \tau \in [0, 1]. \tag{2}$$

The refinements or extensions of inequalities take place extensively in inequality theory. The refinements or extensions of inequalities take place extensively in inequality theory. There are many techniques to obtain new versions of inequalities and studies on the equivalences of inequalities, see [11,12]. Some of these approaches are based on the convexity of functions, as in Jensen type and Hermite-Hadamard type inequalities. Many new inequalities and their versions, such as integral, fractional integral, and Hermite-Hadamard type inequalities, have been derived for various classes of convexity by different authors; see [13–18].

In this article, we give the Lah-Ribarič and Fejér type inequalities for refined m-convex and the refinements of the Giaccardi and Petrović's inequalities for different classes of convexity.

Definition 4. ([19]) A non-negative function $\mathcal{K} : [0, b] \to \mathbb{R}$, $b > 0$, is called refined (α, m)-convex, where $\alpha, m \in (0, 1]$, if

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \leq \tau^{\alpha}(1-\tau^{\alpha})(\mathcal{K}(\theta) + m\mathcal{K}(w)), \quad \forall \theta, w \in [0, \infty), \quad \tau \in (0,1). \tag{3}$$

Remark 1. Now we discuss the cases when α and m have different values in (3).
1. Taking $m = 1$, gives the refined α-convex functions.
2. If one take $\alpha = 1$, one has refined m-convex functions.
3. For $m = 1 = \alpha$, one gets the refined convex funtions.

Remark 2. Every refined convex function is convex, but the converse is not true in general. For example, $\mathcal{K}(\theta) = \theta^2 + 1$ is convex but it is not refined convex.

Definition 5. ([19]) A non-negative function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ is called refined (s, m)-convex, where $s, m \in (0, 1]$, if

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \leq \tau^{s}(1-\tau)^{s}(\mathcal{K}(\theta) + m\mathcal{K}(w)), \quad \forall \theta, w \in [0, \infty), \quad \tau \in (0,1). \tag{4}$$

Remark 3. For different values of s and m in (4), one gets different results.
1. If one take $s = 1$, one has refined m-convex functions.
2. By taking $m = 1$, one gets the definition of refined s-convex functions.
3. For $s = 1 = m$, one gets the refined convex funtions.

The following theorem contains the result given by Giaccardi in [20].

Theorem 1. *Let $[a, b] \subseteq \mathbb{R}$ be an interval, $\theta_0 \in [a, b]$, $(\theta_1, ..., \theta_n) \in [a, b]^n$ and $(w_1, ..., w_n) \in \mathbb{R}^n_+$ $(n \geq 2)$ such that*

$$\tilde{\theta}_n := \sum_{\tau=1}^{n} w_\tau \theta_\tau \in [a, b] \text{ and } (\theta_\tau - \theta_0)(\tilde{\theta}_n - \theta_\tau) \geq 0 \text{ for } \tau = 1, ..., n. \tag{5}$$

If $\mathcal{K} : [a, b] \to \mathbb{R}$ is a convex function, then

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \frac{\sum_{\tau=1}^{n} w_\tau(\theta_\tau - \theta_0)}{\tilde{\theta}_n - \theta_0} \mathcal{K}(\tilde{\theta}_n) + \frac{\tilde{\theta}_n \left(\sum_{\tau=1}^{n} w_\tau - 1 \right)}{\tilde{\theta}_n - \theta_0} \mathcal{K}(\theta_0). \tag{6}$$

Iqbal et al. [21] introduced the Giaccardi inequality for (α, m)-convex function is given in the following theorem.

Theorem 2. *Assuming the conditions of Theorem 1 are satisfied. If a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ is an (α, m)-convex, where $\alpha, m \in (0, 1]$. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\theta_\tau - m\theta_0}{\tilde{\theta}_n - m\theta_0} \right)^{\alpha} \mathcal{K}(\tilde{\theta}_n) + m \sum_{\tau=1}^{n} w_\tau \left(1 - \left(\frac{\theta_\tau - m\theta_0}{\tilde{\theta}_n - m\theta_0} \right)^{\alpha} \right) \mathcal{K}(\theta_0). \tag{7}$$

Andrić and Pečarić [22] gave the Giaccardi inequality for (s, m)-convex function is given in the following theorem.

Theorem 3. *Let the conditions of Theorem 1 be satisfied. If a function $\mathcal{K} : [0,\infty) \to \mathbb{R}$ is an (s,m)-convex, where $s, m \in (0,1]$. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \mathcal{K}(\theta_0) \sum_{\tau=1}^{n} w_\tau \frac{(\tilde{\theta}_n - \theta_\tau)^s}{(\tilde{\theta}_n - \theta_0)^s} + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m}\right) \sum_{\tau=1}^{n} w_\tau \frac{(\theta_\tau - \theta_0)^s}{(\tilde{\theta}_n - \theta_0)^s}. \tag{8}$$

The Lah-Ribariĉ type inequality for m-convex function is given by Andrić and Pečarić in [22].

Theorem 4. *Let $\sigma : [0,b] \to \mathbb{R}$ be an integrable function with $0 \leq b < \infty$, $\mu \leq \sigma(\theta) \leq M$ for all $\theta \in [0,b]$, $\mu < M$, and let $w : [0,b] \to \mathbb{R}$ be a non-negative function. Let $\mathcal{K}$ be a non-negative m-convex function on $[0,\infty)$ such that $[\mu, M] \subseteq [0,\infty)$ with $m \in (0,1]$ and $\mathcal{K} \in L^1[0,b]$. Then*

$$\frac{1}{\int_0^b w(\theta)d\theta} \int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta \leq \frac{M-\bar{\sigma}}{M-\mu}\mathcal{K}(\mu) + m\frac{\bar{\sigma}-\mu}{M-\mu}\mathcal{K}\left(\frac{M}{m}\right), \tag{9}$$

where

$$\bar{\sigma} = \frac{\int_0^b w(\theta)\sigma(\theta)d\theta}{\int_0^b w(\theta)d\theta}. \tag{10}$$

This paper is organized as follows:

In second section, we give the refinement of Lah-Ribarič type inequality and find the Fejér type inequality for refined m-convex functions. Also, some special cases are discussed.

In third section, we define the Giaccardi and Petrović's inequalities for (α, m)-convex function. Also, we introduce the refinements of these inequalities with the help of refined (α, m)-convex function and other related functions.

In fourth section, we consider the special cases of refined $(\alpha, h-m)$-convex functions defined by Jung et al. in [23] and find the Giaccardi and Petrović's inequalities. Also, we find the refinements of these inequalities for (s,m)-convex functions and for other related function for different values of s and m.

2. Lah-Ribarič and Fejér type inequalities for refined m-convex functions

In the following theorem, we give the refinement of Lah-Ribarič type inequality with the help of refined m-convex function.

Theorem 5. *Let $\sigma : [0,b] \to \mathbb{R}$ be an integrable function with $0 \leq b < \infty$, $\mu \leq \sigma(\theta) \leq M$ for all $\theta \in [0,b]$, $\mu < M$, and let $w : [0,b] \to \mathbb{R}$ be a non-negative function. Let $\mathcal{K}$ be a non-negative refined m-convex function on $[0,\infty)$ such that $[\mu, M] \subseteq [0,\infty)$ with $m \in (0,1]$ and $\mathcal{K} \in L^1[0,b]$. Then*

$$\begin{aligned}\frac{1}{\int_0^b w(\theta)d\theta} \int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta &\leq \frac{\left(\mathcal{K}(\mu) + m\mathcal{K}\left(\frac{M}{m}\right)\right)}{(M-\mu)^2}((M+\mu)\bar{\sigma} - M\mu - \bar{\sigma}\sigma) \\ &\leq \frac{M-\bar{\sigma}}{M-\mu}\mathcal{K}(\mu) + m\frac{\bar{\sigma}-\mu}{M-\mu}\mathcal{K}\left(\frac{M}{m}\right),\end{aligned} \tag{11}$$

where $\bar{\sigma}$ is defined in (10).

Proof. One can note that $t(1-t)(\mathcal{K}(\mu) + m\mathcal{K}\left(\frac{M}{m}\right) \leq t\mathcal{K}(\mu) + m(1-t)\mathcal{K}\left(\frac{M}{m}\right)$, where $t \in (0,1)$ and $m \in (0,1]$. Since

$$\mathcal{K}(\theta) = \mathcal{K}\left(\frac{M-\theta}{M-\mu}\mu + m\frac{\theta-\mu}{M-\mu}\frac{M}{m}\right), \forall\theta \in [0,b].$$

Using the refined m-convexity of $\mathcal{K}$, one has

$$\mathcal{K}(\theta) \leq \left(\frac{M-\theta}{M-\mu}\right)\left(\frac{\theta-\mu}{M-\mu}\right)\left(\mathcal{K}(\mu) + m\mathcal{K}\left(\frac{M}{m}\right)\right) \leq \left(\frac{M-\theta}{M-\mu}\right)\mathcal{K}(\mu) + m\left(\frac{\theta-\mu}{M-\mu}\right)\mathcal{K}\left(\frac{M}{m}\right).$$

Replace θ with $\sigma(\theta)$, multiply the above inequality by $w(\theta)$ and integrate on $[0, b]$, then

$$\begin{aligned}\int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta &\leq \int_0^b w(\theta)\left(\frac{M-\sigma(\theta)}{M-\mu}\right)\left(\frac{\sigma(\theta)-\mu}{M-\mu}\right)\left(\mathcal{K}(\mu)+m\mathcal{K}\left(\frac{M}{m}\right)\right)d\theta \\ &\leq \int_0^b w(\theta)\left\{\left(\frac{M-\sigma(\theta)}{M-\mu}\right)\mathcal{K}(\mu)+m\left(\frac{\sigma(\theta)-\mu}{M-\mu}\right)\mathcal{K}\left(\frac{M}{m}\right)\right\}d\theta.\end{aligned}$$

This implies

$$\begin{aligned}\int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta &\leq \frac{\left(\mathcal{K}(\mu)+m\mathcal{K}\left(\frac{M}{m}\right)\right)}{(M-\mu)^2}\int_0^b w(\theta)(M-\sigma(\theta))(\sigma(\theta)-\mu)d\theta \\ &\leq \frac{1}{M-\mu}\mathcal{K}(\mu)\int_0^b w(\theta)(M-\sigma(\theta))d\theta \\ &\quad + \frac{m}{M-\mu}\mathcal{K}\left(\frac{M}{m}\right)\int_0^b w(\theta)(\sigma(\theta)-\mu)d\theta. \qquad (12)\end{aligned}$$

Now

$$\int_0^b w(\theta)(M-\sigma(\theta))(\sigma(\theta)-\mu)d\theta = (M+\mu)\int_0^b w(\theta)\sigma(\theta)d\theta - M\mu\int_0^b w(\theta)d\theta - \int_0^b w(\theta)\sigma^2(\theta)d\theta,$$

$$\int_0^b w(\theta)(M-\sigma(\theta))d\theta = M\int_0^b w(\theta)d\theta - \int_0^b w(\theta)(\sigma(\theta))d\theta,$$

and

$$\int_0^b w(\theta)(\sigma(\theta)-\mu)d\theta = \int_0^b w(\theta)(\sigma(\theta))d\theta - \mu\int_0^b w(\theta)d\theta.$$

Putting the above values in (12), one has

$$\begin{aligned}\int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta \leq &\frac{\left(\mathcal{K}(\mu)+m\mathcal{K}\left(\frac{M}{m}\right)\right)}{(M-\mu)^2}\left((M+\mu)\int_0^b w(\theta)\sigma(\theta)d\theta - M\mu\int_0^b w(\theta)d\theta - \int_0^b w(\theta)\sigma^2(\theta)d\theta\right) \\ \leq &\frac{M}{M-\mu}\mathcal{K}(\mu)\int_0^b w(\theta)d\theta - \frac{1}{M-\mu}\mathcal{K}(\mu)\int_0^b w(\theta)(\sigma(\theta))d\theta \\ &+ \frac{m}{M-\mu}\mathcal{K}\left(\frac{M}{m}\right)\int_0^b w(\theta)(\sigma(\theta))d\theta - \frac{m\mu}{M-\mu}\mathcal{K}\left(\frac{M}{m}\right)\int_0^b w(\theta)d\theta.\end{aligned}$$

Dividing the above inequality by $\int_0^b w(\theta)d\theta$ on both sides and using the notation $\bar{\sigma}$ given in (10), one gets the required result. □

With the help of refined convex function, the refinement of Lah-Ribarič type inequality is given in the following corollary, by taking $m = 1$ in (11).

Corollary 1. *Under the assumption of Theorem 5, let $\mathcal{K}$ be a non-negative refined convex function on $[0, b]$, then*

$$\begin{aligned}\frac{1}{\int_0^b w(\theta)d\theta}\int_0^b w(\theta)\mathcal{K}(\sigma(\theta))d\theta &\leq \frac{(\mathcal{K}(\mu)+\mathcal{K}(M))}{(M-\mu)^2}((M+\mu)\bar{\sigma} - M\mu - \bar{\sigma}\sigma) \\ &\leq \frac{M-\bar{\sigma}}{M-\mu}\mathcal{K}(\mu) + \frac{\bar{\sigma}-\mu}{M-\mu}\mathcal{K}(M),\end{aligned}$$

where $\bar{\sigma}$ is defined in (10).

Bracamonte et al. gave the Fejér type inequalities for m-convex functions in [24].

The Fejér-type inequality involving refined m-convex functions is given in the following theorem.

Theorem 6. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined m-convex, which is integrable in $[a, b] \subset [0, \infty)$ and let $g : [a, b] \to \mathbb{R}$ be a non-negative and integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$\int_a^b \mathcal{K}(\theta)g(\theta)d\theta \leq \frac{1}{2(b-a)^2}\left(\mathcal{K}(a) + \mathcal{K}(b) + m\left(\mathcal{K}\left(\frac{a}{m}\right) + \mathcal{K}\left(\frac{b}{m}\right)\right)\right)\int_a^b (b-\theta)(\theta-a)g(\theta)d\theta. \tag{13}$$

Proof. Due to the symmetry of g around $\frac{a+b}{2}$:

$$\begin{aligned}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta =& \frac{1}{2}\left[\int_a^b \mathcal{K}(\theta)g(\theta)d\theta + \int_a^b \mathcal{K}(a+b-\theta)g(a+b-\theta)d\theta\right]\\ =& \frac{1}{2}\int_a^b [\mathcal{K}(\theta) + \mathcal{K}(a+b-\theta)]g(\theta)d\theta\\ =& \frac{1}{2}\int_a^b \left[\mathcal{K}\left(a\left(\frac{b-\theta}{b-a}\right) + b\left(\frac{\theta-a}{b-a}\right)\right)\right.\\ &\left. + \mathcal{K}\left(a\left(\frac{\theta-a}{b-a}\right) + b\left(\frac{\theta-a}{b-a}\right)\right)\right] g(\theta)d\theta.\end{aligned}$$

Using the refined m-convexity of $\mathcal{K}$, one has

$$\begin{aligned}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta \leq& \frac{1}{2}\int_a^b \left\{\left(\frac{b-\theta}{b-a}\right)\left(\frac{\theta-a}{b-a}\right)\left(\mathcal{K}(a) + m\mathcal{K}\left(\frac{b}{m}\right)\right)\right.\\ &\left. + \left(\frac{b-\theta}{b-a}\right)\left(\frac{\theta-a}{b-a}\right)\left(m\mathcal{K}\left(\frac{a}{m}\right) + \mathcal{K}(b)\right)\right\} g(\theta)d\theta,\end{aligned}$$

which is equivalent to (13). □

The Fejér type inequality for refined convex functions is given in the following corollary, by setting $m = 1$ in (13).

Corollary 2. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined convex, which is integrable in $[a, b]$, where $a, b \in [0, \infty)$, $a < b$ and let $g : [a, b] \to \mathbb{R}$ be a non-negative and integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$\int_a^b \mathcal{K}(\theta)g(\theta)d\theta \leq \frac{(\mathcal{K}(a) + \mathcal{K}(b))}{(b-a)^2}\int_a^b (b-\theta)(\theta-a)g(\theta)d\theta. \tag{14}$$

Here, we introduce an important lemma that is essential for proving the upcoming results. This lemma provides valuable insights and plays a crucial role in establishing the logical connections and conclusions we want to demonstrate.

Lemma 1. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined m-convex, then for all $\theta \in [a, b] \subset [0, \infty)$, there is $\tau \in [0, 1]$ such that*

$$\mathcal{K}(a+b-\theta) \leq 2\tau(1-\tau)\left(\mathcal{K}(a) + m\mathcal{K}\left(\frac{b}{m}\right)\right) - \mathcal{K}(\theta), \text{ where } m \in (0, 1]. \tag{15}$$

Proof. Let $\theta = \tau a + (1-\tau)b$ and $a + b - \theta = (1-\tau)a + \tau b$, one has

$$\mathcal{K}(a+b-\theta) = \mathcal{K}((1-\tau)a + \tau b).$$

Using the refined m-convexity of $\mathcal{K}$, one has

$$\mathcal{K}(a+b-\theta) \leq \tau(1-\tau)\left(\mathcal{K}(a) + m\mathcal{K}\left(\frac{b}{m}\right)\right).$$

After addition and subtraction of $m\tau(1-\tau)\mathcal{K}(b)$ and $\tau(1-\tau)\mathcal{K}(a)$, one has

$$\begin{aligned}\mathcal{K}(a+b-\theta) &\leq 2\tau(1-\tau)\left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)-\tau(1-\tau)\left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)\\ &\leq 2\tau(1-\tau)\left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)-\mathcal{K}(\theta).\end{aligned}$$

□

Theorem 7. *Under the condition of Theorem 6, one has*

$$\int_a^b \mathcal{K}(\theta)g(\theta)d\theta \leq \left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)\int_a^b g(\theta)d\theta. \tag{16}$$

Proof. Since g is symmetric about $\frac{a+b}{2}$, one has

$$\begin{aligned}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta &= \frac{1}{2}\int_a^b \mathcal{K}(a+b-\theta)g(a+b-\theta)d\theta+\frac{1}{2}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta\\ &= \frac{1}{2}\int_a^b \mathcal{K}(a+b-\theta)g(\theta)d\theta+\frac{1}{2}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta.\end{aligned}$$

Using Lemma 1, one has

$$\begin{aligned}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta \leq &\frac{1}{2}\int_a^b \left[2\tau(1-\tau)\left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)-\mathcal{K}(\theta)\right]g(\theta)d\theta+\frac{1}{2}\int_a^b \mathcal{K}(\theta)g(\theta)d\theta\\ =&\int_a^b \left[\tau(1-\tau)\left(\mathcal{K}(a)+m\mathcal{K}\left(\frac{b}{m}\right)\right)\right]g(\theta)d\theta.\end{aligned}$$

After simplification and neglecting the negative terms, one gets (16). □

3. Giaccardi inequality for (α, m) and refined (α, m)-convex functions

Throughout the paper, we assume that $[0,b] \subseteq \mathbb{R}$ be an interval, $(\alpha,m),(s,m) \in (0,1)^2$, $\theta_0 \in [0,b]$, $(\theta_1,...,\theta_n) \in [0,b]^n$ and $(w_1,...,w_n) \in \mathbb{R}_+^n$ $(n \geq 2)$ such that

$$\tilde{\theta_n} := \sum_{\tau=1}^{n} w_\tau\theta_\tau \in [0,b] \text{ and } (\theta_\tau-\theta_0)(\tilde{\theta_n}-\theta_\tau) \geq 0 \text{ for } \tau=1,...,n. \tag{17}$$

In the following theorem, a Giaccardi inequality for (α,m)-convex functions is given. W. Iqbal and A. U. Rehman have also obtained a similar result in [21, Remark 2.3] through the utilization of an increasing function.

Theorem 8. *If a function $\mathcal{K}:[0,\infty)\to\mathbb{R}$ is an (α,m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau\mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^\alpha\mathcal{K}(\theta_0)+m\sum_{\tau=1}^{n} w_\tau\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^\alpha\right)\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right). \tag{18}$$

Proof. Since $\mathcal{K}$ is an (α,m)-convex function, so by taking $\tau=\frac{p}{p+q}$ and $1-\tau=\frac{q}{p+q}$ in (1), one has

$$\mathcal{K}\left(\frac{p}{p+q}v+m\frac{q}{p+q}w\right) \leq \left(\frac{p}{p+q}\right)^\alpha\mathcal{K}(\mu)+m\left(1-\left(\frac{p}{p+q}\right)^\alpha\right)\mathcal{K}(w). \tag{19}$$

Let $p=\tilde{\theta_n}-\theta_\tau$, $q=\theta_\tau-\theta_0$, $\theta=\theta_0$ and $w=\frac{\tilde{\theta_n}}{m}$, for $\tau=1,...,n$. Then

$$\mathcal{K}\left(\frac{p}{p+q}+m\frac{q}{p+q}w\right)=\mathcal{K}\left(\frac{(\tilde{\theta_n}-\theta_\tau)\theta_0+m(\theta_\tau-\theta_0)\frac{\tilde{\theta_n}}{m}}{\tilde{\theta_n}-\theta_\tau+\theta_\tau-\theta_0}\right)=\mathcal{K}(\theta_\tau),$$

and

$$\left(\frac{p}{p+q}\right)^{\alpha}\mathcal{K}(\mu)+m\left(1-\left(\frac{p}{p+q}\right)^{\alpha}\right)\mathcal{K}(w)=\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\mathcal{K}(\theta_0)+m\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\right)\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right).$$

Putting these values in (19), one has

$$\mathcal{K}(\theta_\tau)\leq\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\mathcal{K}(\theta_0)+m\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\right)\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right).$$

By multiplying the aforementioned inequality by p_τ and summing it from $\tau=1,...,n$, we obtain equation (18). □

For the particular values of α and m in Theorem 8, one has different results.

For $m=1$ in (18), one gets a Giaccardi inequality for $(\alpha,1)$-convex functions. A similar result has been obtained in [25, Corollary 12] by considering the increasing function.

Corollary 3. *If a function $\mathcal{K}:[0,\infty)\to\mathbb{R}$ is an $(\alpha,1)$-convex. Then*

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\sum_{\tau=1}^{n}w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\mathcal{K}(\theta_0)+\sum_{\tau=1}^{n}w_\tau\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)^{\alpha}\right)\mathcal{K}(\tilde{\theta_n}).$$

Remark 4. By taking $\alpha=1$ in (18), one gets the Giaccardi inequality for m-convex functions given by M. Andrić and J. E. Pečarić in [22] as:

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\sum_{\tau=1}^{n}w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)\mathcal{K}(\theta_0)+m\sum_{\tau=1}^{n}w_\tau\left(\frac{\theta_\tau-\theta_0}{\tilde{\theta_n}-\theta_0}\right)\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right).$$

Setting $\alpha=1=m$ in (18), one gets the Giaccardi inequality given in Theorem 1 (i.e [26, Theorem 1.1]):

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\sum_{\tau=1}^{n}w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}-\theta_0}\right)\mathcal{K}(\theta_0)+\sum_{\tau=1}^{n}w_\tau\left(\frac{\theta_\tau-\theta_0}{\tilde{\theta_n}-\theta_0}\right)\mathcal{K}(\tilde{\theta_n}).$$

Giaccardi inequality is the generalized form of the Petrović's inequality. In the following theorem, we give Petrović's inequality involving (α,m)-convex function.

Theorem 9. *Let a function $\mathcal{K}:[0,\infty)\to\mathbb{R}$ be a (α,m)-convex. Then*

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\sum_{\tau=1}^{n}w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}}\right)^{\alpha}\mathcal{K}(0)+m\sum_{\tau=1}^{n}w_\tau\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}}\right)^{\alpha}\right)\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right).\tag{20}$$

Proof. By taking $\theta_0=0$ in (7), one gets (20). □

One can get Petrović's inequality for $(\alpha,1)$-convex functions by taking $m=1$ in (20).

Corollary 4. *Let a function $\mathcal{K}:[0,\infty)\to\mathbb{R}$ be a $(\alpha,1)$-convex. Then*

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\sum_{\tau=1}^{n}w_\tau\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}}\right)^{\alpha}\mathcal{K}(0)+\sum_{\tau=1}^{n}w_\tau\left(1-\left(\frac{\tilde{\theta_n}-\theta_\tau}{\tilde{\theta_n}}\right)^{\alpha}\right)\mathcal{K}(\tilde{\theta_n}).$$

Some already published results can also be obtained for different values of α and m in Theorem 9.

Remark 5. For $\alpha=1=m$ in (20), one gets the inequality for convex functions given in [27, Theorem A] as:

$$\sum_{\tau=1}^{n}w_\tau\mathcal{K}(\theta_\tau)\leq\left(\sum_{\tau=1}^{n}w_\tau-1\right)\mathcal{K}(0)+\mathcal{K}(\tilde{\theta_n}).$$

By taking $\alpha = 1$ in (20), one gets the inequality for m-convex functions given in [22] as:

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \left(\sum_{\tau=1}^{n} w_\tau - 1 \right) \mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m} \right).$$

The Giaccardi inequality for refined (α, m)-convex functions is given in the following theorem.

Theorem 10. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined (α, m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right)^{\alpha} \left(1 - \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right)^{\alpha} \right) \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m} \right) \right). \tag{21}$$

Proof. The proof is similar to Theorem 8. □

By setting $m = 1$ in Theorem 10, one can get the Giaccardi inequality for refined $(\alpha, 1)$-convex functions.

Corollary 5. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined $(\alpha, 1)$-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right)^{\alpha} \left(1 - \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right)^{\alpha} \right) (\mathcal{K}(\theta_0) + \mathcal{K}(\tilde{\theta}_n)).$$

The Giaccardi inequality for refined m-convex functions can be obtained by taking $\alpha = 1$ in Theorem 10.

Corollary 6. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined m-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right) \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta}_n - \theta_0} \right) \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m} \right) \right).$$

To get a Giaccardi inequality for refined convex functions, take $\alpha = 1 = m$ in Theorem 10.

Corollary 7. *Let a function $\mathcal{K} : [0, b] \to \mathbb{R}$ be a refined convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n - \theta_0} \right) \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta}_n - \theta_0} \right) (\mathcal{K}(\theta_0) + \mathcal{K}(\tilde{\theta}_n)).$$

In the following theorem, the Petrović's inequality for refined (α, m)-convex functions is given.

Theorem 11. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined (α, m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n} \right)^{\alpha} \left(1 - \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n} \right)^{\alpha} \right) \left(\mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m} \right) \right). \tag{22}$$

Proof. By taking $\theta_0 = 0$ in (21), one gets (22). □

The Petrović's inequality for refined $(\alpha, 1)$-convex functions can be obtained by setting $m = 1$ in Theorem 11.

Corollary 8. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be a refined $(\alpha, 1)$-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n} \right)^{\alpha} \left(1 - \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n} \right)^{\alpha} \right) (\mathcal{K}(0) + \mathcal{K}(\tilde{\theta}_n)).$$

By setting $\alpha = 1$ in Theorem 11, one gets the Petrović's inequality for refined m-convex functions.

Corollary 9. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined m-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \left(\frac{\tilde{\theta}_n - \theta_\tau}{\tilde{\theta}_n} \right) \left(\mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta}_n}{m} \right) \right).$$

One can get the Petrović's inequality for refined convex functions by taking $\alpha = 1 = m$ in Theorem 11.

Corollary 10. *Let a function $\mathcal{K} : [0, b] \to \mathbb{R}$ be refined convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}} \right) (\mathcal{K}(0) + \mathcal{K}(\tilde{\theta_n}).$$

The refinement of Giaccardi inequality is given in the following theorem.

Theorem 12. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (α, m)-convex. Then*

$$\begin{aligned} \sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0} \right)^\alpha \left(1 - \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0} \right)^\alpha \right) \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m} \right) \right) \\ &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0} \right)^\alpha \mathcal{K}(\theta_0) + m \sum_{\tau=1}^{n} w_\tau \left(1 - \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0} \right)^\alpha \right) \mathcal{K}\left(\frac{\tilde{\theta_n}}{m} \right). \end{aligned} \tag{23}$$

Proof. Since

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \leq \tau^\alpha(1-\tau^\alpha)(\mathcal{K}(\mu) + m\mathcal{K}(w)) \leq \tau^\alpha \mathcal{K}(\mu) + m(1-\tau^\alpha)\mathcal{K}(w), \tag{24}$$
$$\forall \theta, w \in [0, \infty), \quad \alpha, m \text{ and } \tau \in (0, 1).$$

Let $\tau = \frac{p}{p+q}$ and $1 - \tau = \frac{q}{p+q}$. By substituting $p = \tilde{\theta_n} - \theta_\tau$, $q = \theta_\tau - \theta_0$, $\theta = \theta_0$ and $w = \frac{\tilde{\theta_n}}{m}$, for $\tau = 1, ..., n$ in (24), and follows the steps as done in Theorem 2, one gets (23). □

Refinement of Petrović's inequality is given in the following theorem.

Theorem 13. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (α, m)-convex. Then*

$$\begin{aligned} \sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}} \right)^\alpha \left(1 - \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}} \right)^\alpha \right) \left(\mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m} \right) \right) \\ &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}} \right)^\alpha \mathcal{K}(0) + m \sum_{\tau=1}^{n} w_\tau \left(1 - \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}} \right)^\alpha \right) \mathcal{K}\left(\frac{\tilde{\theta_n}}{m} \right). \end{aligned} \tag{25}$$

Proof. The required result can be obtained by taking $\theta_0 = 0$ in (23). □

Remark 6. Refinements of Giaccardi and Petrović's inequalities for other functions can be stated by choosing different value of α and m in (23) and (25) respectively. However, the details are omitted.

4. Giaccardi inequality for refined (s, m)-convex functions

A Giaccardi inequality for refined (s, m)-convex functions is given in the following theorem.

Theorem 14. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (s, m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0} \right)^s \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0} \right)^s \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m} \right) \right). \tag{26}$$

Proof. Take $\tau = \frac{p}{p+q}$ and $1 - \tau = \frac{q}{p+q}$ in (4), then

$$\mathcal{K}\left(\frac{p}{p+q} + m\frac{q}{p+q}w \right) \leq \left(\frac{p}{p+q} \right)^s \left(\frac{q}{p+q} \right)^s (\mathcal{K}(\theta) + m\mathcal{K}(w)). \tag{27}$$

Let $p = \tilde{\theta_n} - \theta_\tau$, $q = \theta_\tau - \theta_0$, $\theta = \theta_0$ and $w = \frac{\tilde{\theta_n}}{m}$, for $i = 1, ..., n$. Then one has

$$\mathcal{K}\left(\frac{p}{p+q} + m\frac{q}{p+q}w \right) = \mathcal{K}(\theta_\tau),$$

and

$$\left(\frac{p}{p+q}\right)^s \left(\frac{q}{p+q}\right)^s (\mathcal{K}(\theta) + m\mathcal{K}(w)) = \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0}\right)^s \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0}\right)^s \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right)\right).$$

Putting these values in (27), one has

$$\mathcal{K}(\theta_\tau) \leq \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0}\right)^s \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0}\right)^s \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right)\right). \tag{28}$$

By multiplying the aforementioned inequality by p_τ and summing it from $\tau = 1, ..., n$, one can obtain the required result. □

The Giaccardi inequality for refined s-convex functions can be obtained by taking $m = 1$ in Theorem 14.

Corollary 11. *Let a function $\mathcal{K} : [0, b] \to \mathbb{R}$ be refined s-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0}\right)^s \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0}\right)^s (\mathcal{K}(\theta_0) + \mathcal{K}(\tilde{\theta_n})).$$

For different values of s and m in Theorem 14, one gets different results.

Setting $s = 1$, yields the Giaccardi inequality for refined m-convex functions as presented in Corollary 6. Also, one can get the result given in Corollary 7, by setting $m = 1 = s$.

In the following theorem, the Petrović's inequality for refined (s, m)-convex function is given .

Theorem 15. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (s, m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}}\right)^s \left(\frac{\theta_\tau}{\tilde{\theta_n}}\right)^s \left(\mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right)\right). \tag{29}$$

Proof. One can get the required result by taking $\theta_0 = 0$ in (26). □

The Petrović's inequality for refined s-convex function can be obtained by taking $m = 1$ in Theroem 15.

Corollary 12. *Let a function $\mathcal{K} : [0, b] \to \mathbb{R}$ be refined s-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}}\right)^s \left(\frac{\theta_\tau}{\tilde{\theta_n}}\right)^s (\mathcal{K}(\theta_0) + \mathcal{K}(\tilde{\theta_n})).$$

For the particular values of s and m in Theroem 15, one has the results given in the previous section such as:

The result given in the Corollary 9 can be obtained by setting $s = 1$. Also, by taking $s = 1 = m$, one gets the Petrović's inequality for refined convex function given in the Corollary 10.

The refinement of Giaccardi inequality for (s, m)-convex function is given in the following theorem.

Theorem 16. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (s, m)-convex. Then*

$$\begin{aligned} \sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0}\right)^s \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0}\right)^s \left(\mathcal{K}(\theta_0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right)\right) \\ &\leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n} - \theta_0}\right)^s \mathcal{K}(\theta_0) + m \sum_{\tau=1}^{n} w_\tau \left(\frac{\theta_\tau - \theta_0}{\tilde{\theta_n} - \theta_0}\right)^s \mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right). \end{aligned} \tag{30}$$

Proof. As

$$\mathcal{K}(\tau\theta + m(1-\tau)w) \leq \tau^s(1-\tau)^s(\mathcal{K}(\theta) + m\mathcal{K}(w)) \leq \tau^s\mathcal{K}(\theta) + m(1-\tau)^s\mathcal{K}(w), \quad \forall \theta, w \in [0, \infty), \quad s, m \text{ and } \tau \in (0, 1). \tag{31}$$

By substituting $p = \tilde{\theta_n} - \theta_\tau$, $q = \theta_\tau - \theta_0$, $\theta = \theta_0$ and $w = \frac{\tilde{\theta_n}}{m}$, for $\tau = 1, ..., n$ in (31), and follows the steps as done in Theorem 8, one gets (30). □

The refinement of Petrović's inequality can be obtained by taking $\theta_0 = 0$ in (30) given in the following theorem.

Theorem 17. *Let a function $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be refined (s, m)-convex. Then*

$$\sum_{\tau=1}^{n} w_\tau \mathcal{K}(\theta_\tau) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}}\right)^s \left(\frac{\theta_\tau}{\tilde{\theta_n}}\right)^s \left(\mathcal{K}(0) + m\mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right)\right) \leq \sum_{\tau=1}^{n} w_\tau \left(\frac{\tilde{\theta_n} - \theta_\tau}{\tilde{\theta_n}}\right)^s \mathcal{K}(0) + m \sum_{\tau=1}^{n} w_\tau \left(\frac{\theta_\tau}{\tilde{\theta_n}}\right)^s \mathcal{K}\left(\frac{\tilde{\theta_n}}{m}\right). \tag{32}$$

Remark 7. Corresponding inequalities can be stated by choosing different value of α and m in (30) and (32) respectively. However, the details are omitted.

5. Conclusions

In this article, we derived refinements of the Lah-Ribarič, Giaccardi, and Petrovi'c inequalities by employing various classes of convex functions. Furthermore, we have established a Fejér type inequality with the help of refined m-convex functions, as well as extending Giaccardi and Petrovi'c's inequalities to (α, m)-convex functions and their variants.

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