

Article

Unifying Hardy-Hilbert-type integral inequalities

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Abstract: The Hardy-Hilbert integral inequality is a classic result in mathematical analysis that has inspired numerous generalizations and modifications. In this article, we present two comprehensive frameworks that unify many of these developments. Our approach introduces kernel functions that take into account both the maximum and the product of the variables, controlled by three independent, adjustable parameters. Although the kernels are primarily inhomogeneous and somewhat complicated, they offer much greater generality. We provide detailed proofs, together with thorough discussions and context within the wider mathematical literature. In addition, several intermediate integral results emerge naturally from our framework and may serve as useful tools for further exploration.

Keywords: Hardy-Hilbert-type integral inequalities, inhomogeneous kernel functions, beta function, integral norms

MSC: 26D15, 33E20

1. Introduction

This introductory section is divided into five subsections: (1) the context of the study, (2) the presentation of two theorems established by W.T. Sulaiman in [1], (3) the statement of a recent theorem by C. Chesneau in [2], (4) a summary of the main contributions, and (5) the overall structure of the article.

1.1. Context

The starting point of this article is a classical result of G.H. Hardy in [3]. It is known as the Hardy-Hilbert integral inequality (HHII). An exact statement of this inequality is given below. Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have

$$\iint_0^{+\infty} \frac{1}{x+y} f(x)g(y) dx dy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy \right]^{1/q},$$

where the integrals on the right-hand side must converge. The constant factor is shown to be optimal, and the inequality is strict for all non-zero functions. The kernel function in the double integral is explicitly given by

$$k_{ern}(x, y) = \frac{1}{x+y}.$$

For any $\lambda > 0$, we clearly have $k_{ern}(\lambda x, \lambda y) = \lambda^{-1} k_{ern}(x, y)$, meaning that $k_{ern}(x, y)$ is homogeneous of degree -1 . The HHII is a pillar of mathematical analysis, and plays a key role in the development of functional analysis, particularly in operator theory. As a result, it has garnered considerable attention, leading to numerous extensions, improvements and modifications. The following list of references gives a complete overview of the subject: [2,4–21].

The contributions of this article are based on three theorems concerning extended and modified forms of the HHII. As mentioned above, two of these theorems were established by W.T. Sulaiman in [1], and the third

by C. Chesneau in [2]. In this section, we present all three theorems in full, accompanied by discussion and comments.

1.2. Two theorems in [1]

As a preliminary step, we introduce a special function that plays a central role in many modifications of the HHII: the beta function. It is defined as

$$B_{\text{eta}}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt,$$

where $a, b > 0$. An equivalent representation, involving a different integration interval, is given by

$$B_{\text{eta}}(a, b) = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt. \quad (1)$$

Among its notable properties, there are $B_{\text{eta}}(a, b) = B_{\text{eta}}(b, a)$, $B_{\text{eta}}(a, 1) = 1/a$, $B_{\text{eta}}(a, 2) = 1/[a(a+1)]$, $B_{\text{eta}}(a, 3) = 2/[a(a+1)(a+2)]$, $B_{\text{eta}}(a, a) = 2^{1-2a} B_{\text{eta}}(1/2, a)$ and $B_{\text{eta}}(a, 1-a) = \pi/\sin(\pi a)$. Further properties and formulas can be found in [22]. In the context of integral inequalities, the beta function often appears in the expression of the optimal constant factors. This role is clearly emphasized in the book [4].

The first theorem of interest, [1, Theorem 1], can now be presented.

Theorem 1. [1, Theorem 1] Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\gamma > 0$, we set

$$\mathcal{A}(\gamma) = \iint_0^{+\infty} \frac{1}{[x+y+\max(x,y)]^\gamma} f(x)g(y) dx dy.$$

Then, depending on the values of γ , the inequalities below hold.

1. For any $\gamma > 1$, we have

$$\mathcal{A}(\gamma) \leq \frac{1}{\gamma-1} \left(\frac{1}{2^{\gamma-1}} - \frac{1}{2 \times 3^{\gamma-1}} \right) \left[\int_0^{+\infty} x^{1-\gamma} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{1-\gamma} g^q(y) dy \right]^{1/q}.$$

2. For $\gamma = 1$, we have

$$\mathcal{A}(\gamma) \leq 2\sqrt{2} \arctan \left[\frac{1}{\sqrt{2}} \right] \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}.$$

3. For any $\gamma \in (0, 1)$, we have

$$\mathcal{A}(\gamma) \leq \Lambda(\gamma) \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} g^q(y) dy \right]^{1/q},$$

where

$$\Lambda(\gamma) = \frac{1}{2^{\gamma/2}} \left[B_{\text{eta}} \left(\frac{\gamma}{2}, \frac{\gamma}{2} \right) - \int_{1/2}^2 \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right].$$

In each case, the integrals on the right-hand side must converge.

There are a few comments to be made about this result. First, Theorem 1 simplifies the original setting of [1, Theorem 1]; it replaces the two auxiliary functions with the variables x and y . Second, we correct a typo in the constant factor into $\Lambda(\gamma)$. The original version used the factor $1/\sqrt{2}$, whereas careful development yields $1/2^{\gamma/2}$, reflecting a more important effect of γ in $\Lambda(\gamma)$. This theorem also extends [23, Theorem 2.2], which only deals with the case $\gamma = 1$. Theorem 1 is therefore a one-parameter generalization of this result. It is of great interest, showing how the parameter γ affects the overall functional structure. We also note the emergence of the beta function at the particular values $\gamma/2$ and $\gamma/2$ for the case $\gamma \in (0, 1)$. As a side note, we mention that the kernel function associated with the double integral is given by

$$k_{ern}(x, y) = \frac{1}{[x + y + \max(x, y)]^\gamma}.$$

For any $\lambda > 0$ we clearly have $k_{ern}(\lambda x, \lambda y) = \lambda^{-\gamma} k_{ern}(x, y)$, which means that $k_{ern}(x, y)$ is homogeneous of degree $-\gamma$. This is the best known case in the context of modifications of the HHIII.

The next result of interest is [1, Theorem 2]. It can be described as a minimum-type version of [1, Theorem 1].

Theorem 2. [1, Theorem 2] Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\gamma > 0$, we set

$$\mathcal{B}(\gamma) = \iint_0^{+\infty} \frac{1}{[x + y + \min(x, y)]^\gamma} f(x)g(y) dx dy.$$

Then, depending on the values of γ , the inequalities below hold.

1. For any $\gamma > 1$, we have

$$\mathcal{B}(\gamma) \leq \frac{1}{2(\gamma-1)} (1 + 3^{1-\gamma}) \left[\int_0^{+\infty} x^{1-\gamma} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{1-\gamma} g^q(y) dy \right]^{1/q}.$$

2. For $\gamma = 1$, we have

$$\mathcal{B}(\gamma) \leq 2\sqrt{2} \arctan[\sqrt{2}] \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}.$$

3. For any $\gamma \in (0, 1)$, we have

$$\mathcal{B}(\gamma) \leq \Delta(\gamma) \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} g^q(y) dy \right]^{1/q},$$

where

$$\Delta(\gamma) = \frac{1}{2^{\gamma/2}} \left[B_{\text{eta}}\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) + \int_{1/2}^2 \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right].$$

In each case, the integrals on the right-hand side must converge.

Again, there are a few comments to make about this theorem. First, compared to [1, Theorem 1], the minimum has replaced the maximum. Second, Theorem 2 simplifies the original setup by choosing the auxiliary functions x and y . This mainly affects the constant factors, i.e., $\Delta(\gamma)$ has replaced $\Lambda(\lambda)$; the integral norms of f and g are unchanged. We have also corrected a typo in the definition of the constant $\Delta(\gamma)$. Specifically, the constant factor 1 from the original definition is now correctly written as $1/2^{\gamma/2}$. This shows

a greater importance of the parameter γ in the definition of $\Delta(\gamma)$. As a side note, we mention that the kernel function associated with the double integral is given by

$$k_{ern}(x, y) = \frac{1}{[x + y + \min(x, y)]^\gamma}.$$

For any $\lambda > 0$, we clearly have $k_{ern}(\lambda x, \lambda y) = \lambda^{-\gamma} k_{ern}(x, y)$, meaning that $k_{ern}(x, y)$ is homogeneous with the parameter $-\gamma$. To the best of our knowledge, [1, Theorem 2], is the first result exploring such a ratio-polynomial-minimum kernel function in the context of modifications of the HHIII.

1.3. One theorem in [2]

To complete our presentation, we now recall [2, Theorem 1], which has the property of dealing with an original kernel function that is mainly inhomogeneous (as discussed later).

Theorem 3. [2, Theorem 1] Let $p > 1$, $q = p/(p - 1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\alpha \geq 0$ and $\gamma > 0$, we set

$$\mathcal{C}(\alpha, \gamma) = \iint_0^{+\infty} \frac{1}{(x + y + \alpha xy)^\gamma} f(x)g(y) dx dy.$$

Then, depending on the values of γ , the inequalities below hold.

1. For any $\gamma > 1$, we have

$$\mathcal{C}(\alpha, \gamma) \leq \frac{1}{\gamma - 1} \left[\int_0^{+\infty} x^{1-\gamma} \frac{1}{1 + \alpha x} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{1-\gamma} \frac{1}{1 + \alpha y} g^q(y) dy \right]^{1/q}.$$

2. For $\gamma = 1$, we have

$$\mathcal{C}(\alpha, \gamma) \leq \pi \left[\int_0^{+\infty} x^{p/2-1} \frac{1}{\sqrt{1 + \alpha x}} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} \frac{1}{\sqrt{1 + \alpha y}} g^q(y) dy \right]^{1/q}.$$

3. For any $\gamma \in (0, 1)$, we have

$$\begin{aligned} \mathcal{C}(\alpha, \gamma) &\leq B_{eta} \left(\frac{\gamma}{2}, \frac{\gamma}{2} \right) \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} \frac{1}{(1 + \alpha x)^{\gamma/2}} f^p(x) dx \right]^{1/p} \\ &\times \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} \frac{1}{(1 + \alpha y)^{\gamma/2}} g^q(y) dy \right]^{1/q}. \end{aligned}$$

In each case, the integrals on the right-hand side must converge.

The main interest of this theorem lies in its complexity and flexibility. Note that the HHIII is covered by setting $\alpha = 0$, $\gamma = 1$ and $p = 2$. In full generality, the associated kernel function is given by

$$k_{ern}(x, y) = \frac{1}{(x + y + \alpha xy)^\gamma}.$$

Note that, for any $\alpha, \lambda > 0$, we cannot find a $\epsilon \in \mathbb{R}$ such that $k_{ern}(\lambda x, \lambda y) = \lambda^\epsilon k_{ern}(x, y)$, meaning that $k_{ern}(x, y)$ is inhomogeneous, unlike the kernel functions considered in [1, Theorems 1 and 2]. This inhomogeneity is mainly due to the product xy , which is determined by the parameter α . The structural complexity of this kernel function also explains the form of the unweighted integral norms of f and g used in the upper bounds, which differ significantly from those in [1, Theorems 1 and 2]. The counterpart of this

complexity is an interesting degree of flexibility, due to the presence of two adjustable parameters that can be adapted to different mathematical contexts. We also note that the constant factor has the beta function evaluated at $\gamma/2$ and $\gamma/2$ for the case $\gamma \in (0, 1)$.

1.4. Contributions

Motivated by the three theorems above, we propose two new results that significantly extend their scopes. The first theorem unifies the frameworks of [1, Theorem 1] and [2, Theorem 1], while the second theorem unifies those of [1, Theorem 2] and [2, Theorem 1]. This leads us to consider the following two kernel functions:

$$k_{ern}(x, y) = \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma}, \quad (2)$$

and

$$k_{ern}(x, y) = \frac{1}{[x + y + \alpha xy + \beta \min(x, y)]^\gamma}, \quad (3)$$

where $\alpha, \beta \geq 0$ and $\gamma > 0$ are parameters. If we take $\alpha = 0$ and $\beta = 1$, these kernel functions correspond to those in [1, Theorems 1 and 2]. If we take $\beta = 0$, then they reduce to the same expression, which is the one considered in [2, Theorem 1]. All other values for α and β lead to an unexplored case. We also emphasize the role of β in modulating the maximum or minimum of the variables, which is also a novelty; it is not possible to modulate them in [1, Theorems 1 and 2].

Our aim is therefore to derive sharp bounds on the double integrals associated with these kernel functions. They must logically capture the underlying complexity and generality. We follow the spirit of the classical HHII, along with the inequalities in [1, Theorems 1 and 2] and [2, Theorem 1]. In doing so, we contribute to the unification and flexibility of modifications of the HHII. The proofs rely on an appropriate decomposition of the integrands, the use of the Hölder integral inequality, new integral formulas, and standard mathematical techniques. All arguments are detailed; the article is completely self-contained. We also include discussions and comparisons between the new results, cite[Theorems 1 and 2]sulava and [2, Theorem 1].

1.5. Organization

The article is organized as follows: §2 presents some intermediate integral results. The first theorem is stated in §3, together with its detailed proof. §4 covers the second theorem in a similar way. Finally, §5 concludes the article with a summary and suggestions for future work.

2. Key integral results

2.1. Max-type integral formulas

The proposition below is an integral result that will be at the center of the proof of our first main theorem. It implicitly involves the kernel function presented in Eq. (2).

Proposition 1. For any $\alpha, \beta \geq 0$, $\gamma > 1$ and $x > 0$, we have

$$\int_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} dy = \frac{1}{(\gamma - 1)(1 + \alpha x)} \left[\frac{1}{(\beta + 1)^{\gamma-1}} - \frac{\beta}{(\beta + 1 + \alpha x)(\beta + 2 + \alpha x)^{\gamma-1}} \right] x^{1-\gamma}.$$

Proof. Using the definition of the maximum, the Chasles integral formula and the standard rules for primitives, we have

$$\begin{aligned} \int_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} dy &= \int_0^x \frac{1}{[(\beta + 1)x + (1 + \alpha x)y]^\gamma} dy + \int_x^{+\infty} \frac{1}{[x + (\beta + 1 + \alpha x)y]^\gamma} dy \\ &= \left[\frac{1}{1 - \gamma} \times \frac{1}{1 + \alpha x} \times \frac{1}{[(\beta + 1)x + (1 + \alpha x)y]^{\gamma-1}} \right]_{y=0}^{y=x} \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{1}{1-\gamma} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{[x+(\beta+1+\alpha x)y]^{\gamma-1}} \right]_{y=x}^{y \rightarrow +\infty} \\
& = \frac{1}{\gamma-1} \times \frac{1}{1+\alpha x} \times \left[\frac{1}{[(\beta+1)x]^{\gamma-1}} - \frac{1}{[(\beta+1)x+(1+\alpha x)x]^{\gamma-1}} \right] \\
& \quad + \frac{1}{\gamma-1} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{[x+(\beta+1+\alpha x)x]^{\gamma-1}} \\
& = \frac{1}{\gamma-1} \times \frac{1}{1+\alpha x} \times \frac{1}{x^{\gamma-1}} \times \frac{1}{(\beta+1)^{\gamma-1}} \\
& \quad - \frac{1}{\gamma-1} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{1+\alpha x} \times \frac{1}{x^{\gamma-1}} \times \frac{\beta}{(\beta+2+\alpha x)^{\gamma-1}} \\
& = \frac{1}{(\gamma-1)(1+\alpha x)} \left[\frac{1}{(\beta+1)^{\gamma-1}} - \frac{\beta}{(\beta+1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} \right] x^{1-\gamma}.
\end{aligned}$$

This concludes the proof of Proposition 1. $\square$

The proposition below is a modification of Proposition 1, which also considers the unexamined case $\gamma = 1$.

Proposition 2. For any $\alpha, \beta \geq 0$ and $x > 0$, we have

$$\begin{aligned}
& \int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy+\beta \max(x,y)} dy \\
& = 2 \left\{ \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] + \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha x}} \right] \right\} x^{-1/2}.
\end{aligned}$$

Proof. Using the definition of the maximum, the Chasles integral formula, the standard rules for primitives and the formula $\arctan(a) + \arctan(1/a) = \pi/2$ for $a > 0$, we have

$$\begin{aligned}
& \int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy+\beta \max(x,y)} dy = \int_0^x \frac{y^{-1/2}}{(\beta+1)x+(1+\alpha x)y} dy + \int_x^{+\infty} \frac{y^{-1/2}}{x+(\beta+1+\alpha x)y} dy \\
& = \left[\frac{2}{\sqrt{(\beta+1)(1+\alpha x)x}} \arctan \left[\sqrt{\frac{(1+\alpha x)y}{(\beta+1)x}} \right] \right]_{y=0}^{y=x} \\
& \quad + \left[\frac{2}{\sqrt{(\beta+1+\alpha x)x}} \arctan \left[\sqrt{\frac{(\beta+1+\alpha x)y}{x}} \right] \right]_{y=x}^{y \rightarrow +\infty} \\
& = \frac{2}{\sqrt{(\beta+1)(1+\alpha x)x}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] \\
& \quad + \frac{2}{\sqrt{(\beta+1+\alpha x)x}} \left\{ \frac{\pi}{2} - \arctan \left[\sqrt{\beta+1+\alpha x} \right] \right\} \\
& = 2 \left\{ \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] + \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha x}} \right] \right\} x^{-1/2}.
\end{aligned}$$

The proof of Proposition 2 is concluded. $\square$

A more general version than Propositions 1 and 2 is given below, in which a new parameter θ is introduced.

Proposition 3. For any $\alpha, \beta \geq 0$, $\gamma > 0$, $\theta \in \mathbb{R}$ and $x > 0$, we have

$$\int_0^{+\infty} \frac{y^\theta}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} dy$$

$$= \left\{ \frac{(\beta+1)^{1+\theta-\gamma}}{(1+\alpha x)^{\theta+1}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^\theta}{(1+t)^\gamma} dt + \frac{1}{(\beta+1+\alpha x)^{\theta+1}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^\theta}{(1+t)^\gamma} dt \right\} x^{1+\theta-\gamma},$$

where the integrals on the right-hand side must converge.

Proof. Using the definition of the maximum, the Chasles integral formula and the standard rules for primitives, we have

$$\begin{aligned} \int_0^{+\infty} \frac{y^\theta}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} dy &= \int_0^x \frac{y^\theta}{(x+y+\alpha xy+\beta x)^\gamma} dy + \int_x^{+\infty} \frac{y^\theta}{(x+y+\alpha xy+\beta y)^\gamma} dy \\ &= \frac{1}{[(\beta+1)x]^\gamma} \int_0^x \frac{y^\theta}{\{1+(1+\alpha x)y/[(\beta+1)x]\}^\gamma} dy + \frac{1}{x^\gamma} \int_x^{+\infty} \frac{y^\theta}{[1+(\beta+1+\alpha x)y/x]^\gamma} dy. \end{aligned}$$

Making the change of variables $y = u(\beta+1)x/(1+\alpha x)$ in the first integral, and $y = vx/(\beta+1+\alpha x)$ in the second, we obtain

$$\begin{aligned} &\frac{1}{[(\beta+1)x]^\gamma} \int_0^x \frac{y^\theta}{\{1+(1+\alpha x)y/[(\beta+1)x]\}^\gamma} dy + \frac{1}{x^\gamma} \int_x^{+\infty} \frac{y^\theta}{[1+(\beta+1+\alpha x)y/x]^\gamma} dy \\ &= \frac{1}{[(\beta+1)x]^\gamma} \left[\frac{(\beta+1)x}{1+\alpha x} \right]^{\theta+1} \int_0^{(1+\alpha x)/(\beta+1)} \frac{u^\theta}{(1+u)^\gamma} du + \frac{1}{x^\gamma} \left[\frac{x}{\beta+1+\alpha x} \right]^{\theta+1} \int_{\beta+1+\alpha x}^{+\infty} \frac{v^\theta}{(1+v)^\gamma} dv \\ &= \left\{ \frac{(\beta+1)^{1+\theta-\gamma}}{(1+\alpha x)^{\theta+1}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^\theta}{(1+t)^\gamma} dt + \frac{1}{(\beta+1+\alpha x)^{\theta+1}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^\theta}{(1+t)^\gamma} dt \right\} x^{1+\theta-\gamma}. \end{aligned}$$

This ends the proof of Proposition 3. $\square$

We can eventually introduce the beta function in the final expression under the form in Eq. (1), but sophisticated integral terms remain. We will use this method in more specific cases.

2.2. Min-type integral formulas

The proposition below is the analogue of Proposition 1, but with the minimum of the variables instead of the maximum. It will be at the center of the proof of our second main theorem. It implicitly includes the kernel function presented in Eq. (3).

Proposition 4. For any $\alpha, \beta \geq 0$, $\gamma > 1$ and $x > 0$, we have

$$\int_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dy = \frac{1}{(\gamma-1)(\beta+1+\alpha x)} \left[1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} \right] x^{1-\gamma}.$$

Proof. Using the definition of the minimum, the Chasles integral formula and the standard rules for primitives, we have

$$\begin{aligned} \int_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dy &= \int_0^x \frac{1}{[x+(\beta+1+\alpha x)y]^\gamma} dy + \int_x^{+\infty} \frac{1}{[(\beta+1)x+(1+\alpha x)y]^\gamma} dy \\ &= \left[\frac{1}{1-\gamma} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{[x+(\beta+1+\alpha x)y]^{\gamma-1}} \right]_{y=0}^{y=x} \\ &\quad + \left[\frac{1}{1-\gamma} \times \frac{1}{1+\alpha x} \times \frac{1}{[(\beta+1)x+(1+\alpha x)y]^{\gamma-1}} \right]_{y=x}^{y \rightarrow +\infty} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-\gamma} \times \frac{1}{\beta+1+\alpha x} \times \left[\frac{1}{[x+(\beta+1+\alpha x)x]^{\gamma-1}} - \frac{1}{x^{\gamma-1}} \right] \\
&\quad + \frac{1}{\gamma-1} \times \frac{1}{1+\alpha x} \times \frac{1}{[(\beta+1)x+(1+\alpha x)x]^{\gamma-1}} \\
&= \frac{1}{\gamma-1} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{x^{\gamma-1}} \\
&\quad + \frac{1}{\gamma-1} \times \frac{1}{1+\alpha x} \times \frac{1}{\beta+1+\alpha x} \times \frac{1}{x^{\gamma-1}} \times \frac{\beta}{(\beta+2+\alpha x)^{\gamma-1}} \\
&= \frac{1}{(\gamma-1)(\beta+1+\alpha x)} \left[1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} \right] x^{1-\gamma}.
\end{aligned}$$

This concludes the proof of Proposition 4. $\square$

The proposition below is the analogue of Proposition 2, but with the minimum of the variables instead of the maximum.

Proposition 5. For any $\alpha, \beta \geq 0$ and $x > 0$, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy+\beta \min(x,y)} dy \\
&= 2 \left\{ \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right] \right\} x^{-1/2}.
\end{aligned}$$

Proof. Using the definition of the minimum, the Chasles integral formula, the standard rules for primitives and the formula $\arctan(a) + \arctan(1/a) = \pi/2$ for $a > 0$, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy+\beta \min(x,y)} dy = \int_0^x \frac{y^{-1/2}}{x+(\beta+1+\alpha x)y} dy + \int_x^{+\infty} \frac{y^{-1/2}}{(\beta+1)x+(1+\alpha x)y} dy \\
&= \left[\frac{2}{\sqrt{(\beta+1+\alpha x)x}} \arctan \left[\sqrt{\frac{(\beta+1+\alpha x)y}{x}} \right] \right]_{y=0}^{y=x} \\
&\quad + \left[\frac{2}{\sqrt{(\beta+1)(1+\alpha x)x}} \arctan \left[\sqrt{\frac{(1+\alpha x)y}{(\beta+1)x}} \right] \right]_{y=x}^{y \rightarrow +\infty} \\
&= \frac{2}{\sqrt{(\beta+1+\alpha x)x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] \\
&\quad + \frac{2}{\sqrt{(\beta+1)(1+\alpha x)x}} \left\{ \frac{\pi}{2} - \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] \right\} \\
&= 2 \left\{ \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right] \right\} x^{-1/2}.
\end{aligned}$$

The proof of Proposition 5 is achieved. $\square$

The proposition below is the analogue of Proposition 3, but with the minimum of the variables instead of the maximum. Compared to Propositions 4 and 5, the adjustable parameter θ adds a new degree of flexibility.

Proposition 6. For any $\alpha, \beta \geq 0$, $\gamma > 0$, $\theta \in \mathbb{R}$ and $x > 0$, we have

$$\int_0^{+\infty} \frac{y^\theta}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dy$$

$$= \left\{ \frac{1}{(\beta+1+\alpha x)^{\theta+1}} \int_0^{\beta+1+\alpha x} \frac{t^\theta}{(1+t)^\gamma} dt + \frac{(\beta+1)^{1+\theta-\gamma}}{(1+\alpha x)^{\theta+1}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^\theta}{(1+t)^\gamma} dt \right\} x^{1+\theta-\gamma},$$

where the integrals on the right-hand side must converge.

Proof. Using the definition of the minimum, the Chasles integral formula and the standard rules for primitives, we have

$$\begin{aligned} \int_0^{+\infty} \frac{y^\theta}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dy &= \int_0^x \frac{y^\theta}{(x+y+\alpha xy+\beta y)^\gamma} dy + \int_x^{+\infty} \frac{y^\theta}{(x+y+\alpha xy+\beta x)^\gamma} dy \\ &= \frac{1}{x^\gamma} \int_0^x \frac{y^\theta}{[1+(\beta+1+\alpha x)y/x]^\gamma} dy + \frac{1}{[(\beta+1)x]^\gamma} \int_x^{+\infty} \frac{y^\theta}{\{1+(1+\alpha x)y/[(\beta+1)x]\}^\gamma} dy. \end{aligned}$$

Making the change of variables $y = ux/(\beta+1+\alpha x)$ in the first integral, and $y = v(\beta+1)x/(1+\alpha x)$ in the second, we obtain

$$\begin{aligned} &\frac{1}{x^\gamma} \int_0^x \frac{y^\theta}{[1+(\beta+1+\alpha x)y/x]^\gamma} dy + \frac{1}{[(\beta+1)x]^\gamma} \int_x^{+\infty} \frac{y^\theta}{\{1+(1+\alpha x)y/[(\beta+1)x]\}^\gamma} dy \\ &= \frac{1}{x^\gamma} \left[\frac{x}{\beta+1+\alpha x} \right]^{\theta+1} \int_0^{\beta+1+\alpha x} \frac{u^\theta}{(1+u)^\gamma} du + \frac{1}{[(\beta+1)x]^\gamma} \left[\frac{(\beta+1)x}{1+\alpha x} \right]^{\theta+1} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{v^\theta}{(1+v)^\gamma} dv \\ &= \left\{ \frac{1}{(\beta+1+\alpha x)^{\theta+1}} \int_0^{\beta+1+\alpha x} \frac{t^\theta}{(1+t)^\gamma} dt + \frac{(\beta+1)^{1+\theta-\gamma}}{(1+\alpha x)^{\theta+1}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^\theta}{(1+t)^\gamma} dt \right\} x^{1+\theta-\gamma}. \end{aligned}$$

The proof of Proposition 6 is complete. $\square$

As for Proposition 3, it is possible to introduce the beta function in the final expression under the form in Eq. (1), but sophisticated integral terms remain. We will use this method for more specific configurations.

We are now in a position to state our main theorems, starting with the first in the section below.

3. First theorem

3.1. Main statement

The theorem below proposes a unification of the frameworks of [1, Theorem 1] and [2, Theorem 1], as recalled in Subsections 1.2 and 1.3. It is based on the kernel function described in Eq. (2).

Theorem 4. Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\alpha, \beta \geq 0$ and $\gamma > 0$, we set

$$\mathfrak{A}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} f(x)g(y) dx dy.$$

Then, depending on the values of γ , the inequalities below hold.

1. For any $\gamma > 1$, we have

$$\mathfrak{A}(\alpha, \beta, \gamma) \leq \frac{1}{\gamma-1} \left[\int_0^{+\infty} \frac{1}{1+\alpha x} x^{1-\gamma} \Omega(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \times \left[\int_0^{+\infty} \frac{1}{1+\alpha y} y^{1-\gamma} \Omega(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q},$$

where

$$\Omega(x; \alpha, \beta, \gamma) = \frac{1}{(\beta + 1)^{\gamma-1}} - \frac{\beta}{(\beta + 1 + \alpha x)(\beta + 2 + \alpha x)^{\gamma-1}}.$$

2. For $\gamma = 1$, we have

$$\mathfrak{A}(\alpha, \beta, \gamma) \leq 2 \left[\int_0^{+\infty} x^{p/2-1} \Xi(x; \alpha, \beta) f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} \Xi(y; \alpha, \beta) g^q(y) dy \right]^{1/q},$$

where

$$\Xi(x; \alpha, \beta) = \frac{1}{\sqrt{(\beta + 1)(1 + \alpha x)}} \arctan \left[\sqrt{\frac{1 + \alpha x}{\beta + 1}} \right] + \frac{1}{\sqrt{\beta + 1 + \alpha x}} \arctan \left[\frac{1}{\sqrt{\beta + 1 + \alpha x}} \right].$$

3. For any $\gamma \in (0, 1)$, we have

$$\mathfrak{A}(\alpha, \beta, \gamma) \leq \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} Y(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \times \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} Y(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q},$$

where

$$Y(x; \alpha, \beta, \gamma) = \frac{1}{(\beta + 1)^{\gamma/2}(1 + \alpha x)^{\gamma/2}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt + \frac{1}{(\beta + 1 + \alpha x)^{\gamma/2}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt.$$

In each case, the integrals on the right-hand side must converge.

Proof of Theorem 4. Each of the three items is proved in turn below.

1. We assume that $\gamma > 1$. Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned} \mathfrak{A}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^{\gamma/p}} f(x) \times \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^{\gamma/q}} g(y) dx dy \\ &\leq \mathfrak{B}^{1/p}(\alpha, \beta, \gamma) \mathfrak{C}^{1/q}(\alpha, \beta, \gamma), \end{aligned} \quad (4)$$

where

$$\mathfrak{B}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} f^p(x) dx dy,$$

and

$$\mathfrak{C}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{B}(\alpha, \beta, \gamma)$ and $\mathfrak{C}(\alpha, \beta, \gamma)$ in turn.

For $\mathfrak{B}(\alpha, \beta, \gamma)$, using successively an exchange of the order of integration (by the Fubini-Tonelli integral theorem), Proposition 1 which arises naturally, and the definition of $\Omega(x; \alpha, \beta, \gamma)$, we have

$$\begin{aligned} \mathfrak{B}(\alpha, \beta, \gamma) &= \int_0^{+\infty} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} dy \right\} dx \\ &= \int_0^{+\infty} f^p(x) \left\{ \frac{1}{(\gamma - 1)(1 + \alpha x)} \left[\frac{1}{(\beta + 1)^{\gamma-1}} \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{\beta}{(\beta + 1 + \alpha x)(\beta + 2 + \alpha x)^{\gamma-1}} \Big] x^{1-\gamma} \Big\} dx \\
& = \frac{1}{\gamma - 1} \int_0^{+\infty} \frac{1}{1 + \alpha x} x^{1-\gamma} \Omega(x; \alpha, \beta, \gamma) f^p(x) dx.
\end{aligned} \tag{5}$$

Similarly, for $\mathfrak{E}(\alpha, \beta, \gamma)$, we have

$$\begin{aligned}
\mathfrak{E}(\alpha, \beta, \gamma) &= \int_0^{+\infty} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{[x + y + \alpha xy + \beta \max(x, y)]^\gamma} dx \right\} dy \\
&= \int_0^{+\infty} g^q(y) \left\{ \frac{1}{(\gamma - 1)(1 + \alpha y)} \left[\frac{1}{(\beta + 1)^{\gamma-1}} \right. \right. \\
&\quad \left. \left. - \frac{\beta}{(\beta + 1 + \alpha y)(\beta + 2 + \alpha y)^{\gamma-1}} \right] y^{1-\gamma} \right\} dy \\
&= \frac{1}{\gamma - 1} \int_0^{+\infty} \frac{1}{1 + \alpha y} y^{1-\gamma} \Omega(y; \alpha, \beta, \gamma) g^q(y) dy.
\end{aligned} \tag{6}$$

Joining Eqs. (4), (5) and (6) and using the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}
\mathfrak{A}(\alpha, \beta, \gamma) &\leq \left[\frac{1}{\gamma - 1} \int_0^{+\infty} \frac{1}{1 + \alpha x} x^{1-\gamma} \Omega(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \\
&\quad \times \left[\frac{1}{\gamma - 1} \int_0^{+\infty} \frac{1}{1 + \alpha y} y^{1-\gamma} \Omega(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q} \\
&= \frac{1}{\gamma - 1} \left[\int_0^{+\infty} \frac{1}{1 + \alpha x} x^{1-\gamma} \Omega(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \\
&\quad \times \left[\int_0^{+\infty} \frac{1}{1 + \alpha y} y^{1-\gamma} \Omega(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q}.
\end{aligned}$$

The stated inequality is achieved.

2. We assume that $\gamma = 1$, so that

$$\mathfrak{A}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{x + y + \alpha xy + \beta \max(x, y)} f(x) g(y) dx dy.$$

Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned}
\mathfrak{A}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{x^{1/(2q)} y^{-1/(2p)}}{[x + y + \alpha xy + \beta \max(x, y)]^{1/p}} f(x) \times \frac{x^{-1/(2q)} y^{1/(2p)}}{[x + y + \alpha xy + \beta \max(x, y)]^{1/q}} g(y) dx dy \\
&\leq \mathfrak{D}^{1/p}(\alpha, \beta) \mathfrak{E}^{1/q}(\alpha, \beta),
\end{aligned} \tag{7}$$

where

$$\mathfrak{D}(\alpha, \beta) = \iint_0^{+\infty} \frac{x^{p/(2q)} y^{-1/2}}{x + y + \alpha xy + \beta \max(x, y)} f^p(x) dx dy,$$

and

$$\mathfrak{E}(\alpha, \beta) = \iint_0^{+\infty} \frac{x^{-1/2} y^{q/(2p)}}{x + y + \alpha xy + \beta \max(x, y)} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{D}(\alpha, \beta)$ and $\mathfrak{E}(\alpha, \beta)$ in turn.

For $\mathfrak{D}(\alpha, \beta)$, using successively an exchange of the order of integration, the identity $p/q = p - 1$, Proposition 2 which arises naturally, and the definition of $\Xi(x; \alpha, \beta)$, we have

$$\begin{aligned} \mathfrak{D}(\alpha, \beta) &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{y^{-1/2}}{x + y + \alpha xy + \beta \max(x, y)} dy \right] dx \\ &= \int_0^{+\infty} x^{p/2-1/2} f^p(x) \left[2 \left\{ \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha x}} \right] \right\} x^{-1/2} \right] dx \\ &= 2 \int_0^{+\infty} x^{p/2-1} f^p(x) \Xi(x; \alpha, \beta) dx. \end{aligned} \quad (8)$$

Similarly, for $\mathfrak{E}(\alpha, \beta)$, we have

$$\begin{aligned} \mathfrak{E}(\alpha, \beta) &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{x^{-1/2}}{x + y + \alpha xy + \beta \max(x, y)} dx \right] dy \\ &= \int_0^{+\infty} y^{q/2-1/2} g^q(y) \left[2 \left\{ \frac{1}{\sqrt{(\beta+1)(1+\alpha y)}} \arctan \left[\sqrt{\frac{1+\alpha y}{\beta+1}} \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{\beta+1+\alpha y}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha y}} \right] \right\} y^{-1/2} \right] dy \\ &= 2 \int_0^{+\infty} y^{q/2-1} g^q(y) \Xi(y; \alpha, \beta) dy. \end{aligned} \quad (9)$$

Joining Eqs. (7), (8) and (9) and using the identity $1/p + 1/q = 1$, we get

$$\begin{aligned} \mathfrak{A}(\alpha, \beta, \gamma) &\leq \left[2 \int_0^{+\infty} x^{p/2-1} f^p(x) \Xi(x; \alpha, \beta) dx \right]^{1/p} \left[2 \int_0^{+\infty} y^{q/2-1} g^q(y) \Xi(y; \alpha, \beta) dy \right]^{1/q} \\ &= 2 \left[\int_0^{+\infty} x^{p/2-1} f^p(x) \Xi(x; \alpha, \beta) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) \Xi(y; \alpha, \beta) dy \right]^{1/q}. \end{aligned}$$

The stated inequality is achieved.

3. We assume that $\gamma \in (0, 1)$. Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned} \mathfrak{A}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{x^{-(\gamma/2-1)/q} y^{(\gamma/2-1)/p}}{[x + y + \alpha xy + \beta \max(x, y)]^{\gamma/p}} f(x) \times \frac{x^{(\gamma/2-1)/q} y^{-(\gamma/2-1)/p}}{[x + y + \alpha xy + \beta \max(x, y)]^{\gamma/q}} g(y) dx dy \\ &\leq \mathfrak{F}^{1/p}(\alpha, \beta, \gamma) \mathfrak{G}^{1/q}(\alpha, \beta, \gamma), \end{aligned} \quad (10)$$

where

$$\mathfrak{F}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{x^{-(\gamma/2-1)p/q} y^{\gamma/2-1}}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} f^p(x) dx dy,$$

and

$$\mathfrak{G}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{x^{\gamma/2-1} y^{-(\gamma/2-1)q/p}}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{F}(\alpha, \beta, \gamma)$ and $\mathfrak{G}(\alpha, \beta, \gamma)$ in turn.

For $\mathfrak{F}(\alpha, \beta, \gamma)$, using successively an exchange of the order of integration, Proposition 3 with $\theta = \gamma/2 - 1$ which arises naturally, and the definition of $Y(x; \alpha, \beta, \gamma)$, we have

$$\begin{aligned} \mathfrak{F}(\alpha, \beta, \gamma) &= \int_0^{+\infty} x^{-(\gamma/2-1)p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{y^{\gamma/2-1}}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} dy \right\} dx \\ &= \int_0^{+\infty} x^{-(\gamma/2-1)(p-1)} f^p(x) \left[\left\{ \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right. \right. \\ &\quad \left. \left. + \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right\} x^{-\gamma/2} \right] dx \\ &= \int_0^{+\infty} x^{(1-\gamma/2)p-1} Y(x; \alpha, \beta, \gamma) f^p(x) dx. \end{aligned} \quad (11)$$

Similarly, for $\mathfrak{G}(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} \mathfrak{G}(\alpha, \beta, \gamma) &= \int_0^{+\infty} y^{-(\gamma/2-1)q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{x^{\gamma/2-1}}{[x+y+\alpha xy+\beta \max(x,y)]^\gamma} dx \right\} dy \\ &= \int_0^{+\infty} y^{-(\gamma/2-1)(q-1)} g^q(y) \left[\left\{ \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha y)^{\gamma/2}} \int_0^{(1+\alpha y)/(\beta+1)} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right. \right. \\ &\quad \left. \left. + \frac{1}{(\beta+1+\alpha y)^{\gamma/2}} \int_{\beta+1+\alpha y}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right\} y^{-\gamma/2} \right] dy \\ &= \int_0^{+\infty} y^{(1-\gamma/2)q-1} Y(y; \alpha, \beta, \gamma) g^q(y) dy. \end{aligned} \quad (12)$$

Joining Eqs. (10), (11) and (12) and using the identity $1/p + 1/q = 1$, we get

$$\mathfrak{A}(\alpha, \beta, \gamma) \leq \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} Y(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \times \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} Y(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q}.$$

The stated inequality is achieved.

This concludes the proof of Theorem 4. $\square$

3.2. Discussion

Let us first discuss the new theorem in the light of [1, Theorem 1], as recalled in §1.2. In the special case of $\alpha = 0$ and $\beta = 1$, the remarks below apply.

1. For any $\gamma > 1$, we have

$$\Omega(x; \alpha, \beta, \gamma) = \frac{1}{(\beta+1)^{\gamma-1}} - \frac{\beta}{(\beta+1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} = \frac{1}{2^{\gamma-1}} - \frac{1}{2 \times 3^{\gamma-1}}.$$

Using it, the corresponding upper bound obtained is the one in [1, Theorem 1].

2. For $\gamma = 1$, we have

$$\begin{aligned} \Xi(x; \alpha, \beta) &= \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] + \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha x}} \right] \\ &= \frac{1}{\sqrt{2}} \arctan \left[\frac{1}{\sqrt{2}} \right] + \frac{1}{\sqrt{2}} \arctan \left[\frac{1}{\sqrt{2}} \right] = \sqrt{2} \arctan \left[\frac{1}{\sqrt{2}} \right]. \end{aligned}$$

As expected, the corresponding upper bound obtained is the one in [1, Theorem 1].

3. For any $\gamma \in (0, 1)$, using the Chasles integral formula, we have

$$\begin{aligned} Y(x; \alpha, \beta, \gamma) &= \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \\ &= \frac{1}{2^{\gamma/2}} \int_0^{1/2} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{2^{\gamma/2}} \int_{1/2}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \\ &= \frac{1}{2^{\gamma/2}} \left[\text{Beta} \left(\frac{\gamma}{2}, \frac{\gamma}{2} \right) - \int_{1/2}^2 \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \right]. \end{aligned}$$

Using it, the corresponding upper bound obtained is the one in [1, Theorem 1].

Let us now discuss the new theorem in the light of [2, Theorem 1], as recalled in §1.3. In the special case of $\beta = 0$, the remarks below apply.

1. For any $\gamma > 1$, we have

$$\Omega(x; \alpha, \beta, \gamma) = \frac{1}{(\beta+1)^{\gamma-1}} - \frac{\beta}{(\beta+1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} = 1 - 0 = 1.$$

As expected, the corresponding upper bound obtained is the one in [2, Theorem 1].

2. For $\gamma = 1$, using the formula $\arctan(a) + \arctan(1/a) = \pi/2$ for $a > 0$, we have

$$\begin{aligned} \Xi(x; \alpha, \beta) &= \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{1+\alpha x}{\beta+1}} \right] + \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\frac{1}{\sqrt{\beta+1+\alpha x}} \right] \\ &= \frac{1}{\sqrt{1+\alpha x}} \arctan \left[\sqrt{1+\alpha x} \right] + \frac{1}{\sqrt{1+\alpha x}} \arctan \left[\frac{1}{\sqrt{1+\alpha x}} \right] \\ &= \frac{1}{\sqrt{1+\alpha x}} \times \frac{\pi}{2}. \end{aligned}$$

Using it, the corresponding upper bound obtained is the one in [2, Theorem 1].

3. For any $\gamma \in (0, 1)$, using the Chasles integral formula, we have

$$\begin{aligned} Y(x; \alpha, \beta, \gamma) &= \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_0^{(1+\alpha x)/(\beta+1)} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_{\beta+1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \\ &= \frac{1}{(1+\alpha x)^{\gamma/2}} \int_0^{1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{(1+\alpha x)^{\gamma/2}} \int_{1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \end{aligned}$$

$$= \frac{1}{(1+\alpha x)^{\gamma/2}} B_{\text{eta}}\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right).$$

As expected, the corresponding upper bound obtained is the one in [2, Theorem 1].

Theorem 4 thus succeeds in proposing a unification of [1, Theorem 1] and [2, Theorem 1]. The undiscussed values of α and β lead to new configurations that can be used in functional analysis and in operator theory in particular.

4. Second theorem

4.1. Main statement

The theorem below proposes a unification of the frameworks of [1, Theorem 2] and [2, Theorem 1], as recalled in Subsections 1.2 and 1.3. It is based on the kernel function described in Eq. (3).

Theorem 5. Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\alpha, \beta \geq 0$ and $\gamma > 0$, we set

$$\mathfrak{H}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x, y)]^\gamma} f(x)g(y) dx dy.$$

Then, depending on the values of γ , the inequalities below hold.

1. For any $\gamma > 1$, we have

$$\begin{aligned} \mathfrak{H}(\alpha, \beta, \gamma) &\leq \frac{1}{\gamma-1} \left[\int_0^{+\infty} \frac{1}{\beta+1+\alpha x} x^{1-\gamma} \Phi(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \\ &\quad \times \left[\int_0^{+\infty} \frac{1}{\beta+1+\alpha y} y^{1-\gamma} \Phi(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q}, \end{aligned}$$

where

$$\Phi(x; \alpha, \beta, \gamma) = 1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}}.$$

2. For $\gamma = 1$, we have

$$\mathfrak{H}(\alpha, \beta, \gamma) \leq 2 \left[\int_0^{+\infty} x^{p/2-1} \Psi(x; \alpha, \beta) f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} \Psi(y; \alpha, \beta) g^q(y) dy \right]^{1/q},$$

where

$$\Psi(x; \alpha, \beta) = \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right].$$

3. For any $\gamma \in (0, 1)$, we have

$$\mathfrak{H}(\alpha, \beta, \gamma) \leq \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} \mathfrak{N}(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \times \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} \mathfrak{N}(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q},$$

where

$$\aleph(x; \alpha, \beta, \gamma) = \frac{1}{(\beta + 1 + \alpha x)^{\gamma/2}} \int_0^{\beta+1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt + \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt.$$

In each case, the integrals on the right-hand side must converge.

Proof of Theorem 5. Each of the three items is proved in turn below.

1. We assume that $\gamma > 1$. Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned} \mathfrak{H}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma/p}} f(x) \times \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma/q}} g(y) dx dy \\ &\leq \mathfrak{I}^{1/p}(\alpha, \beta, \gamma) \mathfrak{J}^{1/q}(\alpha, \beta, \gamma), \end{aligned} \quad (13)$$

where

$$\mathfrak{I}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} f^p(x) dx dy,$$

and

$$\mathfrak{J}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{I}(\alpha, \beta, \gamma)$ and $\mathfrak{J}(\alpha, \beta, \gamma)$ in turn.

For $\mathfrak{I}(\alpha, \beta, \gamma)$, using successively an exchange of the order of integration, Proposition 4 which arises naturally, and the definition of $\Phi(x; \alpha, \beta, \gamma)$, we have

$$\begin{aligned} \mathfrak{I}(\alpha, \beta, \gamma) &= \int_0^{+\infty} f^p(x) \left\{ \int_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dy \right\} dx \\ &= \int_0^{+\infty} f^p(x) \left\{ \frac{1}{(\gamma-1)(\beta+1+\alpha x)} \left[1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} \right] x^{1-\gamma} \right\} dx \\ &= \frac{1}{\gamma-1} \int_0^{+\infty} \frac{1}{\beta+1+\alpha x} x^{1-\gamma} \Phi(x; \alpha, \beta, \gamma) f^p(x) dx. \end{aligned} \quad (14)$$

Similarly, for $\mathfrak{J}(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} \mathfrak{J}(\alpha, \beta, \gamma) &= \int_0^{+\infty} g^q(y) \left\{ \int_0^{+\infty} \frac{1}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dx \right\} dy \\ &= \int_0^{+\infty} g^q(y) \left\{ \frac{1}{(\gamma-1)(\beta+1+\alpha y)} \left[1 + \frac{\beta}{(1+\alpha y)(\beta+2+\alpha y)^{\gamma-1}} \right] y^{1-\gamma} \right\} dy \\ &= \frac{1}{\gamma-1} \int_0^{+\infty} \frac{1}{\beta+1+\alpha y} y^{1-\gamma} \Phi(y; \alpha, \beta, \gamma) g^q(y) dy. \end{aligned} \quad (15)$$

Joining Eqs. (13), (14) and (15) and using the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}\mathfrak{H}(\alpha, \beta, \gamma) &\leq \left[\frac{1}{\gamma-1} \int_0^{+\infty} \frac{1}{\beta+1+\alpha x} x^{1-\gamma} \Phi(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \\ &\quad \times \left[\frac{1}{\gamma-1} \int_0^{+\infty} \frac{1}{\beta+1+\alpha y} y^{1-\gamma} \Phi(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q} \\ &= \frac{1}{\gamma-1} \left[\int_0^{+\infty} \frac{1}{\beta+1+\alpha x} x^{1-\gamma} \Phi(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \\ &\quad \times \left[\int_0^{+\infty} \frac{1}{\beta+1+\alpha y} y^{1-\gamma} \Phi(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q}.\end{aligned}$$

The stated inequality is achieved.

2. We assume that $\gamma = 1$, so that

$$\mathfrak{H}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{1}{x+y+\alpha xy+\beta \min(x,y)} f(x)g(y) dx dy.$$

Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned}\mathfrak{H}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{x^{1/(2q)} y^{-1/(2p)}}{[x+y+\alpha xy+\beta \min(x,y)]^{1/p}} f(x) \times \frac{x^{-1/(2q)} y^{1/(2p)}}{[x+y+\alpha xy+\beta \min(x,y)]^{1/q}} g(y) dx dy \\ &\leq \mathfrak{K}^{1/p}(\alpha, \beta) \mathfrak{L}^{1/q}(\alpha, \beta),\end{aligned}\tag{16}$$

where

$$\mathfrak{K}(\alpha, \beta) = \iint_0^{+\infty} \frac{x^{p/(2q)} y^{-1/2}}{x+y+\alpha xy+\beta \min(x,y)} f^p(x) dx dy$$

and

$$\mathfrak{L}(\alpha, \beta) = \iint_0^{+\infty} \frac{x^{-1/2} y^{q/(2p)}}{x+y+\alpha xy+\beta \min(x,y)} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{K}(\alpha, \beta)$ and $\mathfrak{L}(\alpha, \beta)$ in turn.

For $\mathfrak{K}(\alpha, \beta)$, using successively an exchange of the order of integration, the identity $p/q = p-1$, Proposition 5 which arises naturally, and the definition of $\Psi(x; \alpha, \beta)$, we have

$$\begin{aligned}\mathfrak{K}(\alpha, \beta) &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy+\beta \min(x,y)} dy \right] dx \\ &= \int_0^{+\infty} x^{p/2-1/2} f^p(x) \left[2 \left\{ \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right] \right\} x^{-1/2} \right] dx \\ &= 2 \int_0^{+\infty} x^{p/2-1} f^p(x) \Psi(x; \alpha, \beta) dx.\end{aligned}\tag{17}$$

Similarly, for $\mathfrak{L}(\alpha, \beta)$, we have

$$\begin{aligned}\mathfrak{L}(\alpha, \beta) &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{x^{-1/2}}{x+y+\alpha xy+\beta \min(x,y)} dx \right] dy \\ &= \int_0^{+\infty} y^{q/2-1/2} g^q(y) \left[2 \left\{ \frac{1}{\sqrt{\beta+1+\alpha y}} \arctan \left[\sqrt{\beta+1+\alpha y} \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{\sqrt{(\beta+1)(1+\alpha y)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha y}} \right] \right\} y^{-1/2} \right] dy \\ &= 2 \int_0^{+\infty} y^{q/2-1} g^q(y) \Psi(y; \alpha, \beta) dy.\end{aligned}\quad (18)$$

Joining Eqs. (16), (17) and (18) and using the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}\mathfrak{H}(\alpha, \beta, \gamma) &\leq \left[2 \int_0^{+\infty} x^{p/2-1} f^p(x) \Psi(x; \alpha, \beta) dx \right]^{1/p} \left[2 \int_0^{+\infty} y^{q/2-1} g^q(y) \Psi(y; \alpha, \beta) dy \right]^{1/q} \\ &= 2 \left[\int_0^{+\infty} x^{p/2-1} f^p(x) \Psi(x; \alpha, \beta) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) \Psi(y; \alpha, \beta) dy \right]^{1/q}.\end{aligned}$$

The stated inequality is achieved.

3. We assume that $\gamma \in (0, 1)$. Based on the identity $1/p + 1/q = 1$, an appropriate decomposition of the integrand and the Hölder integral inequality at p and q , we get

$$\begin{aligned}\mathfrak{H}(\alpha, \beta, \gamma) &= \iint_0^{+\infty} \frac{x^{-(\gamma/2-1)/q} y^{(\gamma/2-1)/p}}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma/p}} f(x) \times \frac{x^{(\gamma/2-1)/q} y^{-(\gamma/2-1)/p}}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma/q}} g(y) dx dy \\ &\leq \mathfrak{M}^{1/p}(\alpha, \beta, \gamma) \mathfrak{N}^{1/q}(\alpha, \beta, \gamma),\end{aligned}\quad (19)$$

where

$$\mathfrak{M}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{x^{-(\gamma/2-1)p/q} y^{\gamma/2-1}}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma}} f^p(x) dx dy,$$

and

$$\mathfrak{N}(\alpha, \beta, \gamma) = \iint_0^{+\infty} \frac{x^{\gamma/2-1} y^{-(\gamma/2-1)q/p}}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma}} g^q(y) dx dy.$$

Let us examine the integrals $\mathfrak{M}(\alpha, \beta, \gamma)$ and $\mathfrak{N}(\alpha, \beta, \gamma)$ in turn.

For $\mathfrak{M}(\alpha, \beta, \gamma)$, using successively an exchange of the order of integration, Proposition 6 with $\theta = \gamma/2 - 1$ which arises naturally, and the definition of $\mathfrak{N}(x; \alpha, \beta, \gamma)$, we have

$$\begin{aligned}\mathfrak{M}(\alpha, \beta, \gamma) &= \int_0^{+\infty} x^{-(\gamma/2-1)p/q} f^p(x) \left\{ \int_0^{+\infty} \frac{y^{\gamma/2-1}}{[x+y+\alpha xy+\beta \min(x,y)]^{\gamma}} dy \right\} dx \\ &= \int_0^{+\infty} x^{-(\gamma/2-1)(p-1)} f^p(x) \left[\left\{ \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_0^{\beta+1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \right. \right. \\ &\quad \left. \left. + \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \right\} x^{-\gamma/2} \right] dx\end{aligned}$$

$$= \int_0^{+\infty} x^{(1-\gamma/2)p-1} \aleph(x; \alpha, \beta, \gamma) f^p(x) dx. \quad (20)$$

Similarly, for $\aleph(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} \aleph(\alpha, \beta, \gamma) &= \int_0^{+\infty} y^{-(\gamma/2-1)q/p} g^q(y) \left\{ \int_0^{+\infty} \frac{x^{\gamma/2-1}}{[x+y+\alpha xy+\beta \min(x,y)]^\gamma} dx \right\} dy \\ &= \int_0^{+\infty} y^{-(\gamma/2-1)(q-1)} g^q(y) \left[\left\{ \frac{1}{(\beta+1+\alpha y)^{\gamma/2}} \int_0^{\beta+1+\alpha y} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right. \right. \\ &\quad \left. \left. + \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha y)^{\gamma/2}} \int_{(1+\alpha y)/(\beta+1)}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \right\} y^{-\gamma/2} \right] dy \\ &= \int_0^{+\infty} y^{(1-\gamma/2)q-1} \aleph(y; \alpha, \beta, \gamma) g^q(y) dy. \end{aligned} \quad (21)$$

Joining Eqs. (19), (20) and (21) and using the identity $1/p + 1/q = 1$, we get

$$\aleph(\alpha, \beta, \gamma) \leq \left[\int_0^{+\infty} x^{(1-\gamma/2)p-1} \aleph(x; \alpha, \beta, \gamma) f^p(x) dx \right]^{1/p} \times \left[\int_0^{+\infty} y^{(1-\gamma/2)q-1} \aleph(y; \alpha, \beta, \gamma) g^q(y) dy \right]^{1/q}.$$

The stated inequality is achieved.

This concludes the proof of Theorem 5. $\square$

4.2. Discussion

Let us first discuss the new theorem in the light of [1, Theorem 2], as recalled in §1.2. In the special case of $\alpha = 0$ and $\beta = 1$, the remarks below apply.

1. For any $\gamma > 1$, we have

$$\Phi(x; \alpha, \beta, \gamma) = 1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} = 1 + \frac{1}{3^{\gamma-1}}.$$

Using it, the corresponding upper bound obtained is the one in [1, Theorem 2].

2. For $\gamma = 1$, we have

$$\begin{aligned} \Psi(x; \alpha, \beta) &= \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right] \\ &= \frac{1}{\sqrt{2}} \arctan \left[\sqrt{2} \right] + \frac{1}{\sqrt{2}} \arctan \left[\sqrt{2} \right] = \sqrt{2} \arctan \left[\sqrt{2} \right]. \end{aligned}$$

As expected, the corresponding upper bound obtained is the one in [1, Theorem 2].

3. For any $\gamma \in (0, 1)$, using the Chasles integral formula, we have

$$\begin{aligned} \aleph(x; \alpha, \beta, \gamma) &= \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_0^{\beta+1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt + \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \\ &= \frac{1}{2^{\gamma/2}} \int_0^2 \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt + \frac{1}{2^{\gamma/2}} \int_{1/2}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^\gamma} dt \end{aligned}$$

$$= \frac{1}{2^{\gamma/2}} \left[B_{\text{eta}} \left(\frac{\gamma}{2}, \frac{\gamma}{2} \right) + \int_{1/2}^2 \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \right].$$

Using it, the corresponding upper bound obtained is the one in [1, Theorem 2].

Let us now discuss the new theorem in the light of [2, Theorem 1], as recalled in §1.3. In the special case of $\beta = 0$, the remarks below apply.

1. For any $\gamma > 1$, we have

$$\Phi(x; \alpha, \beta, \gamma) = 1 + \frac{\beta}{(1+\alpha x)(\beta+2+\alpha x)^{\gamma-1}} = 1 + 0 = 1.$$

As expected, the corresponding upper bound obtained is the one in [2, Theorem 1].

2. For $\gamma = 1$, using the formula $\arctan(a) + \arctan(1/a) = \pi/2$ for $a > 0$, we have

$$\begin{aligned} \Psi(x; \alpha, \beta) &= \frac{1}{\sqrt{\beta+1+\alpha x}} \arctan \left[\sqrt{\beta+1+\alpha x} \right] + \frac{1}{\sqrt{(\beta+1)(1+\alpha x)}} \arctan \left[\sqrt{\frac{\beta+1}{1+\alpha x}} \right] \\ &= \frac{1}{\sqrt{1+\alpha x}} \arctan \left[\sqrt{1+\alpha x} \right] + \frac{1}{\sqrt{1+\alpha x}} \arctan \left[\frac{1}{\sqrt{1+\alpha x}} \right] = \frac{\pi}{2} \times \frac{1}{\sqrt{1+\alpha x}}. \end{aligned}$$

As expected, the corresponding upper bound obtained is the one in [2, Theorem 1].

3. For any $\gamma \in (0, 1)$, using the Chasles integral formula, we have

$$\begin{aligned} \aleph(x; \alpha, \beta, \gamma) &= \frac{1}{(\beta+1+\alpha x)^{\gamma/2}} \int_0^{\beta+1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{(\beta+1)^{\gamma/2}(1+\alpha x)^{\gamma/2}} \int_{(1+\alpha x)/(\beta+1)}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \\ &= \frac{1}{(1+\alpha x)^{\gamma/2}} \int_0^{1+\alpha x} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt + \frac{1}{(1+\alpha x)^{\gamma/2}} \int_{1+\alpha x}^{+\infty} \frac{t^{\gamma/2-1}}{(1+t)^{\gamma}} dt \\ &= \frac{1}{(1+\alpha x)^{\gamma/2}} B_{\text{eta}} \left(\frac{\gamma}{2}, \frac{\gamma}{2} \right). \end{aligned}$$

As expected, the corresponding upper bound obtained is the one in [2, Theorem 1].

Like Theorem 4, Theorem 5 thus succeeds in proposing a unification of two existing theorems, i.e., [1, Theorem 2] and [2, Theorem 1]. The undiscussed values of α and β lead to new configurations that can be used in functional analysis and in operator theory in particular.

5. Conclusion

In conclusion, we have developed two new and unified integral inequality frameworks based on [1, Theorems 1 and 2] and [2, Theorem 1]. They are characterized by the use of flexible kernel functions with three adjustable parameters. In particular, the maximum and the product of variables are considered, opening up new directions in mathematical analysis. The relative complexity of these frameworks is compensated by a high degree of generality. We provide full proofs and contextualize our results within the current state of the art.

Possible extensions of this work include the study of inequalities for double integrals defined with kernel functions that modify those in Eqs. (2) and (3), such as

$$k_{\text{ern}}(x, y) = \frac{1}{[\omega + x + y + \alpha xy + \beta \min(x, y)]^{\gamma}},$$

$$k_{\text{ern}}(x, y) = \frac{1}{[\omega + x + y + \alpha xy + \beta \max(xy, \omega)]^{\gamma}},$$

$$k_{ern}(x, y) = \frac{1}{[x + y + \alpha xy + \beta \min(xy, \omega)]^\gamma},$$

and

$$k_{ern}(x, y) = \frac{1}{[x + y + \alpha xy + \beta \max(xy, \omega)]^\gamma},$$

with $\alpha, \beta, \omega \geq 0$ and $\gamma > 0$. The main novelty is the presence of ω , which modulates different quantities in each of these kernels. The following generalized kernel function is also in line with our research:

$$k_{ern}(x, y) = \frac{1}{[x + y + \alpha xy + \beta \min(x, y) + \omega \max(x, y)]^\gamma},$$

still with $\alpha, \beta, \omega \geq 0$ and $\gamma > 0$. It is characterized by the simultaneous activation of xy , $\min(x, y)$ and $\max(x, y)$, which remains an unexplored mathematical setting. This is also true for the not necessarily ratio form.

In addition, we can think of examining the three-dimensional cases, with the following candidate kernel functions:

$$k_{ern}(x, y, z) = \frac{1}{[x + y + z + \alpha xyz + \beta \max(x, y, z)]^\gamma},$$

and

$$k_{ern}(x, y, z) = \frac{1}{[x + y + z + \alpha xyz + \beta \min(x, y, z)]^\gamma},$$

with $\alpha, \beta \geq 0$ and $\gamma > 0$. The addition of the variable z significantly complicates the situation and requires special treatment when dealing with triple integrals.

These ideas are valuable, but they require further effort, which we will leave to the future.

Conflicts of Interest: The author has no conflicts of interest to declare.

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