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Vector bundle construction via Monads on multiprojective spaces

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Abstract: In this paper we construct indecomposable vector bundles associated to monads on multiprojective spaces. Specifically we establish the existence of monads on $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and on $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$. We prove stability of the kernel bundle which is a dual of a generalized Schwarzenberger bundle associated to the monads on $X = \mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and prove that the cohomology vector bundle which is simple, a generalization of special instanton bundles. We also prove stability of the kernel bundle and that the cohomology vector bundle associated to the monad on $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ is simple. Lastly, we construct explicitly the morphisms that establish the existence of monads on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$.

Keywords: Monads, multiprojective spaces, simple vector bundles

MSC: 14F05, 14J60, 18G60.

1. Introduction

The existence of indecomposable low rank vector bundles on algebraic varieties in comparison with the ambient space has been a fertile area in algebraic geometry for the last 45 years. Regardless it remains intriguing, fascinating and exciting to construct new examples of indecomposable low rank vector bundles. Some of the remarkable works in this regard are: the famous Horrocks-Mumford bundle of rank 2 over $\mathbf{P}^4$ [1], the Horrocks vector bundle of rank 3 on $\mathbf{P}^5$ [2] the Tango bundles [3] of rank $n-1$ on $\mathbf{P}^n$ for $n \geq 3$ and the rank 2 vector bundle on $\mathbf{P}^5$ in characteristic 2 by Tango [4] are all obtained as cohomologies of certain monads.

This (monads) is one of the techniques used to construct these vector bundles. They appear in many contexts within algebraic geometry. and were first introduced by Horrocks [5] where he proved that all vector bundles E on $\mathbf{P}^3$ could be obtained as the cohomology bundle of a given monad. In vector bundle construction via monads on a given algebraic variety, the first task is to show the existence of monads. Fløystad [6] gave a theorem on the existence of monads over projective spaces. Costa and Miro-Roig [7] extended these results to smooth quadric hypersurfaces of dimension at least 3. Marchesi, Marques and Soares [8] generalized Fløystad's theorem to a larger set of varieties. Maingi [9–12] proved the existence of monads on $\mathbf{P}^n \times \mathbf{P}^m$, $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1}$, $\mathbf{P}^{a_1} \times \mathbf{P}^{a_1} \times \mathbf{P}^{a_2} \times \mathbf{P}^{a_2} \times \cdots \times \mathbf{P}^{a_n} \times \mathbf{P}^{a_n}$ and on $\mathbf{P}^n \times \mathbf{P}^n \times \mathbf{P}^m \times \mathbf{P}^m$ respectively and proved simplicity of the cohomology bundles associated.

A natural and efficient technique to construct monads and hence more examples of vector bundles is to vary the ambient variety and choose a different polarisation. In Section three of the paper, we first generalize the work of Maingi [10] by construction of monads on $\mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ for a rank $\beta - \alpha - \gamma$. We then prove stability of the kernel bundle which is a generalization of the dual of Schwarzenberger (steiner) bundles. Next we prove simplicity of the cohomolgy vector bundle. Specifically we establish the existence of monads

$$0 \longrightarrow \mathcal{O}_X(-1,\cdots,-1)^{\oplus\alpha} \xrightarrow[f]{} \oplus\mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(1,\cdots,1)^{\oplus\gamma} \longrightarrow 0,$$

on $X = \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$. We shall call the monad above Type I in this paper.

Next, in Section four we establish the existence of monads on $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ for the polarisation $\mathscr{L} = \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)$. This is a generalization of the results of Maingi [9, Theorem 3.2] where he gave a conditional variant theorem for the existence of a monad on $\mathbf{P}^n \times \mathbf{P}^m$.

Specifically we establish the existence of monads

$$0 \longrightarrow \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\alpha} \xrightarrow[f]{} \oplus\mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus\gamma} \longrightarrow 0,$$

on $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ which we shall call monad Type II. We then prove stability of the kernel bundle $\ker g$ and finally prove that the cohomology vector bundle, $E = \ker g / \operatorname{im} f$ is simple.

Lastly, in Section five we construct the morphisms that establish the existence of monads

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \longrightarrow 0,$$

on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ which are matrices whose entries are multidegree monomials.

2. Preliminaries

In this work we give generalizations for previous results by several authors. To be specific we build upon results by Maingi [9–11] therefore the definitions, notation, the methods applied are quite similar and the trend follows the paper by Ancona and Ottaviani [13]. In this section we define and give notation in order to set up for the main results. Most of the definitions are from chapter two of the book by Okonek, Schneider and Spindler [14].

Definition 1. Let X be a nonsingular projective variety.

1. A *monad* on X is a complex of vector bundles:

$$0 \to M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2 \to 0,$$

which is exact at M_0 and at M_2 i.e. α is injective and β surjective.

2. A monad as defined above has a display diagram of short exact sequences as shown below:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_0 & \longrightarrow & \ker\beta & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_0 & \xrightarrow[\alpha]{} & M_1 & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\
 & & & & {\scriptstyle\beta}\downarrow & & \downarrow & & \\
 & & & & M_2 & = & M_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

3. The kernel of the map β, $F = \ker\beta$ and the cokernel of α, $\operatorname{coker}\alpha$ for the given monad are also vector bundles and the vector bundle $E = \ker(\beta) / \operatorname{im}(\alpha)$ and is called the cohomology bundle of the monad.

Definition 2. Let X be a nonsingular projective variety, let $\mathscr{L}$ be a very ample line sheaf, and V, W, U be finite dimensional k-vector spaces. A linear monad on X is a complex of sheaves,

$$M_\bullet : \ 0 \longrightarrow V \otimes \mathscr{L}^{-1} \xrightarrow{A} W \otimes \mathscr{O}_X \xrightarrow{B} U \otimes \mathscr{L} \longrightarrow 0 \ ,$$

where $A \in \operatorname{Hom}(V, W) \otimes H^0\mathscr{L}$ is injective and $B \in \operatorname{Hom}(W, U) \otimes H^0\mathscr{L}$ is surjective.

The existence of the monad $M_\bullet$ is equivalent to: A and B being of maximal rank and BA being the zero matrix.

Definition 3. Let X be a non-singular irreducible projective variety of dimension d and let $\mathscr{L}$ be an ample line bundle on X. For a torsion-free sheaf F on X we define

1. the degree of F relative to $\mathscr{L}$ as $\deg_{\mathscr{L}} F := c_1(F) \cdot \mathscr{L}^{d-1}$, where $c_1(F)$ is the first Chern class of F,
2. the slope of F as $\mu_{\mathscr{L}}(F) := \frac{\deg_{\mathscr{L}} F}{rk(F)}$.

2.1. Hoppe's criterion over polycyclic varieties

Suppose that the Picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}^l$ where $l \geq 2$ is an integer then X is a polycyclic variety. Given a divisor B on X we define $\delta_{\mathscr{L}}(B) := \deg_{\mathscr{L}} \mathcal{O}_X(B)$. Then one has the following stability criterion [15, Theorem 3]:

Theorem 1 (Generalized Hoppe criterion)**.** *Let $G \to X$ be a holomorphic vector bundle of rank $r \geq 2$ over a polycyclic variety X equiped with a polarisation $\mathscr{L}$ if*

$$H^0(X, (\wedge^s G) \otimes \mathcal{O}_X(B)) = 0,$$

for all $B \in \mathrm{Pic}(X)$ and $s \in \{1, \ldots, r-1\}$ such that $\delta_{\mathscr{L}}(B) < -s\mu_{\mathscr{L}}(G)$ then G is stable and if $\delta_{\mathscr{L}}(B) \leq -s\mu_{\mathscr{L}}(G)$ then G is semi-stable.

Conversely if then G is (semi-)stable then

$$H^0(X, G \otimes \mathcal{O}_X(B)) = 0,$$

for all $B \in \mathrm{Pic}(X)$ and all $s \in \{1, \ldots, r-1\}$ such that $(\delta_{\mathscr{L}}(B) \leq)\ \delta_{\mathscr{L}}(B) < -s\mu_{\mathscr{L}}(G)$.

Notation 1. *Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$.*

We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.

Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := p_1{}^*\mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes p_n{}^*\mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.

For any line bundle $\mathscr{L} = \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ on X and a vector bundle E, we write $E(g_1, g_2, \cdots, g_n) = E \otimes \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ and $(g_1, g_2, \cdots, g_n) := g_1[h_1 \times \mathbf{P}^{a_1}] + \cdots + g_n[\mathbf{P}^{a_n} \times h_n]$ representing its corresponding divisor.

The normalization of E on X with respect to $\mathscr{L}$ is defined as follows:

Set $d = \deg_{\mathscr{L}}(\mathcal{O}_X(1, 0, \cdots, 0))$, since $\deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) = \deg_{\mathscr{L}}(E) - nk \cdot \mathrm{rank}(E)$ there is a unique integer $k_E := \lceil \mu_{\mathscr{L}}(E)/d \rceil$ such that $1 - d.\mathrm{rank}(E) \leq \deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) \leq 0$. The twisted bundle $E_{\mathscr{L}-norm} := E(-k_E, 0, \cdots, 0)$ is called the $\mathscr{L}$-normalization of E.

Lastly, the linear functional $\delta_{\mathscr{L}}$ on $\mathbb{Z}^n$ is defined as $\delta_{\mathscr{L}}(p_1, p_2, \cdots, p_n) := \deg_{\mathscr{L}} \mathcal{O}_X(p_1, p_2, \cdots, p_n)$.

For the q^{-th} cohomology group we use the notation $H^q(\mathscr{F})$ in place of $H^q(X, \mathscr{F})$, for the sake of brevity.

The following proposition is actually a corollary of Theorem 1 above, a special case of the generalized Hoppe criterion on stability.

Proposition 1. *Let X be a polycyclic variety with Picard number n, let $\mathscr{L}$ be an ample line bundle and let E be a rank $r > 1$ holomorphic vector bundle over X. If $H^0(X, (\bigwedge^q E)_{\mathscr{L}-norm}(p_1, \cdots, p_n)) = 0$ for $1 \leq q \leq r-1$ and every $(p_1, \cdots, p_n) \in \mathbb{Z}^n$ such that $\delta_{\mathscr{L}} \leq 0$ then E is $\mathscr{L}$-stable.*

Proposition 2. *Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then we have the following exact sequence involving exterior and symmetric powers*

$$0 \longrightarrow \bigwedge^q E \longrightarrow \bigwedge^q F \longrightarrow \bigwedge^{q-1} F \otimes G \longrightarrow \cdots \longrightarrow F \otimes S^{q-1}G \longrightarrow S^q G \longrightarrow 0.$$

Theorem 2 (Künneth formula)**.** *Let X and Y be projective varieties over a field k. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X and Y respectively. Let $\mathscr{F} \boxtimes \mathscr{G}$ denote $p_1^*(\mathscr{F}) \otimes p_2^*(\mathscr{G})$ then $H^m(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) \cong \bigoplus_{p+q=m} H^p(X, \mathscr{F}) \otimes H^q(Y, \mathscr{G})$.*

Lemma 1. *Let* $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ *then*

$$H^t(X, \mathcal{O}_X(p_1, \cdots, p_n)) \cong \bigoplus_{\sum_{q_i=1}^{t}} H^{q_1}(\mathbf{P}^{a_1}, \mathcal{O}_{\mathbf{P}^{a_1}}(p_1)) \otimes H^{q_2}(\mathbf{P}^{a_2}, \mathcal{O}_{\mathbf{P}^{a_2}}(p_2)) \otimes \cdots \otimes H^{q_n}(\mathbf{P}^{a_n}, \mathcal{O}_{\mathbf{P}^{a_n}}(p_n)).$$

Theorem 3 ([16], Theorem 4.1)**.** *Let $n \geq 1$ be an integer and d be an integer. We denote by S_d the space of homogeneous polynomials of degree d in $n+1$ variables (conventionally if $d < 0$ then $S_d = 0$). Then the following statements are true:*

1. $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = S_d$ *for all* d.
2. $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = 0$ *for* $1 < i < n$ *and for all* d.
3. $H^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \cong H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-d-n-1))$.

Lemma 2. *If* $\sum_{i=1}^{n} p_i > 0$ *then* $h^p(X, \mathcal{O}_X(-p_1, \cdots, -p_n)^{\oplus k}) = 0$ *where* $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ *and for* $0 \leq p < \dim(X) - 1$*, for k a positive integer.*

Lemma 3. *Let A and B be vector bundles canonically pulled back from A' on $\mathbf{P}^n$ and B' on $\mathbf{P}^m$ then*

$$H^q\left(\bigwedge^s (A \otimes B)\right) = \sum_{k_1+\cdots+k_s=q} \left\{ \bigoplus_{i=1}^{s} \left(\sum_{j=0}^{s} \sum_{m=0}^{k_i} H^m(\wedge^j(A)) \otimes (H^{k_i-m}(\wedge^{s-j}(B))) \right) \right\}.$$

Proof. The proof follows from the following standard identities:

1. $$H^q(A_1 \oplus \cdots \oplus A_s) = \sum_{k_1+\cdots+k_s=q} \left\{ \bigoplus_{i=1}^{s} H_i^k(A_i) \right\}.$$
2. $$H^q(A \otimes B) = \sum_{m=0}^{q} H^m(A) \otimes H^{q-m}(B).$$
3. $$\wedge^s(A \otimes B) = \sum_{j=0}^{s} \wedge^j(A) \otimes \wedge^{s-j}(B).$$

□

Lemma 4 ([6], Main theorem)**.** *Let $k \geq 1$. There exists monads on $\mathbf{P}^k$ whose maps are matrices of linear forms,*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^k}(-1)^{\oplus a} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^k}^{\oplus b} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^k}(1)^{\oplus c} \longrightarrow 0,$$

if and only if at least one of the following is fulfilled;
(1) $b \geq 2c + k - 1$ *and* $b \geq a + c$,
(2) $b \geq a + c + k$.

Lemma 5 ([10], Theorem 3.9)**.** *Let n and k be positive integers and A and B be morphisms of linear forms as in*

$$B := \left(\begin{array}{ccc|ccc} x_0 \cdots & x_n & & y_0 \cdots & y_n & \\ & \ddots & \ddots & & \ddots & \ddots \\ & & x_0 \cdots x_n & & y_0 \cdots & y_n \end{array} \right),$$

and

$$A := \left(\begin{array}{cccc} -y_0 \cdots & -y_n & & \\ & \ddots & \ddots & \\ & & -y_0 \cdots & -y_n \\ \hline x_0 \cdots & x_n & & \\ & \ddots & \ddots & \\ & & x_0 \cdots & x_n \end{array} \right),$$

then there exists a linear monad of the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0.$$

Lemma 6 ([10], Theorem 3.2)**.** *Let $X = \mathbf{P}^n \times \mathbf{P}^m$ and let $\mathscr{L} = \mathcal{O}_X(\rho,\sigma)$ be an ample line bundle on X. Denote by $N = h^0(\mathcal{O}_X(\rho,\sigma)) - 1$. Let α, β, γ be positive integers such that at least one of the following conditions holds*

(1) $\beta \geq 2\gamma + N - 1$, *and* $\beta \geq \alpha + \gamma$,

(2) $\beta \geq \alpha + \gamma + N$.

Then, there exists a linear monad on X of the form

$$0 \longrightarrow \mathcal{O}_X(-\rho,-\sigma)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_X(\rho,\sigma)^{\oplus\gamma} \longrightarrow 0.$$

Definition 4. Let X be a projective variety. A sheaf S on X is a steiner bundle if has short exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus a} \longrightarrow \mathcal{O}_X^{\oplus b} \longrightarrow S \longrightarrow 0.$$

They were first defined by Dolgachev and Kapranov [17].

Definition 5. [18] Let $k \geq 0$ the exact sequence of sheaves on $\mathbf{P}^{2n+1}$

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus k} \xrightarrow[\phi]{} \mathcal{O}_X^{\oplus 2n+2k} \longrightarrow S \longrightarrow 0,$$

where ϕ is given by the matrix

$$\left[\begin{array}{cccc|cccc} x_0 \cdots & x_n & & & y_0 \cdots & y_n & & \\ & \ddots & \ddots & & \ddots & \ddots & & \\ & & x_0 \cdots x_n & & & y_0 \cdots & y_n & \end{array}\right],$$

defines a $2n+k-$bundle S on $\mathbf{P}^{2n+1}$ called a (generalized) Schwarzenberger bundle.

The display of the monad in Lemma 5 is

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} & \longrightarrow & S^* := \ker(B) & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} & \xrightarrow[A]{} & \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} & \longrightarrow & S := \operatorname{coker}(A) & \longrightarrow & 0 \\
 & & & & {\scriptstyle B}\downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} & = & \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

A special instanton bundle on $\mathbf{P}^{2n+1}$ of quantum number k is defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \longrightarrow S^* := \ker(B) \longrightarrow E \longrightarrow 0,$$

which is exactly the way Spindler and Trautmann remarkably described [18] where S is a Schwarzenberger bundle of rank $2n+k$ which is defined by the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \longrightarrow S := \operatorname{coker}(A) \longrightarrow 0,$$

and they were proved by Ancona and Ottaviani [13], Theorem 2.2 to be stable and in Theorem 2.8 they proved that E is simple. Independently, Bohnhorst and Spindler [19] proved the stability of rank n Schwarzenberger bundles on $\mathbf{P}^n$.

In the next section we are going to establish the existence of monads on a more general space namely $\mathbf{P}^{2n+1} \times \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and prove stability of the kernel bundle T and simplicity of the cohomology vector bundle E.

3. Monad type I and associated vector bundles

The goal of this section is to construct monads over a multiprojective space of m copies of $\mathbf{P}^{2n+1}$. More specifically we generalize the results of Maingi [10] by varying the ambient space. We rely on methods similar to those used in [11]. The kernel bundle T is a more generalized version of the dual of a Schwarzenberger vector bundle and we prove that it is stable and consequently we prove that the cohomology vector bundle E associated to the monad on X is simple. The vector bundle E is a generalized version of an instanton bundle.

Theorem 4. *Let $X = \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and $\mathscr{L} = \mathcal{O}_X(1, \cdots, 1)$ an ample line bundle. Denote by $N = h^0(\mathcal{O}_X(1, \cdots, 1)) - 1$. Then there exists a linear monad $M_\bullet$ on X of the form*

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1, \cdots, -1)^{\oplus \alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus \beta} \xrightarrow[g]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus \gamma} \longrightarrow 0,$$

if atleast one of the following is satified

1. *$\beta \geq 2\gamma + N - 1$, and $\beta \geq \alpha + \gamma$,*
2. *$\beta \geq \alpha + \gamma + N$, where α, β, γ be positive integers.*

Proof. For the ample line bundle $\mathscr{L} = \mathcal{O}_X(1, \ldots, 1)$ we have the Segre embedding

$$i^* : X = \mathbb{P}^{2n+1} \times \cdots \times \mathbb{P}^{2n+1} \hookrightarrow \mathbb{P}\big(H^0(X, \mathcal{O}_X(1, \ldots, 1))\big) \cong \mathbb{P}^{N:=(2n+2)^m - 1},$$

such that $i^*(\mathcal{O}_X(1)) \simeq \mathscr{L}$.

Suppose that one of the conditions of Lemma 4 is satified and we have $a = \alpha$, $b = \beta$, $c = \gamma$ and $k = 2n + 1$ thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus \alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus \beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus \gamma} \longrightarrow 0,$$

on $\mathbf{P}^{2n+1}$ whose morphisms are matrices A and B with entries monomials of degree one where

$$A \in \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus \alpha}, \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus \beta}) \cong H^0(\mathbf{P}^{2n+1}, \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus \alpha\beta}),$$
$$B \in \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus \beta}, \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus \gamma}) \cong H^0(\mathbf{P}^{2n+1}, \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus \beta\gamma}).$$

Thus, A and B induce a monad on X,

$$0 \longrightarrow \mathscr{L}^{-1 \oplus \alpha} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus \beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus \gamma} \longrightarrow 0,$$

where whose morphisms are matrices $\bar{A}$ and $\bar{B}$ with entries multidegree monomials such that

$$\bar{A} \in \mathrm{Hom}(\mathcal{O}_X(-1, \ldots, -1)^{\oplus \alpha}, \mathcal{O}_X^{\oplus \beta}),$$

and

$$\bar{B} \in \mathrm{Hom}(\mathcal{O}_X^{\oplus \beta}, \mathcal{O}_X(1, \ldots, 1)^{\oplus \gamma}).$$

□

The kernel bundle T of the above monad is a generalization of the dual of Schwarzenberger vector bundles [19] which we now proceed to prove that it is stable.

Lemma 7. *Let T be a vector bundle on $X = \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ defined by the sequence*

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus \beta} \longrightarrow \mathcal{O}_X(1, \cdots, 1)^{\oplus \gamma} \longrightarrow 0,$$

then T is stable.

Proof. We show that $H^0(X, \bigwedge^q T(-p_1, \cdots, -p_m)) = 0$ for all $\sum_i^m p_i > 0$ and $1 \leq q \leq \text{rank}(T)$.

Consider the ample line bundle $\mathscr{L} = \mathcal{O}_X(1, \cdots, 1) = \mathcal{O}(L)$.

Its class in

$$\text{Pic}(X) = \langle [h_1 \times \mathbf{P}^{2n+1}], \cdots, [\mathbf{P}^{2n+1} \times h_m] \rangle,$$

corresponds to the class

$$\sum_{i=1}^{m} 1 \cdot [h_i \times \mathbf{P}^{2n+1}],$$

where h_i, $i = 1, \cdots, n$ are hyperplanes of $\mathbf{P}^{2n+1}$ with the intersection product induced by $h_i^{2n+1} = 1$ and $h_i^{2n+2} = 0$.

Now from the display diagram of the monad we get

$$\begin{aligned} c_1(T) &= c_1(\mathcal{O}_X^{\beta}) - c_1(\mathcal{O}_X(1, \cdots, 1)^{\oplus \gamma}) \\ &= \beta(0, \cdots, 0) - \gamma(1, \cdots, 1) \\ &= (-\gamma, \cdots, -\gamma). \end{aligned}$$

Now $L^{(2n+1)m} > 0$ hence , the degree of T is:

$$\begin{aligned} \deg_{\mathscr{L}} T &= -\gamma([h_1 \times \mathbf{P}^{2n+1}] + \cdots + [\mathbf{P}^{2n+1} \times h_m]) \cdot (\sum_{i=1}^{m} 1 \cdot [h_i \times \mathbf{P}^{2n+1}])^{m(2n+1)-1} \\ &= -\gamma L^{m(2n+1)} < 0. \end{aligned}$$

Since $\deg_{\mathscr{L}} T < 0$, then $(\bigwedge^q T)_{\mathscr{L}-norm} = (\bigwedge^q T)$ and it suffices by Proposition 1, to prove that $h^0(\bigwedge^q T(-p_1, \cdots, -p_m)) = 0$ with $\sum_{i=1}^{m} p_i \geq 0$ and for all $1 \leq q \leq \text{rank}(T) - 1$. Next we twist the exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus \beta} \longrightarrow \mathcal{O}_X(1, \cdots, 1)^{\oplus \gamma} \longrightarrow 0,$$

by $\mathcal{O}_X(-p_1, \cdots, -p_m)$ we get,

$$0 \longrightarrow T(-p_1, \cdots, -p_m) \longrightarrow \mathcal{O}_X(-p_1, \cdots, -p_m)^{\oplus \beta} \longrightarrow \mathcal{O}_X(1 - p_1, \cdots, 1 - p_m)^{\oplus \gamma} \longrightarrow 0,$$

and taking the exterior powers of the sequence by Proposition 2 we get

$$0 \longrightarrow \bigwedge^q T(-p_1, \cdots, -p_m) \longrightarrow \bigwedge^q (\mathcal{O}_X(-p_1, \cdots, -p_m)^{\oplus \beta}) \longrightarrow \bigwedge^{q-1} (\mathcal{O}_X(1 - 2p_1, \cdots, 1 - 2p_m)^{\oplus \beta + \gamma}) \cdots .$$

Taking cohomology we have the injection:

$$0 \longrightarrow H^0(X, \bigwedge^q T(-p_1, \cdots, -p_m)) \hookrightarrow H^0(X, \bigwedge^q (\mathcal{O}_X(-p_1, \cdots, -p_m)^{\oplus \beta})).$$

Set $\mathscr{G} = \mathcal{O}_X(-p_1, \cdots, -p_m)^{\beta} = \mathcal{O}_X(-p_1, \cdots, -p_2) \otimes \mathcal{O}_X^{\oplus \beta}$ and using Lemma 2 $H^0(X, \bigwedge^q \mathscr{G})$ expands into $H^0(X, \sum_{j=0}^{q} \wedge^j \mathcal{O}_X(-p_1, \cdots, -p_2) \otimes \mathcal{O}_X^{\oplus \beta})$ and since $\sum_i^m p_i > 0$ by Lemma 3 then

$$h^0(X, \bigwedge^q (\mathcal{O}_X(-p_1, \cdots, -p_m)^{\oplus \beta})) = h^0(X, \bigwedge^q T(-p_1, \cdots, -p_m)) = 0,$$

i.e. $h^0(\bigwedge^q T(-p_1, \cdots, -p_m)) = 0$ and thus T is stable. $\square$

Theorem 5. *Let $X = \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$, then the cohomology vector bundle E associated to the monad*

$$0 \longrightarrow \mathcal{O}_X(-1,\cdots,-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_X(1,\cdots,1)^{\oplus\gamma} \longrightarrow 0,$$

of rank $\beta - \alpha - \gamma$ is simple.

Proof. The display of the monad is

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1,\cdots,-1)^{\oplus\alpha} & \longrightarrow & T & \longrightarrow & E & \longrightarrow & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1,\cdots,-1)^{\oplus\alpha} & \xrightarrow[f]{} & \mathcal{O}_X^{\oplus\beta} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & g\downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(1,\cdots,1)^{\oplus\gamma} & = & \mathcal{O}_X(1,\cdots,1)^{\oplus\gamma} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since E is simple if its only endomorphisms are the homotheties then we need to prove that $\mathrm{Hom}(E,E) = k$ which is equivalent to $h^0(E \otimes E^*)$.

The first step is to take the dual short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1,\cdots,-1)^{\oplus\alpha} \longrightarrow T \longrightarrow E \longrightarrow 0,$$

to get

$$0 \longrightarrow E^* \longrightarrow T^* \longrightarrow \mathcal{O}_X(1,\cdots,1)^{\oplus\alpha} \longrightarrow 0.$$

Tensoring by E we get

$$0 \longrightarrow E \otimes E^* \longrightarrow E \otimes T^* \longrightarrow E(1,\cdots,1)^{\oplus\alpha} \longrightarrow 0.$$

Now taking cohomology gives:

$$0 \longrightarrow H^0(X, E \otimes E^*) \longrightarrow H^0(X, E \otimes T^*) \longrightarrow H^0(E(1,\cdots,1)^{\oplus\alpha}) \longrightarrow \cdots,$$

which implies that

$$h^0(X, E \otimes E^*) \leq h^0(X, E \otimes T^*). \tag{1}$$

Now we dualize the short exact sequence

$$0 \longrightarrow T \longrightarrow \mathcal{O}_X^{\oplus\beta} \longrightarrow \mathcal{O}_X(1,\cdots,1)^{\oplus\gamma} \longrightarrow 0,$$

to get

$$0 \longrightarrow \mathcal{O}_X(-1,\cdots,-1)^{\oplus\gamma} \longrightarrow \mathcal{O}_X^{\oplus\beta} \longrightarrow T^* \longrightarrow 0.$$

Now twisting by $\mathcal{O}_X(-1,\cdots,-1)$ and taking cohomology and get

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{O}_X(-2,\cdots-2)^{\oplus\gamma}) &\longrightarrow H^0(X, \mathcal{O}_X(-1,\cdots,-1)^{\oplus\beta}) \longrightarrow H^0(X, T^*(-1,\cdots,-1)) \longrightarrow \\
\longrightarrow H^1(X, \mathcal{O}_X(-2,\cdots,-2)^{\oplus\gamma}) &\longrightarrow H^1(X, \mathcal{O}_X(-1,\cdots,-1)^{\oplus\beta}) \longrightarrow H^1(X, T^*(-1,\cdots,-1)) \longrightarrow \\
\longrightarrow H^2(X, \mathcal{O}_X(-2,\cdots,-2)^{\oplus\gamma}) &\longrightarrow H^2(X, \mathcal{O}_X(-1,\cdots,-1)^{\oplus\beta}) \longrightarrow H^2(X, T^*(-1,\cdots,-1)) \longrightarrow \cdots,
\end{aligned}$$

from which we deduce $H^0(X, T^*(-1,\cdots,-1)) = 0$ and $H^1(X, T^*(-1,\cdots,-1)) = 0$ from Lemmas 1, 2 and Theorem 3.

Lastly, tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}(-1,\cdots,-1)^{\oplus\alpha} \longrightarrow T \longrightarrow E \longrightarrow 0,$$

by T^* to get

$$0 \longrightarrow T^*(-1,\cdots,-1)^{\oplus\alpha} \longrightarrow T\otimes T^* \longrightarrow E\otimes T^* \longrightarrow 0,$$

and taking cohomology we have

$$\begin{aligned}0 &\longrightarrow H^0(X,T^*(-1,\cdots,-1)^{\oplus\alpha}) \longrightarrow H^0(X,T\otimes T^*) \longrightarrow H^0(X,E\otimes T^*) \longrightarrow \\ &\longrightarrow H^1(X,T^*(-1,\cdots,-1)^{\oplus\alpha}) \longrightarrow \quad \cdots .\end{aligned}$$

But $H^1(X,T^*(-1,\cdots,-1)^{\oplus\alpha} = 0$ for $\alpha > 1$ from above. So we have

$$0 \longrightarrow H^0(X,T^*(-1,\cdots,-1)^{\oplus\alpha}) \longrightarrow H^0(X,T\otimes T^*) \longrightarrow H^0(X,E\otimes T^*) \longrightarrow 0.$$

This implies that

$$h^0(X,T\otimes T^*) \leq h^0(X,E\otimes T^*). \tag{2}$$

Since T is stable then it follows that it is simple which implies $h^0(X,T\otimes T^*) = 1$.
From (1) and now (2) and putting these together, we have

$$1 \leq h^0(X,E\otimes E^*) \leq h^0(X,E\otimes T^*) = h^0(X,T\otimes T^*) = 1.$$

We have $h^0(X,E\otimes E^*) = 1$ and therefore E is simple. □

4. Monad type II and associated vector bundles

The goal of this section is to construct monads over a multiprojectivespace $\mathbf{P}^{a_1}\times\cdots\times\mathbf{P}^{a_n}$. More specifically we generalize the results of Maingi [10] by varying the ambient space and the polarisation $\mathscr{L}$. We prove that the kernel bundle F is stable and thereafter we prove that the cohomology vector bundle E associated to the monad on X is simple.

Theorem 6. *Let $X = \mathbf{P}^{a_1}\cdots\times\mathbf{P}^{a_n}$ and $\mathscr{L} = \mathcal{O}_X(\alpha_1,\cdots,\alpha_t)$ an ample line bundle. Denote by $N = h^0(\mathcal{O}_X(\alpha_1,\cdots,\alpha_t)) - 1$. Then there exists a linear monad $M_\bullet$ on X of the form*

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-\alpha_1,\cdots,-\alpha_t)^{\oplus\alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1,\cdots,\alpha_t)^{\oplus\gamma} \longrightarrow 0,$$

if atleast one of the following is satified

1. *$\beta \geq 2\gamma + N - 1$, and $\beta \geq \alpha+\gamma$,*
2. *$\beta \geq \alpha+\gamma+N$, where α,β,γ be positive integers.*

Proof. For the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1,\ldots,\alpha_t)$ we have the Segre embedding

$$i^* : X = \mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_n} \hookrightarrow \mathbb{P}\big(H^0(X,\mathcal{O}_X(\alpha_1,\ldots,\alpha_t))\big) \cong \mathbb{P}^N \ ,$$

such that $i^*(\mathcal{O}_X(1)) \simeq \mathscr{L}$ and where $N = \left(\binom{a_1+\alpha_1}{\alpha_1}\binom{a_2+\alpha_2}{\alpha_2}\cdots\binom{a_n+\alpha_t}{\alpha_t}\right) - 1$.

Suppose that one of the conditions of Lemma 4 is satified thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0,$$

on $\mathbf{P}^N$ whose morphisms are matrices A and B with entries monomials of degree one where

$$\begin{aligned}A &\in \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha},\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}) \cong H^0(\mathbf{P}^N,\mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\alpha\beta}),\\ B &\in \mathrm{Hom}(\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta},\mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma}) \cong H^0(\mathbf{P}^N,\mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\beta\gamma}).\end{aligned}$$

Thus, A and B induce a monad on X,

$$0 \longrightarrow \mathscr{L}^{-1 \oplus \alpha} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus \beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus \gamma} \longrightarrow 0,$$

where whose morphisms are matrices $\bar{A}$ and $\bar{B}$ with entries multidegree monomials such that

$$\bar{A} \in \mathrm{Hom}(\mathcal{O}_X(-\alpha_1, \ldots, -\alpha_t)^{\oplus \alpha}, \mathcal{O}_X^{\oplus \beta}),$$

and

$$\bar{B} \in \mathrm{Hom}(\mathcal{O}_X^{\oplus \beta}, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)^{\oplus \gamma}).$$

□

Theorem 7. *Let F be a vector bundle on $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ defined by the short exact sequence*

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus \beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus \gamma} \longrightarrow 0,$$

then F is stable for an ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)$.

Proof. We are going to show that $H^0(X, \bigwedge^q F(-p_1, \cdots, -p_n)) = 0$ for all $\sum_{i=1}^{n} p_i > 0$ and $1 \leq q \leq \mathrm{rank}(F) - 1$.

Consider the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \cdots, \alpha_t) = \mathcal{O}(L)$. Its class in $\mathrm{Pic}(X) = \langle [h_i \times \mathbf{P}^{a_i}], i = 1, \ldots, n] \rangle$ corresponds to $\sum_{i=1}^{n} 1.[h_i \times \mathbf{P}^{a_i}]$ where each h_i is a hyperplane in $\mathbf{P}^{a_i}$ with intersection product induced by $h_i^{a_i} = 1$ and $h_i^{a_i+1} = 0$ for $i = 1, \ldots, n$.

From the display of the monad we get

$$c_1(F) = c_1(\mathcal{O}_X^{\oplus \beta}) - c_1(\mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus \gamma}) = (-\gamma\alpha_1, \cdots, -\gamma\alpha_t).$$

Since $L^{a_1 + \cdots + a_n} > 0$, the degree of F is $\deg_{\mathscr{L}} F = c_1(T) \cdot \mathscr{L}^{d-1}$ that is

$$\deg_{\mathscr{L}} F = -\gamma n \sum_{i=1}^{t} \alpha_i ([h_1 \times \mathbf{P}^{a_1}] + \cdots + [\mathbf{P}^{a_n} \times h_n]) \left(\sum_{i=1}^{n} 1 \cdot [h_i \times \mathbf{P}^{a_i}] \right)^{\sum_{i=1}^{n} a_i - 1} = -\gamma n \sum_{i=1}^{t} \alpha_i L^{(a_1 + \cdots + a_n)} < 0.$$

Since $\deg_{\mathscr{L}} F < 0$, then $(\bigwedge^q F)_{\mathscr{L}-norm} = (\bigwedge^q F)$ and it suffices by the generalized Hoppe Criterion (Proposition 1), to prove that $h^0(\bigwedge^q F(-p_1, -p_2, \cdots, -p_n)) = 0$ with $\sum_{i=1}^{n} p_i > 0$ and for all $1 \leq q \leq \mathrm{rank}(F) - 1$.

Next consider the exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus \beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus \gamma} \longrightarrow 0,$$

on twisting it by $\mathcal{O}_X(-p_1, \cdots, -p_n)$ one gets,

$$0 \longrightarrow F(-p_1, \cdots, -p_n) \longrightarrow \mathcal{O}_X^{\oplus \beta}(-p_1, \cdots, -p_n) \xrightarrow[g]{} \mathcal{O}_X(\alpha_1 - p_1, \cdots, \alpha_t - p_n)^{\oplus \gamma} \longrightarrow 0,$$

and taking the exterior powers of the sequence by Proposition 2 one gets

$$0 \longrightarrow \bigwedge^q F(-p_1, \cdots, -p_n) \longrightarrow \bigwedge^q (\mathcal{O}_X(-p_1, \cdots, -p_n)^{\oplus \beta}) \longrightarrow \bigwedge^{q-1} (\mathcal{O}_X(\alpha_1 - 2p_1, \cdots, \alpha_t - 2p_n)^{\oplus \gamma}) \longrightarrow \cdots.$$

Taking cohomology we have the injection:

$$0 \longrightarrow H^0(X, \bigwedge^q F(-p_1, \cdots, -p_n)) \hookrightarrow H^0(X, \bigwedge^q (\mathcal{O}_X(-p_1, \cdots, -p_n)^{\oplus \beta}).$$

From here $h^0(X, \bigwedge^q F(-p_1, \cdots, -p_n)) = 0$ is proved in the same way as Lemma 7 the last part and thus F is stable. □

Theorem 8. *Let $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$, then the cohomology vector bundle E associated to the monad*

$$0 \longrightarrow \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus\gamma} \longrightarrow 0,$$

of rank $\beta - \alpha - \gamma$ is simple.

Proof. The display of the monad is

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\alpha} & \longrightarrow & F = \ker g & \longrightarrow & E & \longrightarrow & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\alpha} & \xrightarrow[f]{} & \mathcal{O}_X^{\oplus\beta} & \longrightarrow & Q = \operatorname{coker} f & \longrightarrow & 0 \\
 & & & & {\scriptstyle g}\downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus\gamma} & = & \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus\gamma} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since T is stable from Theorem 7, we prove that the cohomology vector bundle E with rank $2n$ is simple.

On taking the dual of the short exact sequence on the first row of the display diagram and tensoring by E we obtain

$$0 \longrightarrow E \otimes E^* \longrightarrow E \otimes F^* \longrightarrow E(t, \cdots, t)^{\oplus\alpha} \longrightarrow 0.$$

Now taking cohomology gives:

$$0 \longrightarrow H^0(X, E \otimes E^*) \longrightarrow H^0(X, E \otimes F^*) \longrightarrow H^0(E(\alpha_1, \cdots, \alpha_t)^{\oplus\alpha}) \longrightarrow \cdots,$$

which implies that

$$h^0(X, E \otimes E^*) \leq h^0(X, E \otimes F^*). \tag{3}$$

Dualize the short exact sequence on the first column of the display diagram to get

$$0 \longrightarrow \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\gamma} \longrightarrow \mathcal{O}_X^{\beta} \longrightarrow F^* \longrightarrow 0.$$

Now twisting the short exact sequence above by $\mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)$ one obtains the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2\alpha_1, \cdots, -2\alpha_t)^{\oplus\gamma} \longrightarrow \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\beta} \longrightarrow F^*(-\alpha_1, \cdots, -\alpha_t) \longrightarrow 0.$$

Next on taking cohomology one gets

$$\begin{aligned}
&0 \longrightarrow H^0(\mathcal{O}_X(-2\alpha_1, \cdots, -2\alpha_t)^{\oplus\gamma}) \longrightarrow H^0(\mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\beta}) \longrightarrow H^0(F^*(-\alpha_1, \cdots, -\alpha_t)) \longrightarrow \\
&0 \longrightarrow H^1(\mathcal{O}_X(-2\alpha_1, \cdots, -2\alpha_t)^{\oplus\gamma}) \longrightarrow H^1(\mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\beta}) \longrightarrow H^1(F^*(-\alpha_1, \cdots, -\alpha_t)) \longrightarrow \\
&\longrightarrow H^2(\mathcal{O}_X(-2\alpha_1, \cdots, -2\alpha_t)^{\oplus\gamma}) \longrightarrow H^2(\mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\beta}) \longrightarrow H^2(F^*(-\alpha_1, \cdots, -\alpha_t)) \longrightarrow \cdots,
\end{aligned}$$

from which we deduce $H^0(X, F^*(-\alpha_1, \cdots, -\alpha_t)) = 0$ and $H^1(X, F^*(-\alpha_1, \cdots, -\alpha_t)) = 0$ from Lemmas 1, 2 and Theorem 3.

Lastly, tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}(-\alpha_1, \cdots, -\alpha_t)^{\oplus k} \longrightarrow F \longrightarrow E \longrightarrow 0,$$

by F^* to get

$$0 \longrightarrow F^*(-\alpha_1, \cdots, -\alpha_t)^k \longrightarrow F \otimes F^* \longrightarrow E \otimes F^* \longrightarrow 0,$$

and taking cohomology we have

$$0 \longrightarrow H^0(X, F^*(-\alpha_1, \cdots, -\alpha_t)^k) \longrightarrow H^0(X, F \otimes F^*) \longrightarrow H^0(X, E \otimes F^*) \longrightarrow$$
$$\longrightarrow H^1(X, F^*(-\alpha_1, \cdots, -\alpha_t)^k) \longrightarrow \cdots.$$

But since $H^0(X, F^*(-\alpha_1, \cdots, -\alpha_t)) = H^1(X, F^*(-\alpha_1, \cdots, -\alpha_t)) = 0$ from above then it follows $H^1(X, F^*(-\alpha_1, \cdots, -\alpha_t)^k) = 0$ for $k > 1$, so we have

$$0 \longrightarrow H^0(X, F^*(-\alpha_1, \cdots, -\alpha_t)^k) \longrightarrow H^0(X, F \otimes F^*) \longrightarrow H^0(X, E \otimes F^*) \longrightarrow 0.$$

This implies that

$$h^0(X, F \otimes F^*) \leq h^0(X, E \otimes F^*). \tag{4}$$

Since F is stable then it is simple implying $h^0(X, F \otimes F^*) = 1$.

From (3) and (4) and putting these together we have;

$$1 \leq h^0(X, E \otimes E^*) \leq h^0(X, E \otimes F^*) = h^0(X, F \otimes F^*) = 1.$$

We have $h^0(X, E \otimes E^*) = 1$ and therefore E is simple. □

5. Monad construction via morphisms

Let X be a nonsingular projective variety. A monad

$$0 \longrightarrow M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2 \longrightarrow 0$$

on X exists if one can give the morphisms α and β. In this section we establish the existence of monads on $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ by providing an explicit contruction of the morphisms derived from the matrices used by Fløystad [6] and Ancona and Ottaviani [13].

Construction 1. Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:

$$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : x_1 : \cdots : x_n : y_0 : y_2 : \ldots : y_n].$$

First note that since we are taking m copies of $\mathbf{P}^1$ then we have

$$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1,$$

i.e., $N = 2n + 1$ where m and n are positive integers such that $n = 2^{m-1} - 1$.

Thus from Lemma 5, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0,$$

whose morphisms A and B that establish the monad are as given in Lemma 5.

We induce a monad on $X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \longrightarrow 0,$$

by giving the morphisms $\overline{A}$ and $\overline{B}$ with $\overline{B} \cdot \overline{A} = 0$ and $\overline{A}$ and $\overline{B}$ are of maximal rank.

From A and B whose entries are $x_0, \cdots, x_n, y_0, \cdots, y_n$ the homogeneous coordinates on $\mathbf{P}^{2n+1}$ we give the correspondence for the the Segre embedding using the following table:

homog.coord. on $\mathbf{P}^{2n+1}$	*representation homog.coord. on* X
x_0	$a_{0000\cdots0000}$
x_1	$a_{0000\cdots0001}$
x_2	$a_{0000\cdots0010}$
x_3	$a_{0000\cdots0011}$
x_4	$a_{0000\cdots0100}$
$\vdots$	$\vdots$
x_{n-1}	$a_{0111\cdots1110}$
x_n	$a_{0111\cdots1111}$
y_0	$a_{1000\cdots0000}$
y_1	$a_{1000\cdots0001}$
y_2	$a_{1000\cdots0010}$
y_3	$a_{1000\cdots0011}$
y_4	$a_{1000\cdots0100}$
$\vdots$	$\vdots$
y_{n-1}	$a_{1111\cdots1110}$
y_n	$a_{1111\cdots1111}$

where $a_{iiii\cdots iiii}$ for i is 0 or 1 are monomials of multidegree $(1,\ldots,1)$, i.e.,

representation homog.coord. on $\mathbf{P}^{2n+1}$	*homog.coord. on* X
$a_{0000\cdots0000}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m0}$
$a_{0000\cdots0001}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m1}$
$a_{0000\cdots0010}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m0}$
$a_{0000\cdots0011}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m1}$
$a_{0000\cdots0100}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)1}\alpha_{(m-1)0}\alpha_{m0}$
$\vdots$	$\vdots$
$a_{0111\cdots1110}$	$\alpha_{10}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m0}$
$a_{0111\cdots1111}$	$\alpha_{10}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m1}$
$a_{1000\cdots0000}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m0}$
$a_{1000\cdots0001}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m1}$
$a_{1000\cdots0010}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m0}$
$a_{1000\cdots0011}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m1}$
$a_{1000\cdots0100}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)1}\alpha_{(m-1)0}\alpha_{m0}$
$\vdots$	$\vdots$
$a_{1111\cdots1110}$	$\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m0}$
$a_{1111\cdots1111}$	$\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m1}$

Specifically we define $\overline{A}$ and $\overline{B}$ as follows

$$\overline{B} := \left[\begin{array}{cccc|cccc} a_{0000\cdots0000} & \cdots & a_{0111\cdots1111} & & a_{1000\cdots0000} & \cdots & a_{1111\cdots1111} & \\ & \ddots & & \ddots & & \ddots & & \ddots \\ & & a_{0000\cdots0000} & \cdots\ a_{0111\cdots1111} & & & a_{1000\cdots0000} & \cdots\ a_{1111\cdots1111} \end{array}\right],$$

and

$$\overline{A} := \left[\begin{array}{cccc} -a_{1000\cdots0000}\cdots & -a_{1111\cdots1111} & & \\ & \ddots & \ddots & \\ & & -a_{1000\cdots0000}\cdots & -a_{1111\cdots1111} \\ \hline a_{0000\cdots0000}\cdots & a_{0111\cdots1111} & & \\ & \ddots & \ddots & \\ & & a_{0000\cdots0000}\cdots & a_{0111\cdots1111} \end{array}\right].$$

We note that

1. $\overline{B} \cdot \overline{A} = 0$, and
2. The matrices $\overline{B}$ and $\overline{A}$ have maximal rank.

Hence we get the desired monad,

$$M_\bullet : 0 \longrightarrow \mathcal{O}_X(-1,\cdots,-1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1,\cdots,1)^{\oplus k} \longrightarrow 0.$$

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