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A degenerate–q hybrid Heine equation and Beta–Kernel integral representations

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Abstract: We introduce a degenerate–q hybrid coefficient series obtained by multiplying a normalised basic-hypergeometric coefficient array by the degenerate rising factorial defined through the Kim–Kim degenerate Gamma function. The normalisation used here deliberately omits the conventional factor $[(-1)^n q^{n(n-1)/2}]^{1+s-r}$ from the fully standard definition of ${}_r\phi_s$; the precise relationship with the standard convention is stated explicitly. Under explicit nonsingularity assumptions on the q-denominator parameters and the degenerate Gamma factors, we determine the disc of absolute convergence and give a corrected boundary analysis. The coefficient ratio $(\alpha)^*_{\lambda,n+1} \,/\, (\alpha)^*_{\lambda,n}$ is encoded as a rational Euler-transport operator, which leads coefficientwise to a hybrid Heine-type q-difference equation, numerator and denominator contiguity relations, and Jackson q-derivative identities. In the scalar case we construct two Frobenius-type local series and formulate the connection coefficients only within the corresponding nonresonant Frobenius span. For terminating series we prove genuine degenerate-Gamma moment representations; for nonterminating series the corresponding moment formalism is explicitly interpreted as a meromorphic regularisation rather than as a positive-measure integral. We also state a symmetric Beta-kernel identity with separate complex-analytic and real-positivity hypotheses and record carefully scaled limit transitions.

Keywords: degenerate Gamma function, basic hypergeometric series, hybrid q-difference equation, contiguity relations, Jackson q-derivative, integral representations

MSC: 33D15, 33D60, 33C90, 39A13, 33B15.

1. Introduction

Basic hypergeometric (q-hypergeometric) series and their q-difference equations are central in special-function theory; standard references include Andrews' q-series monograph [1], Gasper–Rahman [2], Fine [3], Andrews–Askey–Roy [4], and the NIST DLMF [5]. The Heine (or ${}_2\phi_1$) equation is a prototypical example and underlies many families of q-orthogonal polynomials; see, e.g., [2,6–9].

Recent operational, umbral, and monomiality-based developments provide a complementary polynomial perspective. Representative examples include q-Mittag–Leffler-based Bessel and Tricomi functions constructed by an umbral approach [10], q-Legendre- and q-Laguerre-based Appell systems [11,12], generalized bivariate and extended q-Laguerre families [13,14], Gould–Hopper Sheffer–Appell families studied by operational and Riordan-array methods [15], two-variable q-general-Appell polynomials [16], and generalized Laguerre-based Appell systems [17]. Closely related recent work has also treated q-truncated-exponential-based Hahn–Appell polynomials and two-variable q-Gould–Hopper–Hahn–Appell polynomials within quantum q-calculus [18, 19].

A separate one-parameter deformation is given by the *degenerate* exponential and the associated *degenerate* Gamma function Γ_λ of Kim–Kim [20,21]. This function retains several classical features, including a Beta-function representation and meromorphic continuation, while introducing additional λ-dependent pole restrictions. Related degenerate-Gamma variants and matrix-valued extensions have also been studied in [22,23]. Degenerate analogues of hypergeometric-type functions have been explored in several directions; see, for instance, [24] and the references therein.

Recent extensions of this degenerate framework include a Jackson-integral model for the degenerate q-Sumudu transform [25], degenerate q-derangement numbers and polynomials [26], modified-degenerate operational reformulations of $W_{\alpha,\beta,\nu}$-type exponential, trigonometric, and hyperbolic functions [27], degenerate Mittag–Leffler functions defined through the degenerate Gamma function with an application to fractional Maxwell–Zener viscoelasticity [28], and degenerate two-variable q-Legendre polynomials developed via q-operational calculus together with degenerate Laplace and Sumudu transforms [29].

The purpose of this paper is to combine these two deformations in a single coefficient model. We do not claim to work with the full standard ${}_r\phi_s$ convention in all parameter ranges. Instead, we use the normalised basic coefficient array

$$\frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\frac{1}{(q;q)_n},$$

which coincides with the standard ${}_{s+1}\phi_s$ normalisation when $r = s+1$ (and hence with the standard ${}_2\phi_1$ normalisation when $(r,s) = (2,1)$) but, for general (r,s), differs from the fully standard convention by the omitted factor $[(-1)^n q^{n(n-1)/2}]^{1+s-r}$. This convention is convenient for coefficientwise operator calculations, but it must be kept separate from statements about the usual ${}_r\phi_s$ in full generality. If the standard normalisation is desired, the omitted factor can be absorbed into the coefficient ratio and the operator identities below must be modified accordingly. In this article all displayed hybrid identities refer to the normalised convention just described.

Throughout the analytic part we fix $0 < q < 1$ and $\lambda > 0$. Purely formal coefficient identities remain valid for complex nonzero λ provided all denominators occurring in the rational coefficient ratios are nonzero; however, the degenerate Gamma integral, positivity statements, moment interpretations and asymptotics used below require $\lambda > 0$. For arrays of parameters

$$\mathbf{a} = (a_1,\ldots,a_r), \qquad \mathbf{b} = (b_1,\ldots,b_s),$$

and for $\alpha \in \mathbb{C}$ we combine the standard q-shifted-factorial notation for basic hypergeometric coefficients [2,5] with the Kim–Kim degenerate Gamma function [20,21] and define the *degenerate–q hybrid normalised series*

$$\mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; q,\ \lambda,\ \alpha;\ u\right] := \sum_{n=0}^{\infty} (\alpha)^{*}_{\lambda,n}\, \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\, \frac{u^n}{(q;q)_n}. \tag{1}$$

The associated degenerate rising factorial, following the degenerate-Pochhammer notation used in degenerate hypergeometric constructions [21,24], is

$$(\alpha)^{*}_{\lambda,n} := \frac{\Gamma_\lambda(\alpha+n)}{\Gamma_\lambda(\alpha)}, \qquad n \in \mathbb{N}_0, \tag{2}$$

whenever the quotient is defined by the meromorphic continuation of Γ_λ.

The main contribution is not a wholesale replacement of the classical basic-hypergeometric theory. Rather, it is the following specific mechanism: the degenerate multiplier $(\alpha)^{*}_{\lambda,n}$ has a rational coefficient ratio, and this ratio can be transported from the summation index to the argument variable by a rational functional calculus in the Euler operator. This produces a Heine-type operator adapted to the hybrid series. The same mechanism also clarifies which assertions are formal, which are analytic inside the disc of convergence, and which require termination.

The construction is mathematically useful because it converts a coefficient multiplier that is not itself a basic-hypergeometric factor into a diagonal rational operator in the Euler variable while retaining the q-shift structure of the underlying coefficient array. Thus one can transport classical q-difference calculations to the hybrid setting without disguising the new poles, the changed convergence radius, or the finite-moment obstruction. Relative to standard basic-hypergeometric theory, degenerate Pochhammer symbols, and earlier degenerate hypergeometric constructions, the principal novelty claimed here is the combined *transport-and-domain framework*: the rational Euler-transport equation is derived together with an exact boundary analysis and an explicit separation of formal, analytic, terminating-integral, and meromorphically regularised regimes. The transmutation operator and Beta-kernel formulas are structural consequences of that framework. Accordingly, the paper does not present the mere product of two known coefficient arrays as the main result; the new content lies in the operator mechanism and in the analytic corrections needed to make that mechanism mathematically well posed.

The main results can be summarised as follows:

- We define the hybrid normalised coefficient series (1) and state explicit nonsingularity assumptions for the q-denominator parameters, the degenerate Gamma factors, and the rational Euler-transport denominator.
- In the nonterminating case we prove absolute convergence for $|u| < \lambda$ and divergence for $|u| > \lambda$. On $|u| = \lambda$ we distinguish nonconvergence, absolute divergence, and conditional oscillatory convergence according to λ and the boundary phase.
- We encode the rational ratio $(\alpha)^*_{\lambda,n+1} / (\alpha)^*_{\lambda,n}$ as the transport operator $\mathcal{R}_{\lambda,\alpha}(\Theta)$ and derive the hybrid Heine-type equation first as a formal coefficient identity and then, under convergence, as an analytic identity.
- We establish numerator and denominator parameter-shift formulas. The denominator-shift relation is formulated in its correct nonsymmetric operator form.
- In the scalar ${}_2\mathrm{H}_1$ case we give a recurrence-based construction of two Frobenius-type local series. The associated connection coefficients are stated only for solutions known to lie in the corresponding nonresonant Frobenius span.
- We prove genuine degenerate-Gamma moment formulas for terminating series and give a regularised moment prescription for nonterminating series using meromorphic continuation of Beta/Gamma quotients.
- We derive a symmetric Beta-kernel identity, separating the complex analytic identity from the real positive-kernel statement.
- We record the limits $\lambda \to 0^+$ and $q \to 1^-$ with the necessary coefficient scaling and with a clear distinction between standard basic hypergeometric limits and mixed ordinary–basic limits.

The remainder of the paper is organized as follows. §2 fixes notation and collects the required facts from q-calculus and the degenerate Gamma function. §3 establishes convergence and boundary behaviour. §4 develops the transport operator viewpoint. §5 derives the hybrid Heine equation. §6 studies contiguity relations and ladder operators, while §7 focuses on the scalar case and a brief remark records the immediate base-$q^{1/2}$ substitution. Integral and regularised moment representations are obtained in §8, and limit transitions are recorded in §9. §10 provides a terminating-case illustration, and §11 concludes with a summary and further directions.

2. Preliminaries

This section sets notation and records the admissibility assumptions used throughout the paper.

For $n \in \mathbb{N}_0$, the q-shifted factorial is (see [5, §17.2])

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{m=0}^{n-1}(1-aq^m),$$

and $(a;q)_\infty := \prod_{m\geq 0}(1-aq^m)$ when it converges. We use the shorthand $(a;q)_n = (a;q)_n$ and $(a;q)_\infty = (a;q)_\infty$.

The fully standard basic hypergeometric series contains the convention-dependent factor

$$\left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r}.$$

To prevent confusion with the conventional ${}_r\phi_s$, we reserve that symbol for the fully standard series and write the coefficient array used in this paper as

$$\Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,u) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\,\frac{u^n}{(q;q)_n}. \tag{3}$$

Thus, (3) agrees with the usual ${}_{s+1}\phi_s$ normalisation, and in particular with the classical ${}_2\phi_1$ case, but for general (r,s) it is the distinct normalised series $\Phi^{\mathrm{norm}}_{r,s}$. This is the convention used in Gasper–Rahman [2], Andrews–Askey–Roy [4] and the DLMF [5] when $r = s+1$; outside that case it differs from the fully standard convention by the displayed factor.

For completeness, let $\kappa := 1+s-r$ and let $\mathsf{H}^{\mathrm{std}}$ denote the hybrid series obtained by inserting $[(-1)^n q^{n(n-1)/2}]^\kappa$ into every coefficient of (1). A direct coefficient calculation then replaces (15) by

$$\Big[(1-T_q)\prod_{k=1}^{s}(1-b_kq^{-1}T_q) - (-1)^\kappa u\,\mathcal{R}_{\lambda,\alpha}(\Theta)T_q^\kappa\prod_{j=1}^{r}(1-a_jT_q)\Big]\mathsf{H}^{\mathrm{std}}(u) = 0, \tag{4}$$

where $T_q^\kappa f(u) := f(q^\kappa u)$, also for negative integer κ. When $r = s+1$, one has $\kappa = 0$, so the standard and normalised equations coincide. All subsequent generic (r,s) formulas in this paper use $\Phi^{\mathrm{norm}}_{r,s}$ and the normalised hybrid convention unless explicitly stated otherwise.

For the ordinary hypergeometric notation used in the $q \to 1^-$ limiting discussion, we follow the classical references of Bailey [30] and Slater [31].

We use the standard q-shift and Euler operators on functions of u; see, for example, the operator notation in q-difference theory [32,33]:

$$(T_qf)(u) = f(qu), \qquad \Theta := u\frac{d}{du}.$$

On monomials u^n one has $T_q(u^n) = q^nu^n$ and $\Theta(u^n) = n\,u^n$.

Remark 1. Consistently with the diagonal action of the Euler operator on monomials and the functional-calculus conventions commonly used for q-difference operators [33], let $F = P/Q$ be rational and assume $Q(n) \neq 0$ for every Taylor exponent n under consideration. On formal series we define

$$F(\Theta)\Big(\sum_{n\geq 0}c_nu^n\Big) := \sum_{n\geq 0}F(n)c_nu^n,$$

so identities at this level are identities in $\mathbb{C}[[u]]$. If $f(u) = \sum c_nu^n$ has radius of convergence $R > 0$ and $Q(n) \neq 0$ for all $n \in \mathbb{N}_0$, then $F(n)$ has at most polynomial growth; hence $F(\Theta)f$ converges on the same open disc $|u| < R$ (and has the same radius unless the series terminates or an exceptional cancellation occurs). In particular, for $\mathcal{R}_{\lambda,\alpha}(x) = (\alpha+x)/[1-\lambda(\alpha+x+1)]$, the values $\mathcal{R}_{\lambda,\alpha}(n)$ tend to $-1/\lambda$, so $\mathcal{R}_{\lambda,\alpha}(\Theta)$ preserves the radius of every nonterminating Taylor series on which it is defined. No identity in this paper is to be read as an identity after meromorphic continuation unless that continuation is stated explicitly.

For $\lambda > 0$, the meromorphic continuation of the Kim–Kim degenerate Gamma function [21] motivates the pole set

$$\mathcal{P}_\lambda := \mathbb{Z}_{\leq 0} \cup \left(\frac{1}{\lambda} + \mathbb{Z}_{\geq 0}\right).$$

In the nonterminating case, the standing admissibility assumptions are

$$b_k \notin q^{-\mathbb{N}_0} \quad (1 \leq k \leq s), \qquad \alpha + n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+n+1) \neq 0 \quad (n \in \mathbb{N}_0).$$

For a terminating series of degree at most N, the finite admissibility conditions used throughout are

$$1-b_kq^m \neq 0 \quad (1 \leq k \leq s,\ 0 \leq m \leq N-1),$$

$$\alpha+n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+n+1) \neq 0 \quad (0 \leq n \leq N),$$

unless a later statement explicitly introduces a shifted parameter and therefore states its shifted conditions separately. These explicit assumptions replace the informal phrase that parameters "avoid singularities".

Following Kim–Kim [20,21], for $\lambda > 0$ and complex s in the strip

$$0 < \mathrm{Re}(s) < \frac{1}{\lambda}, \tag{5}$$

the degenerate Gamma function is defined by the absolutely convergent integral

$$\Gamma_\lambda(s) := \int_0^\infty (1+\lambda t)^{-1/\lambda}\, t^{s-1}\, dt. \tag{6}$$

It admits a meromorphic continuation in s with poles inherited from the Gamma factors in the Beta representation below; see [21] for details.

Lemma 1. *Assume $\lambda > 0$ and (5). Then*

$$\Gamma_\lambda(s) = \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right) = \lambda^{-s} \frac{\Gamma(s)\Gamma(\frac{1}{\lambda} - s)}{\Gamma(\frac{1}{\lambda})}. \tag{7}$$

Proof. Start from (6) and substitute $y = 1 + \lambda t$, so $t = (y-1)/\lambda$ and $dt = dy/\lambda$. This yields

$$\Gamma_\lambda(s) = \lambda^{-s} \int_1^\infty y^{-1/\lambda}(y-1)^{s-1}\, dy.$$

Next, set $y = (1-u)^{-1}$ with $u \in (0,1)$. Then $y - 1 = u/(1-u)$ and $dy = (1-u)^{-2}du$, and the integrand becomes $u^{s-1}(1-u)^{1/\lambda - s-1}$. Hence

$$\Gamma_\lambda(s) = \lambda^{-s} \int_0^1 u^{s-1}(1-u)^{1/\lambda-s-1}\, du = \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right),$$

and the Gamma quotient in (7) is Euler's Beta identity. □

Proposition 1. *For fixed $\lambda > 0$, the function $\Gamma_\lambda(s)$ extends meromorphically to $s \in \mathbb{C}$. Its poles are precisely at*

$$s \in \{0, -1, -2, \dots\} \cup \left\{\frac{1}{\lambda}, \frac{1}{\lambda} + 1, \frac{1}{\lambda} + 2, \dots\right\},$$

and it has no zeros in the strip (5).

Proof. By Lemma 1, for $0 < \Re(s) < \frac{1}{\lambda}$ we may write

$$\Gamma_\lambda(s) = \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right) = \lambda^{-s} \frac{\Gamma(s)\Gamma(\frac{1}{\lambda} - s)}{\Gamma(\frac{1}{\lambda})}. \tag{8}$$

The right-hand side is meromorphic on $\mathbb{C}$ as a product/quotient of the classical Gamma function and therefore provides the meromorphic continuation of $\Gamma_\lambda(s)$ to all $s \in \mathbb{C}$.

The factor $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \dots$ and no other singularities, while $\Gamma(\frac{1}{\lambda} - s)$ has simple poles at $\frac{1}{\lambda} - s = 0, -1, -2, \dots$, i.e. at $s = \frac{1}{\lambda}, \frac{1}{\lambda} + 1, \frac{1}{\lambda} + 2, \dots$. Since $\Gamma(\frac{1}{\lambda})$ is finite for $\lambda > 0$, no additional poles occur and the stated pole set follows.

Finally, in the strip (5) both s and $\frac{1}{\lambda} - s$ avoid non-positive integers, hence $\Gamma(s)$ and $\Gamma(\frac{1}{\lambda} - s)$ are finite and non-zero. As the classical Gamma function has no zeros, (8) implies $\Gamma_\lambda(s) \neq 0$ throughout (5). □

Corollary 1. *For $\lambda > 0$ and s such that s and $s+1$ avoid poles,*

$$\Gamma_\lambda(s+1) = \frac{s}{1 - \lambda(s+1)}\, \Gamma_\lambda(s). \tag{9}$$

Proof. Using (7) at s and $s+1$, and cancelling the common factor $\Gamma(\frac{1}{\lambda})^{-1}$, we find

$$\frac{\Gamma_\lambda(s+1)}{\Gamma_\lambda(s)} = \frac{\lambda^{-(s+1)}\Gamma(s+1)\Gamma(\frac{1}{\lambda} - s - 1)}{\lambda^{-s}\Gamma(s)\Gamma(\frac{1}{\lambda} - s)} = \frac{1}{\lambda}\, s\, \frac{\Gamma(\frac{1}{\lambda} - s - 1)}{\Gamma(\frac{1}{\lambda} - s)}.$$

Let $z = \frac{1}{\lambda} - s$. Since $\Gamma(z-1) = \Gamma(z)/(z-1)$,

$$\frac{\Gamma(\frac{1}{\lambda} - s - 1)}{\Gamma(\frac{1}{\lambda} - s)} = \frac{1}{\frac{1}{\lambda} - s - 1}.$$

Substituting this into the previous display yields

$$\frac{\Gamma_\lambda(s+1)}{\Gamma_\lambda(s)} = \frac{s}{\lambda(\frac{1}{\lambda} - s - 1)} = \frac{s}{1 - \lambda(s+1)},$$

which is (9). □

The degenerate rising factorial is defined by (2), in analogy with the ordinary Pochhammer symbol in hypergeometric notation [30,31] and its degenerate variants [24]. The next proposition records the recurrence that drives most computations.

Proposition 2. *Fix $\lambda > 0$. For the infinite sequence, assume for every $n \in \mathbb{N}_0$ that*

$$\alpha + n,\ \alpha + n + 1 \notin \mathcal{P}_\lambda, \qquad 1 - \lambda(\alpha + n + 1) \neq 0.$$

For identities used only through an index $N \in \mathbb{N}_0$, assume instead the same three conditions for each $0 \leq n \leq N$. Then, on the stated index range,

$$(\alpha)^*_{\lambda,0} = 1,$$

$$(\alpha)^*_{\lambda,n+1} = \frac{\alpha + n}{1 - \lambda(\alpha + n + 1)}\,(\alpha)^*_{\lambda,n}, \qquad n \geq 0, \tag{10}$$

$$(\alpha)^*_{\lambda,n} = \prod_{m=0}^{n-1} \frac{\alpha + m}{1 - \lambda(\alpha + m + 1)}. \tag{11}$$

Proof. Recall from (2) that, whenever the ratio is finite,

$$(\alpha)^*_{\lambda,n} = \frac{\Gamma_\lambda(\alpha + n)}{\Gamma_\lambda(\alpha)}.$$

Thus $(\alpha)^*_{\lambda,0} = 1$.
For $n \geq 0$, apply Corollary 1 with $s = \alpha + n$:

$$\Gamma_\lambda(\alpha + n + 1) = \frac{\alpha + n}{1 - \lambda(\alpha + n + 1)}\,\Gamma_\lambda(\alpha + n).$$

Dividing by $\Gamma_\lambda(\alpha)$ gives (10). Iterating (10) from $m = 0$ to $m = n - 1$ yields

$$(\alpha)^*_{\lambda,n} = \prod_{m=0}^{n-1} \frac{\alpha + m}{1 - \lambda(\alpha + m + 1)},$$

which is (11). □

Proposition 3. *Assume that at least one numerator parameter equals $a_j = q^{-N}$ for some $N \in \mathbb{N}_0$ and that*

$$1 - b_k q^m \neq 0 \quad (1 \leq k \leq s,\ 0 \leq m \leq N - 1),$$

$$\alpha + n \notin \mathcal{P}_\lambda \quad (0 \leq n \leq N).$$

Then, as in the standard termination mechanism for basic hypergeometric series [2,5], $(a_j; q)_n = 0$ for all $n \geq N + 1$, and the hybrid series (1), interpreted with the usual termination convention that only the coefficients $0 \leq n \leq N$ are evaluated, truncates to a polynomial in u of degree at most N. In particular, it converges for all $u \in \mathbb{C}$.

Proof. If $a_j = q^{-N}$, then the q-Pochhammer factor

$$(a_j;q)_n = \prod_{m=0}^{n-1}(1-q^{-N}q^m),$$

contains the vanishing term $1-q^{-N}q^N = 0$ whenever $n \geq N+1$. Hence the summand in (1) is identically zero for all $n \geq N+1$ and the series reduces to a finite sum. □

Remark 2. When $\lambda > 0$ and α lies in the strip (5), Lemma 1 gives

$$(\alpha)^*_{\lambda,n} = \lambda^{-n}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\frac{\Gamma(\frac{1}{\lambda}-\alpha-n)}{\Gamma(\frac{1}{\lambda}-\alpha)}.$$

Using $\Gamma(z-n)/\Gamma(z) = (-1)^n/(1-z)_n$, this can be rewritten as

$$(\alpha)^*_{\lambda,n} = \frac{(\alpha)_n}{(-\lambda)^n\,(\alpha+1-\frac{1}{\lambda})_n},$$

where $(\cdot)_n$ denotes the classical rising factorial. This form is convenient for analytic continuation and for asymptotic estimates in n.

3. Convergence of the hybrid series

We first determine the disc of absolute convergence of (1). Since the finite q-Pochhammer factors tend to nonzero limits under the generic nonterminating hypotheses, the radius is controlled by the degenerate ratio $(\alpha)^*_{\lambda,n+1}\,/\,(\alpha)^*_{\lambda,n}$.

Theorem 1. *Assume that (1) is nonterminating and that*

$$b_k \notin q^{-\mathbb{N}_0} \quad (1 \leq k \leq s), \qquad \alpha+n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+n+1) \neq 0 \quad (n \in \mathbb{N}_0).$$

Then the power series in u converges absolutely for $|u| < \lambda$ and diverges for $|u| > \lambda$.

Proof. Write (1) as $\mathsf{H}(u) = \sum_{n\geq 0} t_n(u)$ with

$$t_n(u) := (\alpha)^*_{\lambda,n}\,\frac{\prod_{j=1}^r (a_j;q)_n}{\prod_{k=1}^s (b_k;q)_n}\,\frac{u^n}{(q;q)_n}.$$

Using $(a;q)_{n+1} = (1-aq^n)(a;q)_n$, $(q;q)_{n+1} = (1-q^{n+1})(q;q)_n$, and Proposition 2, we obtain

$$\frac{t_{n+1}(u)}{t_n(u)} = \frac{\alpha+n}{1-\lambda(\alpha+n+1)}\,\frac{\prod_{j=1}^r(1-a_jq^n)}{(1-q^{n+1})\prod_{k=1}^s(1-b_kq^n)}\,u.$$

Because $0 < q < 1$, all finite q-Pochhammer ratio factors tend to 1. Hence

$$\lim_{n\to\infty}\left|\frac{t_{n+1}(u)}{t_n(u)}\right| = \frac{|u|}{\lambda}.$$

The ratio test gives absolute convergence for $|u| < \lambda$ and divergence for $|u| > \lambda$. In the terminating case the series is a polynomial and no convergence issue arises. □

Remark 3. The boundary $|u| = \lambda$ is not decided by the ratio test. Its behaviour depends on λ and on the phase of $u/(-\lambda)$.

Lemma 2. *Assume $\lambda > 0$ and*

$$\alpha+n \notin \mathcal{P}_\lambda \qquad (n \in \mathbb{N}_0),$$

so that $(\alpha)^*_{\lambda,n}$ is finite for every n. Then, as $n \to \infty$,

$$(-\lambda)^n \ (\alpha)^*_{\lambda,n} = \frac{\Gamma(\alpha+1-\frac{1}{\lambda})}{\Gamma(\alpha)} \, n^{\frac{1}{\lambda}-1}\Big(1+O\Big(\frac{1}{n}\Big)\Big). \tag{12}$$

Under the stated all-n pole-avoidance hypothesis, the displayed Gamma quotient is finite and nonzero.

Proof. By Remark 2 we may write, whenever the quotient is defined,

$$(-\lambda)^n \ (\alpha)^*_{\lambda,n} = \frac{(\alpha)_n}{(\alpha+1-\frac{1}{\lambda})_n} = \frac{\Gamma(\alpha+1-\frac{1}{\lambda})}{\Gamma(\alpha)} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1-\frac{1}{\lambda}+n)}.$$

The standard asymptotic expansion for Gamma quotients [43, §5.11(i)] gives

$$\frac{\Gamma(n+\beta)}{\Gamma(n+\gamma)} = n^{\beta-\gamma}\Big(1+O\Big(\frac{1}{n}\Big)\Big),$$

with $\beta = \alpha$ and $\gamma = \alpha+1-\frac{1}{\lambda}$. Moreover, $\Gamma(\alpha)$ is finite and nonzero because $\alpha \notin \mathcal{P}_\lambda$. If $\Gamma(\alpha+1-1/\lambda)$ had a pole, then $\alpha+1-1/\lambda = -m$ for some $m \in \mathbb{N}_0$, which would imply $\alpha+m+1 = 1/\lambda \in \mathcal{P}_\lambda$, contradicting the hypothesis at $n = m+1$. Hence the prefactor is finite and nonzero, and (12) follows. □

Proposition 4. *Assume $\lambda > 0$ and that the hybrid series (1) is nonterminating. Assume*

$$\alpha+n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+n+1) \neq 0 \quad (n \in \mathbb{N}_0),$$

and assume further that $(a_j;q)_\infty \neq 0$ and $(b_k;q)_\infty \neq 0$ for all j,k. Let $|u| = \lambda$ and put $\eta := u/(-\lambda)$.

(a) *If $0 < \lambda \leq 1$, then the series diverges at every boundary point.*
(b) *If $\lambda > 1$, then the series is never absolutely convergent on the boundary. It diverges for the nonoscillatory phase $\eta = 1$ and converges conditionally for $\eta \neq 1$.*

Proof. Let t_n denote the nth term of (1). The genericity assumption gives a nonzero constant

$$K := \frac{\prod_{j=1}^r (a_j;q)_\infty}{\prod_{k=1}^s (b_k;q)_\infty \ (q;q)_\infty},$$

such that

$$t_n = K \ (\alpha)^*_{\lambda,n} \ u^n\big(1+O(q^n)\big).$$

By Lemma 2,

$$t_n = KC_{\lambda,\alpha}\,\eta^n n^{\frac{1}{\lambda}-1}\Big(1+O\Big(\frac{1}{n}\Big)+O(q^n)\Big), \tag{13}$$

where $C_{\lambda,\alpha} = \Gamma(\alpha+1-1/\lambda)/\Gamma(\alpha)$ and $\eta = u/(-\lambda)$.

If $0 < \lambda < 1$, then $1/\lambda - 1 > 0$, so t_n does not tend to zero. If $\lambda = 1$, then $|t_n|$ tends to a nonzero limit. Thus the series diverges for all boundary phases when $0 < \lambda \leq 1$.

Assume now $\lambda > 1$. Then $\beta := 1/\lambda - 1$ lies in $(-1,0)$. Hence $|t_n| \asymp n^\beta$ and $\sum |t_n|$ diverges, so absolute convergence is impossible. If $\eta = 1$, (13) is a nonoscillatory constant multiple of $n^\beta(1+o(1))$, whose partial sums do not converge because $\beta > -1$. If $\eta \neq 1$, then $\sum\limits_{n\geq 1} \eta^n n^\beta$ converges by Dirichlet's test, while the error term in (13) is the sum of an absolutely convergent series $O(n^{\beta-1})$ and an exponentially small series. Therefore the boundary series converges conditionally for $\eta \neq 1$. □

Remark 4. Proposition 4 is deliberately restricted to the generic nonterminating regime. It does not cover a numerator parameter $a_j = q^{-N}$ (termination) or any parameter regime in which the nonzero limiting q-product assumptions fail. In such cases the first nonzero asymptotic term must be recomputed from the exact coefficient ratio. No boundary conclusion of Proposition 4 is asserted outside its stated hypotheses.

4. A transport principle

To derive a Heine-type operator equation for (1), we must handle the extra rational factor in the coefficient ratio coming from $(\alpha)^*_{\lambda,n+1} / (\alpha)^*_{\lambda,n}$. The idea is to move ("transport") this index dependence into an operator acting on the argument variable. We do this using the Euler operator $\Theta = u\frac{d}{du}$ and the functional calculus described in Remark 1.

The classical Heine operator is written in terms of the q-shift T_q. In the hybrid setting, the degenerate factor introduces a rational dependence on the summation index. The next definition packages this dependence as an operator acting on u.

Definition 1. Motivated by the Kim–Kim degenerate-Gamma shift relation [21], define the rational function

$$\mathcal{R}_{\lambda,\alpha}(x) := \frac{\alpha + x}{1 - \lambda(\alpha + x + 1)}.$$

On $\mathbb{C}[[u]]$, $\mathcal{R}_{\lambda,\alpha}(\Theta)$ is defined coefficientwise precisely when $1 - \lambda(\alpha + n + 1) \neq 0$ for every exponent n present in the series. On an analytic Taylor series of radius R, the same formula is analytic on $|u| < R$ by Remark 1. In particular, for the nonterminating hybrid series the common analytic disc used below is $|u| < \lambda$.

Lemma 3. *Assume $\alpha + n, \alpha + n + 1 \notin \mathcal{P}_\lambda$ and $1 - \lambda(\alpha + n + 1) \neq 0$ for the index n under consideration. Then*

$$\mathcal{R}_{\lambda,\alpha}(\Theta)\, u^n = \frac{(\alpha)^*_{\lambda,n+1}}{(\alpha)^*_{\lambda,n}}\, u^n. \tag{14}$$

Proof. For each $n \in \mathbb{N}_0$ one has $\Theta(u^n) = n\, u^n$, hence by the definition of the functional calculus

$$\mathcal{R}_{\lambda,\alpha}(\Theta)\, u^n = \mathcal{R}_{\lambda,\alpha}(n)\, u^n.$$

On the other hand, Proposition 2 gives

$$\frac{(\alpha)^*_{\lambda,n+1}}{(\alpha)^*_{\lambda,n}} = \frac{\alpha + n}{1 - \lambda(\alpha + n + 1)} = \mathcal{R}_{\lambda,\alpha}(n).$$

Combining the two identities proves (14). □

Remark 5. If α satisfies (5), then

$$\int_0^\infty t^n\, \frac{(1 + \lambda t)^{-1/\lambda} t^{\alpha-1}}{\Gamma_\lambda(\alpha)}\, dt = (\alpha)^*_{\lambda,n}\,,$$

holds only when $\alpha + n$ remains in the strip (5). In particular, for fixed $\lambda > 0$ only finitely many moments exist. For complex α this is a normalised complex weight, not a probability measure; it becomes a positive probability measure only for real $0 < \alpha < 1/\lambda$. Genuine moment representations are therefore exact only over the finite coefficient ranges for which the moments exist. For nonterminating series, (2) is interpreted via meromorphic continuation of Γ_λ, not via a single fixed measure with all moments; see [21].

To highlight the role of the degenerate parameter α, write

$$\mathsf{H}_\alpha(u) := \mathsf{H}\left[\begin{matrix}\mathbf{a} \\ \mathbf{b}\end{matrix};\, q,\, \lambda,\, \alpha;\, u\right].$$

Proposition 5. *Assume $\alpha \neq 0$. In the nonterminating case require, for every $n \in \mathbb{N}_0$,*

$$b_k \notin q^{-\mathbb{N}_0}\ (1 \le k \le s), \qquad \alpha + n,\ \alpha + 1 + n \notin \mathcal{P}_\lambda,$$

and

$$1 - \lambda(\alpha + n + 1) \neq 0, \qquad 1 - \lambda(\alpha + n + 2) \neq 0.$$

If both series terminate at degree at most N, assume explicitly

$$1 - b_k q^m \neq 0 \quad (1 \le k \le s,\ 0 \le m \le N-1),$$

$$\alpha + n,\ \alpha + 1 + n \notin \mathcal{P}_\lambda \quad (0 \le n \le N),$$

and

$$1 - \lambda(\alpha + n + 1) \neq 0, \qquad 1 - \lambda(\alpha + n + 2) \neq 0 \quad (0 \le n \le N).$$

Then

$$\mathsf{H}_{\alpha+1}(u) = \frac{1 - \lambda(\alpha+1)}{\alpha}\, \mathcal{R}_{\lambda,\alpha}(\Theta)\, \mathsf{H}_\alpha(u),$$

where $\mathcal{R}_{\lambda,\alpha}$ is the transport operator from Definition 1.

Proof. Using Corollary 1 at $s = \alpha$, we have $\Gamma_\lambda(\alpha+1) = \frac{\alpha}{1-\lambda(\alpha+1)}\Gamma_\lambda(\alpha)$, hence

$$\begin{aligned}(\alpha+1)^*_{\lambda,n} &= \frac{\Gamma_\lambda(\alpha+n+1)}{\Gamma_\lambda(\alpha+1)} \\ &= \frac{1-\lambda(\alpha+1)}{\alpha}\,\frac{\Gamma_\lambda(\alpha+n+1)}{\Gamma_\lambda(\alpha)} = \frac{1-\lambda(\alpha+1)}{\alpha}\,(\alpha)^*_{\lambda,n+1}.\end{aligned}$$

Insert this into (1) and use Lemma 3 to identify $\sum_{n\ge 0} (\alpha)^*_{\lambda,n+1}(\cdots)u^n$ with $\mathcal{R}_{\lambda,\alpha}(\Theta)\mathsf{H}_\alpha(u)$. □

5. A degenerate–q Heine equation

This section contains the main structural result: an operator q-difference equation satisfied by the hybrid series (1). We recall the classical Heine operator for ${}_r\phi_s$, in the standard basic-hypergeometric notation of Gasper–Rahman and the DLMF [2,5], and then insert the transport operator from §4 to obtain a degenerate–q Heine equation for H.

Let $\Phi_{\text{norm}}(u) = \Phi^{\text{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,u)$ denote the normalised basic coefficient series (3). A coefficient calculation gives

$$\Big[(1-T_q)\prod_{k=1}^{s}(1-b_k q^{-1}T_q) - u\prod_{j=1}^{r}(1-a_jT_q)\Big]\Phi_{\text{norm}}(u) = 0.$$

For $r = s+1$ this is the standard Heine-type equation for ${}_{s+1}\phi_s$ [2, Ch. 1]; for general (r,s) it is the operator equation associated with the distinct normalised series $\Phi^{\text{norm}}_{r,s}$, not a claim about the conventional ${}_r\phi_s$. The conventional-factor modification is the explicit $(-1)^\kappa T_q^\kappa$ term in (4). The hybrid series (1) differs from Φ_{norm} by $(\alpha)^*_{\lambda,n}$, and Lemma 3 converts its consecutive coefficient ratio into $\mathcal{R}_{\lambda,\alpha}(\Theta)$.

Theorem 2. *Let H be given by (1). In the nonterminating case assume*

$$b_k \notin q^{-\mathbb{N}_0} \quad (1 \le k \le s), \qquad \alpha + n \notin \mathcal{P}_\lambda, \qquad 1 - \lambda(\alpha+n+1) \neq 0 \quad (n \in \mathbb{N}_0).$$

If the series terminates at degree at most N, assume instead

$$1 - b_k q^m \neq 0 \quad (1 \le k \le s,\ 0 \le m \le N-1),$$

$$\alpha + n \notin \mathcal{P}_\lambda \quad (0 \le n \le N), \qquad 1 - \lambda(\alpha+n+1) \neq 0 \quad (0 \le n \le N).$$

Then

$$\Big[(1-T_q)\prod_{k=1}^{s}(1-b_kq^{-1}T_q) - u\,\mathcal{R}_{\lambda,\alpha}(\Theta)\prod_{j=1}^{r}(1-a_jT_q)\Big]\mathsf{H}(u) = 0. \tag{15}$$

The identity is first a formal identity in $\mathbb{C}[[u]]$. In the nonterminating case it is an analytic identity on $|u| < \lambda$; in the terminating case it is a polynomial identity and no convergence assumption is needed.

Proof. Write $\mathsf{H}(u) = \sum_{n \geq 0} c_n u^n$, where

$$c_n := (\alpha)^*_{\lambda,n} \frac{\prod_{j=1}^r (a_j;q)_n}{\prod_{k=1}^s (b_k;q)_n} \frac{1}{(q;q)_n}.$$

Since $T_q(u^n) = q^n u^n$, we have

$$(1-T_q)u^n = (1-q^n)u^n, \qquad (1-b_k q^{-1}T_q)u^n = (1-b_k q^{n-1})u^n.$$

Thus

$$\Big[(1-T_q)\prod_{k=1}^{s}(1-b_k q^{-1}T_q)\Big]u^n = (1-q^n)\prod_{k=1}^{s}(1-b_k q^{n-1})\,u^n.$$

For $n \geq 1$, Lemma 3 gives

$$u\,\mathcal{R}_{\lambda,\alpha}(\Theta)\prod_{j=1}^{r}(1-a_j T_q)\,u^{n-1} = \frac{(\alpha)^*_{\lambda,n}}{(\alpha)^*_{\lambda,n-1}}\prod_{j=1}^{r}(1-a_j q^{n-1})\,u^n.$$

Collecting the coefficient of u^n for $n \geq 1$ in the left-hand side of (15) gives

$$(1-q^n)\prod_{k=1}^{s}(1-b_k q^{n-1})\,c_n - \prod_{j=1}^{r}(1-a_j q^{n-1})\,\frac{(\alpha)^*_{\lambda,n}}{(\alpha)^*_{\lambda,n-1}}\,c_{n-1},$$

which vanishes by the defining coefficient ratio

$$\frac{c_n}{c_{n-1}} = \frac{(\alpha)^*_{\lambda,n}}{(\alpha)^*_{\lambda,n-1}} \frac{\prod_{j=1}^r (1-a_j q^{n-1})}{(1-q^n)\prod_{k=1}^s (1-b_k q^{n-1})}.$$

For $n = 0$, the first operator contributes $(1-q^0)c_0 = 0$, and the second term contains an explicit factor u; therefore it has no constant coefficient. Hence every coefficient is zero. The analytic assertions follow from Theorem 1 or from termination. □

Remark 6. If some $a_j = q^{-N}$ with $N \in \mathbb{N}_0$, then $(a_j;q)_n = 0$ for $n > N$ and the series terminates. In this case all computations are finite and require no convergence assumptions.

Remark 7. The coefficient-extraction proof of (15) mirrors the standard q-hypergeometric recurrences that underlie creative telescoping and symbolic summation. Recent algorithmic work on unified reduction and creative telescoping for hypergeometric and q-hypergeometric terms appears in [34]. It would be natural to ask whether such reduction frameworks extend to hybrid coefficients involving Γ_λ together with the rational transport operator introduced above.

6. Contiguity relations, Jackson q-derivatives, and ladder operators

We now record parameter-shift identities and Jackson q-derivative formulas that complement the hybrid Heine equation. These relations parallel the classical basic hypergeometric theory. They persist here because the degenerate factor $(\alpha)^*_{\lambda,n}$ does not depend on the base q; its n-dependence is handled by the transport operator.

We use the Jackson derivative (see, e.g., [32,35,36])

$$(D_q f)(u) := \frac{f(u)-f(qu)}{(1-q)u}, \qquad u \neq 0,$$

with $(D_q f)(0) := f'(0)$ when f is differentiable at 0.

Lemma 4. *For $n \in \mathbb{N}$ one has $D_q(u^n) = \frac{1-q^n}{1-q}\,u^{n-1}$, and $D_q(1) = 0$.*

Proof. For $n \geq 1$ we compute directly from the definition:

$$(D_q u^n)(u) = \frac{u^n - (qu)^n}{(1-q)u} = \frac{u^n - q^n u^n}{(1-q)u} = \frac{1-q^n}{1-q}\, u^{n-1}.$$

For $n = 0$ this reduces to $D_q(1) = 0$. □

Proposition 6. *In the nonterminating case assume, for every $n \in \mathbb{N}_0$,*

$$b_k \notin q^{-\mathbb{N}_0} \quad (1 \leq k \leq s), \qquad \alpha + n,\ \alpha + 1 + n \notin \mathcal{P}_\lambda,$$

$$1 - \lambda(\alpha + n + 1) \neq 0, \qquad 1 - \lambda(\alpha + n + 2) \neq 0.$$

For a terminating polynomial of degree at most N with $N \geq 1$, assume explicitly

$$b_k \neq 1 \quad (1 \leq k \leq s), \qquad 1 - b_k q^m \neq 0 \quad (1 \leq k \leq s,\ 0 \leq m \leq N-1),$$

$$\alpha + n \notin \mathcal{P}_\lambda \quad (0 \leq n \leq N), \qquad \alpha + 1 + m \notin \mathcal{P}_\lambda \quad (0 \leq m \leq N-1),$$

and

$$1 - \lambda(\alpha + n + 1) \neq 0 \quad (0 \leq n \leq N), \qquad 1 - \lambda(\alpha + m + 2) \neq 0 \quad (0 \leq m \leq N-1).$$

Then

$$D_q \,\mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; q, \lambda, \alpha; u\right] = \frac{(\alpha)^*_{\lambda,1}}{1-q} \frac{\prod_{j=1}^r (1-a_j)}{\prod_{k=1}^s (1-b_k)}\, \mathsf{H}\left[\begin{matrix}q\mathbf{a}\\ q\mathbf{b}\end{matrix}; q, \lambda, \alpha+1; u\right],$$

where $q\mathbf{a} = (qa_1, \ldots, qa_r)$ and $q\mathbf{b} = (qb_1, \ldots, qb_s)$.

Proof. Write (1) as $\mathsf{H}(u) = \sum\limits_{n \geq 0} c_n u^n$. By Lemma 4,

$$\begin{aligned} D_q \mathsf{H}(u) &= \sum_{n \geq 1} c_n \frac{1-q^n}{1-q}\, u^{n-1} \\ &= \frac{1}{1-q} \sum_{n \geq 1} (\alpha)^*_{\lambda,n} \frac{\prod_{j=1}^r (a_j;q)_n}{\prod_{k=1}^s (b_k;q)_n} \frac{u^{n-1}}{(q;q)_{n-1}}, \end{aligned}$$

since $(1-q^n)/(q;q)_n = 1/(q;q)_{n-1}$. Set $m = n-1$ and use the identities $(a;q)_{m+1} = (1-a)(aq;q)_m$ and $(\alpha)^*_{\lambda,m+1} = (\alpha)^*_{\lambda,1}\,(\alpha+1)^*_{\lambda,m}$ to factor out constants. The remaining series is precisely the hybrid series with parameters $(q\mathbf{a}, q\mathbf{b}, \alpha+1)$. □

Contiguity relations follow from elementary q-Pochhammer manipulations and are classical in the basic hypergeometric setting; see, e.g., [2,8].

Proposition 7. *Fix $\ell \in \{1, \ldots, r\}$ and assume $a_\ell \neq 1$. In the nonterminating case assume $b_k \notin q^{-\mathbb{N}_0}$ for all k and, for every $n \in \mathbb{N}_0$, $\alpha + n \notin \mathcal{P}_\lambda$ and $1 - \lambda(\alpha + n + 1) \neq 0$. In a terminating case of degree at most N, assume explicitly*

$$1 - b_k q^m \neq 0 \quad (1 \leq k \leq s,\ 0 \leq m \leq N-1),$$

$$\alpha + n \notin \mathcal{P}_\lambda, \qquad 1 - \lambda(\alpha + n + 1) \neq 0 \quad (0 \leq n \leq N).$$

Let $\mathbf{a}^{(\ell)}$ be obtained from $\mathbf{a}$ by replacing a_ℓ with qa_ℓ. Then

$$(1-a_\ell)\,\mathsf{H}\left[\begin{matrix}\mathbf{a}^{(\ell)}\\ \mathbf{b}\end{matrix}; q, \lambda, \alpha; u\right] = \mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; q, \lambda, \alpha; u\right] - a_\ell\, \mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; q, \lambda, \alpha; qu\right]. \tag{16}$$

Equivalently,

$$(1-a_\ell)\,\mathsf{H}(\mathbf{a}^{(\ell)}; \mathbf{b}; u) = (1 - a_\ell T_q)\,\mathsf{H}(\mathbf{a}; \mathbf{b}; u). \tag{17}$$

Proof. Write the defining series (1) as

$$\mathsf{H}(\mathbf{a};\mathbf{b};u)=\sum_{n=0}^{\infty}c_n(\mathbf{a},\mathbf{b})\,u^n,\qquad c_n:=(\alpha)^*_{\lambda,n}\,\frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\,\frac{1}{(q;q)_n}.$$

If only the ℓ-th numerator parameter is replaced by qa_ℓ, then

$$\frac{(qa_\ell;q)_n}{(a_\ell;q)_n}=\prod_{m=0}^{n-1}\frac{1-a_\ell q^{m+1}}{1-a_\ell q^m}=\frac{1-a_\ell q^n}{1-a_\ell}.$$

Therefore

$$\begin{aligned}(1-a_\ell)\,\mathsf{H}(\mathbf{a}^{(\ell)};\mathbf{b};u)&=\sum_{n=0}^{\infty}c_n(\mathbf{a},\mathbf{b})\,(1-a_\ell q^n)\,u^n\\&=\sum_{n=0}^{\infty}c_nu^n-a_\ell\sum_{n=0}^{\infty}c_n(q^nu^n).\end{aligned}$$

Since $q^nu^n=(qu)^n$, the right-hand side equals $\mathsf{H}(\mathbf{a};\mathbf{b};u)-a_\ell\,\mathsf{H}(\mathbf{a};\mathbf{b};qu)$, which is (16). The operator form (17) is the same identity written with $T_qf(u)=f(qu)$. □

Proposition 8. *Fix $k\in\{1,\dots,s\}$. In the nonterminating case assume $b_j\notin q^{-\mathbb{N}_0}$ for every $1\le j\le s$ (hence also $b_k\neq 1$ and $qb_k\notin q^{-\mathbb{N}_0}$), and assume for every $n\in\mathbb{N}_0$ that $\alpha+n\notin\mathcal{P}_\lambda$ and $1-\lambda(\alpha+n+1)\neq 0$. If the series terminate at degree at most N, assume explicitly*

$$\begin{aligned}1-b_jq^m&\neq 0 &\qquad&(1\le j\le s,\ 0\le m\le N-1),\\1-b_kq^{m+1}&\neq 0&&(0\le m\le N-1).\end{aligned}$$

and

$$\alpha+n\notin\mathcal{P}_\lambda,\qquad 1-\lambda(\alpha+n+1)\neq 0\quad(0\le n\le N).$$

Let $\mathbf{b}^{(k)}$ be obtained from $\mathbf{b}$ by replacing b_k with qb_k. Then

$$(1-b_kT_q)\,\mathsf{H}\left[\begin{matrix}\mathbf{a}\\\mathbf{b}^{(k)}\end{matrix};q,\lambda,\alpha;u\right]=(1-b_k)\,\mathsf{H}\left[\begin{matrix}\mathbf{a}\\\mathbf{b}\end{matrix};q,\lambda,\alpha;u\right].\tag{18}$$

Equivalently,

$$\mathsf{H}(\mathbf{a};\mathbf{b}^{(k)};u)-b_k\mathsf{H}(\mathbf{a};\mathbf{b}^{(k)};qu)=(1-b_k)\mathsf{H}(\mathbf{a};\mathbf{b};u).\tag{19}$$

Proof. Write

$$\mathsf{H}(\mathbf{a};\mathbf{b};u)=\sum_{n=0}^{\infty}c_n(\mathbf{a},\mathbf{b})\,u^n,\qquad c_n(\mathbf{a},\mathbf{b}):=(\alpha)^*_{\lambda,n}\,\frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{\ell=1}^{s}(b_\ell;q)_n}\,\frac{1}{(q;q)_n}.$$

Using

$$(qb_k;q)_n=(b_k;q)_n\,\frac{1-b_kq^n}{1-b_k},$$

the coefficient $\widetilde{c}_n$ of u^n in $\mathsf{H}(\mathbf{a};\mathbf{b}^{(k)};u)$ satisfies

$$\widetilde{c}_n=c_n(\mathbf{a},\mathbf{b})\,\frac{1-b_k}{1-b_kq^n}.$$

Applying $1-b_kT_q$ multiplies the nth coefficient of $\mathsf{H}(\mathbf{a};\mathbf{b}^{(k)};u)$ by $1-b_kq^n$. Hence

$$(1-b_kq^n)\widetilde{c}_n=(1-b_k)c_n(\mathbf{a},\mathbf{b}),$$

which proves (18) coefficient by coefficient. Formula (19) is the same identity written out in values of the function. □

Introduce the first-order q-shift operator (cf. [33,37])

$$L_c := 1 - cT_q.$$

Then the hybrid Heine Eq. (15) can be written compactly as

$$\Big[(1-T_q)\prod_{k=1}^{s} L_{b_k/q} - u\,\mathcal{R}_{\lambda,\alpha}(\Theta)\prod_{j=1}^{r} L_{a_j}\Big]\mathsf{H}(u) = 0. \tag{20}$$

In particular, the rightmost factor $\prod_{j=1}^{r} L_{a_j}$ can be eliminated in favour of parameter shifts using (17):

Corollary 2. *Assume $a_j \neq 1$ for all j. In the nonterminating case assume $b_k \notin q^{-\mathbb{N}_0}$ for $1 \le k \le s$ and, for every $n \in \mathbb{N}_0$, $\alpha + n \notin \mathcal{P}_\lambda$ and $1 - \lambda(\alpha + n + 1) \neq 0$. If the series terminates at degree at most N, assume $1 - b_k q^m \neq 0$ for $1 \le k \le s$, $0 \le m \le N-1$, and $\alpha + n \notin \mathcal{P}_\lambda$, $1 - \lambda(\alpha+n+1) \neq 0$ for $0 \le n \le N$. Then*

$$\prod_{j=1}^{r} L_{a_j}\,\mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix};q,\lambda,\alpha;u\right] = \Big(\prod_{j=1}^{r}(1-a_j)\Big)\,\mathsf{H}\left[\begin{matrix}q\mathbf{a}\\ \mathbf{b}\end{matrix};q,\lambda,\alpha;u\right].$$

Proof. Recall that $L_{a_j} = 1 - a_j T_q$. Applying (17) with $\ell = 1$ yields

$$L_{a_1}\,\mathsf{H}(\mathbf{a};\mathbf{b};u) = (1-a_1)\,\mathsf{H}(qa_1, a_2, \dots, a_r; \mathbf{b}; u).$$

Apply the same identity to the right-hand side with $\ell = 2$, and continue inductively. After r steps one obtains

$$\Big(\prod_{j=1}^{r} L_{a_j}\Big)\mathsf{H}(\mathbf{a};\mathbf{b};u) = \Big(\prod_{j=1}^{r}(1-a_j)\Big)\,\mathsf{H}(q\mathbf{a};\mathbf{b};u),$$

as claimed. □

Denominator shifts must be treated differently. Proposition 8 gives

$$L_{b_k}\,\mathsf{H}(\mathbf{a};\mathbf{b}^{(k)};u) = (1-b_k)\mathsf{H}(\mathbf{a};\mathbf{b};u),$$

so the operator L_{b_k} naturally acts on the series with the shifted denominator parameter qb_k, not on the unshifted series. Thus numerator and denominator shift eliminations are not symmetric. The factorised form (20) should be used with this distinction in mind.

7. The scalar case $r = s = 1$: Frobenius-type series and connection coefficients

We now specialise to the scalar case $r = s = 1$ with parameters a, b. The equation is second order in the q-shift and contains the diagonal Euler-transport operator. Because $\mathcal{R}_{\lambda,\alpha}(\Theta)$ is a rational functional-calculus operator rather than a finite-order differential operator, we formulate the local theory in a specific Frobenius class, following the usual indicial-exponent philosophy for local q-difference equations [2,33], rather than claiming an unrestricted solution-space theorem.

For $r = s = 1$ the hybrid Heine Eq. (15) reads

$$\Big[(1-T_q)(1-bq^{-1}T_q) - u\,\mathcal{R}_{\lambda,\alpha}(\Theta)(1-aT_q)\Big]f(u) = 0. \tag{21}$$

Definition 2. Fix once and for all a value $\operatorname{Log} b$ of the logarithm of $b \neq 0$, and set

$$\rho := 1 - \frac{\operatorname{Log} b}{\log q}, \qquad q^\rho := e^{\rho \log q} = \frac{q}{b}.$$

Choose a simply connected punctured sector $S_{\delta,I} := \{u : 0 < |u| < \delta,\ \arg u \in I\}$ and one branch $\operatorname{Log} u$ on that sector; throughout the local analysis, u^ρ means $e^{\rho \operatorname{Log} u}$. These choices fix the branch of the second Frobenius exponent and are not changed within a local statement.

Remark 8. On the fixed branch sector from Definition 2, the diagonal functional calculus extends to a Frobenius series $u^\rho g(u) = u^\rho \sum_{n\geq 0} d_n u^n$ by

$$F(\Theta)\left(u^\rho \sum_{n\geq 0} d_n u^n\right) := u^\rho \sum_{n\geq 0} F(\rho+n) d_n u^n,$$

whenever the values $F(\rho+n)$ are defined. In particular,

$$\mathcal{R}_{\lambda,\alpha}(\Theta)(u^\rho g) = u^\rho \mathcal{R}_{\lambda,\alpha+\rho}(\Theta) g.$$

Thus the calculations below use an explicitly defined diagonal action on the prescribed Frobenius class, not an unstated extension of the Taylor-series functional calculus.

Lemma 5. *Seek a Frobenius-type solution $f(u) = u^\rho(1+O(u))$ as $u \to 0$, with a fixed branch of u^ρ. Then the indicial equation associated with (21) is*

$$(1-q^\rho)(1-bq^{\rho-1}) = 0.$$

Thus the chosen exponents are $\rho_0 = 0$ and $\rho_1 = \rho$ from Definition 2.

Proof. Substituting $f(u) = u^\rho$ into the part of (21) not multiplied by u gives

$$(1-T_q)(1-bq^{-1}T_q)u^\rho = (1-q^\rho)(1-bq^{\rho-1})u^\rho.$$

The term $u\mathcal{R}_{\lambda,\alpha}(\Theta)(1-aT_q)u^\rho$ has order $u^{\rho+1}$, so the displayed coefficient must vanish at the lowest order. □

Lemma 6. *Let $f(u) = u^\rho g(u)$ with $g(u) = \sum_{n\geq 0} d_n u^n$ and $d_0 \neq 0$. Assume explicitly that*

$$1-\lambda(\alpha+\rho+n) \neq 0 \qquad (n \geq 1),$$

so that $\mathcal{R}_{\lambda,\alpha+\rho}(n-1)$ is defined. Then substitution into (21) gives

$$(1-q^{\rho+n})(1-bq^{\rho+n-1})d_n = \mathcal{R}_{\lambda,\alpha+\rho}(n-1)(1-aq^{\rho+n-1})d_{n-1} \qquad (n \geq 1). \tag{22}$$

To solve successively for d_n, one additionally requires $(1-q^{\rho+n})(1-bq^{\rho+n-1}) \neq 0$ for every $n \geq 1$. For $n = 0$ the lowest-order condition is exactly the indicial equation of Lemma 5.

Proof. The identities

$$T_q(u^\rho g) = q^\rho u^\rho T_q g, \qquad \Theta(u^\rho g) = u^\rho(\Theta+\rho)g,$$

transform (21) into

$$\Big[(1-q^\rho T_q)(1-bq^{\rho-1}T_q) - u\mathcal{R}_{\lambda,\alpha+\rho}(\Theta)(1-aq^\rho T_q)\Big]g = 0. \tag{23}$$

Taking the coefficient of u^n in (23) gives (22). □

The exponent $\rho_0 = 0$ yields the analytic solution $f_0 = \mathsf{H}$. For the second exponent use the fixed ρ of Definition 2; then $bq^{\rho-1} = 1$, and (23) becomes

$$\Big[(1-q^\rho T_q)(1-T_q) - u\mathcal{R}_{\lambda,\alpha+\rho}(\Theta)(1-aq^\rho T_q)\Big]g = 0.$$

Since the two factors in T_q commute, this has the scalar hybrid form with numerator parameter aq^ρ, denominator parameter $q^{\rho+1}$, and degenerate parameter $\alpha+\rho$.

Theorem 3. *Fix the branches in Definition 2. Assume*

$$b \notin q^{-\mathbb{N}_0}, \qquad \alpha+n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+n+1) \neq 0 \qquad (n \in \mathbb{N}_0),$$

$$q^{\rho+1} \notin q^{-\mathbb{N}_0}, \qquad \alpha+\rho+n \notin \mathcal{P}_\lambda, \qquad 1-\lambda(\alpha+\rho+n+1) \neq 0 \qquad\qquad (n \in \mathbb{N}_0),$$

and impose the precise nonresonance condition

$$q^\rho \notin q^{\mathbb{Z}}.$$

For any $0<\delta<\lambda$, on the fixed sector $S_{\delta,I}$ the Eq. (21) admits the two Frobenius-type local series

$$f_0(u) = \mathsf{H}\left[\begin{matrix} a \\ b \end{matrix}; q, \lambda, \alpha; u\right],$$

$$f_1(u) = u^\rho\, \mathsf{H}\left[\begin{matrix} aq^\rho \\ q^{\rho+1} \end{matrix}; q, \lambda, \alpha+\rho; u\right].$$

They are linearly independent in the explicitly fixed local Frobenius class

$$\mathcal{F}_\rho(S_{\delta,I}) := \{g_0(u) + u^\rho g_1(u) : g_0, g_1 \text{ are holomorphic on } |u|<\delta\}.$$

In the two-dimensional span $\operatorname{span}\{f_0, f_1\} \subset \mathcal{F}_\rho(S_{\delta,I})$*, the coefficients c_0, c_1 are unique. No assertion is made that this span is the full solution space in a larger analytic or meromorphic function class.*

Proof. The first series solves (21) by Theorem 2. The preceding conjugation shows that the second factor g must solve the scalar hybrid equation with parameters $(aq^\rho, q^{\rho+1}, \alpha+\rho)$, so the displayed hybrid series gives a formal solution. Under the same nonterminating convergence hypothesis as in Theorem 1, this series is analytic for $|u|<\lambda$; in the terminating case it is a polynomial. The nonresonance condition separates the leading behaviours $1+O(u)$ and $u^\rho(1+O(u))$, so the two Frobenius series are linearly independent and the coefficients in their span are unique. □

Proposition 9. *Assume all hypotheses of Theorem 3. Let f be a solution known a priori to belong to the local Frobenius span* $\operatorname{span}\{f_0, f_1\}$ *on the fixed sector $S_{\delta,I}$. Fix $u_0 \neq 0$ sufficiently small so that f_0, f_1 are defined at u_0 and qu_0, and define*

$$W(u_0) := f_0(u_0) f_1(qu_0) - f_0(qu_0) f_1(u_0).$$

For all sufficiently small admissible u_0 with $W(u_0) \neq 0$, the coefficients in $f = c_0 f_0 + c_1 f_1$ are

$$c_0 = \frac{f(u_0) f_1(qu_0) - f(qu_0) f_1(u_0)}{W(u_0)}, \qquad c_1 = \frac{f_0(u_0) f(qu_0) - f_0(qu_0) f(u_0)}{W(u_0)}.$$

These c_0, c_1 are local decomposition coefficients in the prescribed Frobenius span; they are not global connection coefficients between analytically continued bases.

Proof. Evaluating $f = c_0 f_0 + c_1 f_1$ at u_0 and qu_0 gives a 2×2 linear system whose determinant is $W(u_0)$. Cramer's rule gives the displayed formulas. Moreover,

$$W(u_0) = u_0^\rho\big(q^\rho - 1 + O(u_0)\big),$$

with the chosen branch of u^ρ. The nonresonance assumption implies $q^\rho \neq 1$, so $W(u_0)$ is nonzero for all sufficiently small u_0 with $u_0, qu_0 \in S_{\delta,I}$ outside possible isolated zeros. □

Remark 9. Proposition 4 shows that, in the generic nonterminating regime, the boundary $|u| = \lambda$ requires phase-sensitive analysis. The local power-series solutions above should therefore be used inside the disc of convergence unless a separate analytic-continuation argument is supplied.

Remark 10 (Base-$q^{1/2}$ substitution)**.** Let $\tilde{q} := q^{1/2} \in (0,1)$ and $(T_{q^{1/2}} f)(u) := f(\tilde{q}u)$. The scalar base-$\tilde{q}$ formula is only the immediate substitution $q \mapsto \tilde{q}$ in Theorem 2, not an additional structural result. Explicitly, if

$$\mathsf{H}_{1,1}(u) = \mathsf{H}\left[\begin{matrix} a \\ b \end{matrix}; \tilde{q}, \lambda, \alpha; u\right],$$

if $b \notin \tilde{q}^{-\mathbb{N}_0}$, and if $\alpha + n \notin \mathcal{P}_\lambda$ and $1 - \lambda(\alpha + n + 1) \neq 0$ for every $n \in \mathbb{N}_0$ in the nonterminating case, then the following identity holds. In a terminating case of degree at most N, replace these hypotheses by $1 - b\tilde{q}^m \neq 0$ for $0 \leq m \leq N-1$ and by $\alpha + n \notin \mathcal{P}_\lambda$, $1 - \lambda(\alpha+n+1) \neq 0$ for $0 \leq n \leq N$:

$$\left(1 - T_{q^{1/2}}\right)\left(1 - b\tilde{q}^{-1} T_{q^{1/2}}\right)\mathsf{H}_{1,1}(u) - u\,\mathcal{R}_{\lambda,\alpha}(\Theta)\left(1 - aT_{q^{1/2}}\right)\mathsf{H}_{1,1}(u) = 0.$$

No additional identity involving fractional powers of $T_{q^{1/2}}$ is claimed.

8. Integral representations

Integral representations are useful both conceptually and technically, but in the present setting they must be separated carefully from formal and regularised coefficient identities. For fixed $\lambda > 0$, the Kim–Kim degenerate-Gamma density has only finitely many finite moments, because the defining integral requires $0 < \mathrm{Re}(\alpha + n) < 1/\lambda$. Consequently, genuine positive-measure moment representations are available without qualification only for finite coefficient ranges, most notably in the terminating case. For nonterminating series we use meromorphic continuation of Beta/Gamma quotients as a regularised coefficient prescription, not as a literal integral against a fixed positive measure. Finally, we obtain a symmetric kernel representation for the degenerate Beta function, with positivity stated only in the real convergent regime.

Assume $\lambda > 0$ and $0 < \mathrm{Re}(\alpha) < 1/\lambda$. For $t > 0$ take $t^{\alpha-1} := e^{(\alpha-1)\log t}$ with the real logarithm and define the normalised weight

$$d\mu_{\lambda,\alpha}(t) = \frac{(1+\lambda t)^{-1/\lambda}\, t^{\alpha-1}}{\Gamma_\lambda(\alpha)}\, dt. \tag{24}$$

Its total mass is 1. If α is real with $0 < \alpha < 1/\lambda$, this is a positive probability density. For nonreal α it is a convergent complex weight (equivalently, a complex measure of finite variation), not a probability measure.

Lemma 7. *Assume $\lambda > 0$, $0 < \mathrm{Re}(\alpha) < 1/\lambda$, and $0 < \mathrm{Re}(\alpha + n) < 1/\lambda$. Then*

$$\int_0^\infty t^n\, d\mu_{\lambda,\alpha}(t) = (\alpha)^*_{\lambda,n}. \tag{25}$$

For real $0 < \alpha < 1/\lambda$ this is an ordinary probability moment; for nonreal α it is a moment of the complex weight (24)*.*

Proof. By the defining integral (6), for $\alpha + n$ in the strip (5) one has

$$\Gamma_\lambda(\alpha+n) = \int_0^\infty (1+\lambda t)^{-1/\lambda}\, t^{\alpha+n-1}\, dt.$$

Dividing by $\Gamma_\lambda(\alpha)$ and using $t^{\alpha+n-1} = t^n t^{\alpha-1}$ yields

$$\frac{\Gamma_\lambda(\alpha+n)}{\Gamma_\lambda(\alpha)} = \int_0^\infty t^n\, \frac{(1+\lambda t)^{-1/\lambda} t^{\alpha-1}}{\Gamma_\lambda(\alpha)}\, dt = \int_0^\infty t^n\, d\mu_{\lambda,\alpha}(t).$$

The left-hand side equals $(\alpha)^*_{\lambda,n}$ by (2), proving (25). □

Theorem 4. *Assume $\lambda > 0$, $0 < \mathrm{Re}(\alpha) < 1/\lambda$, and that* (1) *terminates at $n = N$. Assume $0 < \mathrm{Re}(\alpha + n) < 1/\lambda$ for every $0 \leq n \leq N$ and $1 - b_k q^m \neq 0$ for all $1 \leq k \leq s$ and $0 \leq m \leq N-1$. Then*

$$\mathsf{H}\left[\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}; q, \lambda, \alpha; u\right] = \int_0^\infty \Phi^{\mathrm{norm}}_{r,s}(\mathbf{a}; \mathbf{b}; q, ut)\, d\mu_{\lambda,\alpha}(t), \tag{26}$$

where $\Phi^{\mathrm{norm}}_{r,s}$ denotes the normalised basic coefficient series (3); in the present case it is a polynomial. The integral is against the complex weight (24) in general and against a probability measure only when α is real with $0 < \alpha < 1/\lambda$.

Proof. Since the series terminates at $n = N$, $\Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,ut)$ is a polynomial and can be written as

$$\Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,ut) = \sum_{n=0}^{N} \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\,\frac{(ut)^n}{(q;q)_n}.$$

Therefore,

$$\int_0^\infty \Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,ut)\,d\mu_{\lambda,\alpha}(t) = \sum_{n=0}^{N} \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n}\,\frac{u^n}{(q;q)_n}\int_0^\infty t^n\,d\mu_{\lambda,\alpha}(t),$$

where exchanging summation and integration is justified because the sum is finite. By Lemma 7, $\int_0^\infty t^n\,d\mu_{\lambda,\alpha}(t) = (\alpha)^*_{\lambda,n}$, and the right-hand side coincides termwise with (1). This proves (26). □

Remark 11. For nonterminating series, (26) can still be read *formally* at the level of power-series coefficients, but the integral on the right need not exist: for fixed $\lambda > 0$ the measure $\mu_{\lambda,\alpha}$ has only finitely many moments. See Remark 5.

Remark 12. Integral representations for basic hypergeometric functions and their degenerations have regained attention as tools for analytic continuation, asymptotics, and functional identities. See, for example, Cohl–Costas-Santos [38] in the classical q-setting, Heragy–Mansour–Oraby [39] for q-Riccati approaches to indefinite Jackson integrals, and the multiple Jackson integral representation of Nobukawa [40]. Recent transformation and identity results for q-series, including those of Liu [41] and Krattenthaler–Zudilin [42], further illustrate the continuing role of basic-hypergeometric structure in current work. Our Beta-kernel and degenerate-moment formulas can be viewed as hybrid counterparts in which the measure component is deformed by Γ_λ.

Remark 5 explains why the raw moment method cannot literally produce (26) for a nonterminating hybrid series: the measure $\mu_{\lambda,\alpha}$ has only finitely many finite moments for fixed $\lambda > 0$. Nevertheless, the moment identity can be extended in a standard analytic sense by interpreting the divergent moments through meromorphic continuation of the Beta representation of Γ_λ.

Definition 3. Using the meromorphic Beta/Gamma continuation of the degenerate Gamma function [21] together with the classical Gamma-function notation [43], for $\lambda > 0$ and complex s, define the *regularised* degenerate moment by

$$\mathfrak{M}_\lambda(s) := \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right) = \lambda^{-s}\frac{\Gamma(s)\Gamma(\frac{1}{\lambda} - s)}{\Gamma(\frac{1}{\lambda})},$$

i.e. by the meromorphic continuation of the right-hand side of Lemma 1. Outside the fundamental strip $0 < \mathrm{Re}(s) < 1/\lambda$, the symbol $\mathfrak{M}_\lambda(s)$ is a meromorphic coefficient prescription and is not called an integral moment.

Lemma 8. *For every $n \in \mathbb{N}_0$ such that α and $\alpha + n$ avoid the poles of Γ_λ one has*

$$(\alpha)^*_{\lambda,n} = \frac{\mathfrak{M}_\lambda(\alpha + n)}{\mathfrak{M}_\lambda(\alpha)}.$$

Proof. On the fundamental strip (5), Lemma 1 identifies $\Gamma_\lambda(s)$ with the Beta/Gamma quotient defining $\mathfrak{M}_\lambda(s)$, hence for $\alpha, \alpha + n$ in the strip,

$$(\alpha)^*_{\lambda,n} = \frac{\Gamma_\lambda(\alpha + n)}{\Gamma_\lambda(\alpha)} = \frac{\mathfrak{M}_\lambda(\alpha + n)}{\mathfrak{M}_\lambda(\alpha)}.$$

Both sides are meromorphic in α, and they agree on a non-empty open set. The identity theorem for meromorphic functions therefore extends the equality to all α for which the expressions are defined. □

Theorem 5 (Regularised coefficient prescription)**.** *Assume $\lambda > 0$, $b_k \notin q^{-\mathbb{N}_0}$ for $1 \le k \le s$, and, for every $n \in \mathbb{N}_0$, assume $\alpha + n \notin \mathcal{P}_\lambda$ and $1 - \lambda(\alpha + n + 1) \neq 0$. Define, for $|u| < \lambda$, the coefficient prescription*

$$\mathcal{I}(u) := \frac{1}{\mathfrak{M}_\lambda(\alpha)} \sum_{n=0}^{\infty} \mathfrak{M}_\lambda(\alpha + n) \frac{\prod_{j=1}^{r} (a_j; q)_n}{\prod_{k=1}^{s} (b_k; q)_n} \frac{u^n}{(q; q)_n}. \tag{27}$$

Then $\mathcal{I}(u)$ coincides term-by-term with the hybrid series (1) and hence $\mathcal{I}(u) = \mathsf{H}(\mathbf{a}; \mathbf{b}; u)$ for $|u| < \lambda$. This is a meromorphic regularisation of coefficients; it is a genuine integral representation only in regimes where the required moments exist, in particular in the terminating situation of Theorem 4. No nonterminating ordinary integral is asserted.

Proof. Expanding (27) as a power series in u, the coefficient of u^n equals

$$\frac{\mathfrak{M}_\lambda(\alpha + n)}{\mathfrak{M}_\lambda(\alpha)} \frac{\prod_{j=1}^{r} (a_j; q)_n}{\prod_{k=1}^{s} (b_k; q)_n} \frac{1}{(q; q)_n}.$$

By Lemma 8 this ratio is exactly $(\alpha)^*_{\lambda,n}$, so (27) coincides term-by-term with the defining hybrid series (1). Hence $\mathcal{I}(u) = \mathsf{H}(\mathbf{a}; \mathbf{b}; u)$ for $|u| < \lambda$.

Whenever the genuine moment integral in (26) exists, one may identify $\mathfrak{M}_\lambda(\alpha + n)$ with $\Gamma_\lambda(\alpha + n)$ via Lemma 1. In the terminating case the sum in (1) is finite, so exchanging summation and integration is justified and yields exactly (26) as in Theorem 4. □

Remark 13. Theorem 5 is not meant to assert that a single fixed positive measure produces all moments. Rather, it provides a coherent analytic-continuation principle: whenever the integrals exist, the regularised moments coincide with the genuine ones, and outside that regime (27) supplies the unique meromorphic continuation compatible with the Beta representation of Γ_λ.

8.1. A transmutation operator and kernel transformations

A key structural point—and a source of genuinely new identities beyond coefficient recurrences—is that the hybrid deformation acts as a *diagonal coefficient multiplier* on analytic functions. This makes the hybrid series a transmuted version of the classical ${}_r\phi_s$ and provides a systematic mechanism for importing functional identities (in particular, transformation formulas) from the basic q-theory into the degenerate setting.

Definition 4. In analogy with diagonal coefficient multipliers arising in q-difference operator theory [33,37], let $f(u) = \sum_{n \ge 0} c_n u^n$ be a formal power series and assume $(\alpha)^*_{\lambda,n}$ is finite for every exponent with $c_n \neq 0$. Define the formal coefficient multiplier $\mathcal{J}_{\lambda,\alpha}$ by

$$(\mathcal{J}_{\lambda,\alpha} f)(u) := \sum_{n=0}^{\infty} (\alpha)^*_{\lambda,n}\, c_n\, u^n.$$

As a shorthand for this diagonal spectral action, one may write

$$\mathcal{J}_{\lambda,\alpha} = \frac{\Gamma_\lambda(\alpha + \Theta)}{\Gamma_\lambda(\alpha)}.$$

This notation is coefficientwise and does not invoke the rational functional calculus of Remark 1 beyond its diagonal-action principle.

Lemma 9. *Assume $(\alpha)^*_{\lambda,n}$ is finite for all $n \in \mathbb{N}_0$. Then*

$$\mathcal{J}_{\lambda,\alpha} T_q = T_q \mathcal{J}_{\lambda,\alpha}, \qquad \mathcal{J}_{\lambda,\alpha} \Theta = \Theta \mathcal{J}_{\lambda,\alpha},$$

as identities in $\mathbb{C}[[u]]$. If f is analytic at 0 with radius R and the generic asymptotic constant in Lemma 2 is finite and nonzero, then $\mathcal{J}_{\lambda,\alpha} f$ has radius λR; consequently the displayed commutation identities are analytic identities on *the common disc $|u| < \lambda R$. At exceptional parameters, the same identities remain formal and are analytic only on the common disc where the series on both sides converge.*

Proof. Write the formal series $f(u) = \sum_{n\geq 0} c_n u^n$. By Definition 4,

$$(\mathcal{J}_{\lambda,\alpha} f)(u) = \sum_{n\geq 0} (\alpha)^*_{\lambda,n}\ c_n\, u^n.$$

Applying T_q and using $T_q(u^n) = q^n u^n$ gives

$$(T_q \mathcal{J}_{\lambda,\alpha} f)(u) = \sum_{n\geq 0} (\alpha)^*_{\lambda,n}\ c_n\,(qu)^n = \sum_{n\geq 0} (\alpha)^*_{\lambda,n}\ (q^n c_n)\, u^n = (\mathcal{J}_{\lambda,\alpha} T_q f)(u),$$

so T_q commutes with $\mathcal{J}_{\lambda,\alpha}$.

Similarly, since $\Theta(u^n) = n\,u^n$,

$$(\Theta \mathcal{J}_{\lambda,\alpha} f)(u) = \sum_{n\geq 0} n\ (\alpha)^*_{\lambda,n}\ c_n\, u^n = (\mathcal{J}_{\lambda,\alpha}\Theta f)(u).$$

Thus both commutation identities hold coefficientwise in $\mathbb{C}[[u]]$. For the radius statement, Lemma 2 gives $|\,(\alpha)^*_{\lambda,n}\,|^{1/n} \to 1/\lambda$ in the generic case. Hence

$$\limsup_{n\to\infty} |\,(\alpha)^*_{\lambda,n}\, c_n|^{1/n} = \frac{1}{\lambda} \limsup_{n\to\infty} |c_n|^{1/n},$$

so Cauchy–Hadamard gives radius λR. The coefficientwise equalities therefore represent analytic identities on the stated common disc. □

Proposition 10. *Assume $\lambda > 0$ and $0 < \mathrm{Re}(\alpha) < 1/\lambda$. If $f(u) = \sum_{n=0}^{N} c_n u^n$ is a polynomial and $0 < \mathrm{Re}(\alpha + n) < 1/\lambda$ for $0 \leq n \leq N$, then*

$$(\mathcal{J}_{\lambda,\alpha} f)(u) = \int_0^\infty f(ut)\, d\mu_{\lambda,\alpha}(t),$$

where $d\mu_{\lambda,\alpha}$ is the normalised complex weight (24); it is a probability measure only for real $0 < \alpha < 1/\lambda$.

Proof. Write $f(u) = \sum_{n=0}^{N} c_n u^n$. Then by Definition 4 and Lemma 7,

$$(\mathcal{J}_{\lambda,\alpha} f)(u) = \sum_{n=0}^{N} (\alpha)^*_{\lambda,n}\ c_n\, u^n = \sum_{n=0}^{N} c_n u^n \int_0^\infty t^n\, d\mu_{\lambda,\alpha}(t) = \int_0^\infty f(ut)\, d\mu_{\lambda,\alpha}(t),$$

where exchanging sum and integral is justified because the sum is finite. □

Corollary 3. *Assume the hypotheses of Theorem 4. Then for the terminating normalised series $\Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,\cdot)$ one has*

$$\mathsf{H}\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix};\, q,\, \lambda,\, \alpha;\, u\right] = \big(\mathcal{J}_{\lambda,\alpha}\, \Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,\cdot)\big)(u) = \int_0^\infty \Phi^{\mathrm{norm}}_{r,s}(\mathbf{a};\mathbf{b};q,ut)\, d\mu_{\lambda,\alpha}(t),$$

which recovers (26); the last integral uses a probability measure only in the real positive range $0 < \alpha < 1/\lambda$.

Kernel averaging allows one to push classical transformation formulas through $\mathcal{J}_{\lambda,\alpha}$ and obtain *hybrid* transformation statements that are not accessible by coefficient recurrences alone.

Theorem 6 (Terminating transformations in transmutation form)**.** *Assume $0 < q < 1$, $\lambda > 0$, and that one of a, b equals q^{-N} for some $N \in \mathbb{N}_0$. Assume*

$$1 - cq^m \neq 0 \qquad (0 \leq m \leq N-1), \qquad \alpha + n \notin \mathcal{P}_\lambda \qquad (0 \leq n \leq N).$$

Let $\mathcal{H}(z)$ and $\mathcal{E}(z)$ denote the meromorphic germs at $z = 0$ obtained from the classical Heine and Euler transformed expressions

$$\mathcal{H}(z) := \frac{(b;q)_\infty\,(az;q)_\infty}{(c;q)_\infty\,(z;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} c/b,\ z \\ az \end{matrix};q,b\right),$$
$$\mathcal{E}(z) := \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} c/a,\ c/b \\ c \end{matrix};q,abz/c\right).$$

At parameter values where the displayed products and series converge and are nonsingular near $z = 0$, these are literal analytic germs; elsewhere they are understood only through their meromorphic continuation in the classical transformation identities. Then

$$\mathsf{H}\left[\begin{matrix} a,b \\ c \end{matrix};q,\lambda,\alpha;u\right] = (\mathcal{J}_{\lambda,\alpha}\mathcal{H})(u) = (\mathcal{J}_{\lambda,\alpha}\mathcal{E})(u), \tag{28}$$

where $\mathcal{J}_{\lambda,\alpha}$ is applied to the Taylor germ at 0. Because the original ${}_2\phi_1$ terminates, each germ in (28) equals the same degree-N polynomial before $\mathcal{J}_{\lambda,\alpha}$ is applied. Thus (28) is a finite coefficient identity. It is not a claim that the transformed expressions define ordinary convergent integrals on $t \in [0,\infty)$, nor is it asserted for every $u \in \mathbb{C}$ through the displayed nonterminating product/series representations.

Proof. The classical Heine and Euler transformations identify the terminating polynomial ${}_2\phi_1(a,b;c;q,z)$ with the meromorphic germs $\mathcal{H}(z)$ and $\mathcal{E}(z)$ at $z = 0$. Since the left-hand side is a polynomial of degree at most N, equality of germs forces the Taylor coefficients of each continued right-hand side above degree N to vanish. Applying the diagonal multiplier $\mathcal{J}_{\lambda,\alpha}$ is therefore a finite operation and gives (28). No interchange of a nonterminating transformed series with an integral is used. □

Remark 14. For a nonterminating hybrid series, Theorem 6 supplies no ordinary integral formula. A separate regularised statement may be formed only coefficientwise, using Theorem 5 and a meromorphically continued Taylor germ; such a prescription must not be read as integration of the displayed infinite products or nonterminating ${}_2\phi_1$ terms along $[0,\infty)$.

Following the usual construction of a Beta function from Gamma quotients and the degenerate-Gamma setting of Kim–Kim [21], define the degenerate Beta function by

$$B_\lambda(x,y) := \frac{\Gamma_\lambda(x)\Gamma_\lambda(y)}{\Gamma_\lambda(x+y)}.$$

Using the Beta representation in Lemma 1, one can obtain a symmetric kernel that generalises the classical Euler kernel $t^{x-1}(1-t)^{y-1}$.

Theorem 7. *Let $\lambda > 0$ and let x, y be complex parameters satisfying*

$$0 < \mathrm{Re}(x) < \frac{1}{\lambda}, \qquad 0 < \mathrm{Re}(y) < \frac{1}{\lambda}, \qquad 0 < \mathrm{Re}(x+y) < \frac{1}{\lambda}.$$

Then the complex analytic identity

$$B_\lambda(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,K_\lambda(t;x+y)\,dt, \tag{29}$$

holds, where

$$K_\lambda(t;\nu) := \frac{1}{\Gamma_\lambda(\nu)}\int_0^\infty r^{\nu-1}\,(1+\lambda r t)^{-1/\lambda}\,(1+\lambda r(1-t))^{-1/\lambda}\,dr. \tag{30}$$

If, in addition, x, y and $\nu = x+y$ are real and lie in the same range, then $K_\lambda(t;\nu) > 0$ for $t \in (0,1)$ and $K_\lambda(t;\nu) = K_\lambda(1-t;\nu)$.

Proof. Under the stated assumptions the product integral

$$\Gamma_\lambda(x)\Gamma_\lambda(y) = \int_0^\infty\int_0^\infty (1+\lambda s)^{-1/\lambda}(1+\lambda t)^{-1/\lambda}s^{x-1}t^{y-1}\,ds\,dt,$$

is absolutely convergent: near 0 this follows from $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, and near ∞ from $\mathrm{Re}(x), \mathrm{Re}(y) < 1/\lambda$. We may therefore use the change of variables $(s,t) = (r\xi, r(1-\xi))$, $r > 0$, $0 < \xi < 1$, with Jacobian r. This gives

$$\Gamma_\lambda(x)\Gamma_\lambda(y) = \int_0^1 \xi^{x-1}(1-\xi)^{y-1} \times \left(\int_0^\infty r^{x+y-1}(1+\lambda r\xi)^{-1/\lambda}(1+\lambda r(1-\xi))^{-1/\lambda}\,dr\right)d\xi.$$

For fixed $\xi \in (0,1)$, the inner integral converges near $r = 0$ because $\mathrm{Re}(x+y) > 0$, and near $r = \infty$ because the integrand is $O(r^{\mathrm{Re}(x+y)-1-2/\lambda})$; the latter is integrable under the stronger assumption $\mathrm{Re}(x+y) < 1/\lambda$. The endpoint behaviour in ξ is controlled by the original absolutely convergent double integral after the above change of variables. Dividing by $\Gamma_\lambda(x+y)$ yields (29) with (30).

When x, y, ν are real in the stated range, all factors in the defining integral for $K_\lambda(t;\nu)$ are positive for $0 < t < 1$, so $K_\lambda(t;\nu) > 0$. Symmetry follows immediately by replacing t with $1-t$ in (30). □

Remark 15. The concrete form (30) is chosen to make positivity and symmetry explicit. Other equivalent representations follow from Lemma 1.

The integral representations yield immediate consistency checks in simple limits.

Corollary 4. *Assume $b_k \neq 1$ for $1 \le k \le s$. Assume either the nonterminating conditions $b_k \notin q^{-\mathbb{N}_0}$ for $1 \le k \le s$, $\alpha + n \notin \mathcal{P}_\lambda$, and $1 - \lambda(\alpha+n+1) \neq 0$ for all $n \in \mathbb{N}_0$, or assume that the series terminates at degree at most N and*

$$\begin{aligned} 1 - b_k q^m &\neq 0 && (1 \le k \le s,\ 0 \le m \le N-1), \\ \alpha + n &\notin \mathcal{P}_\lambda && (0 \le n \le N), \\ 1 - \lambda(\alpha+n+1) &\neq 0 && (0 \le n \le N). \end{aligned}$$

Then

$$\mathsf{H}(0) = 1, \qquad \mathsf{H}'(0) = \frac{(\alpha)^*_{\lambda,1}}{1-q}\,\frac{\prod_{j=1}^r (1-a_j)}{\prod_{k=1}^s (1-b_k)}.$$

Proof. From the definition (1) we have

$$\mathsf{H}(u) = \sum_{n=0}^\infty c_n\,u^n, \qquad c_n = (\alpha)^*_{\lambda,n}\,\frac{\prod_{j=1}^r (a_j;q)_n}{\prod_{k=1}^s (b_k;q)_n}\,\frac{1}{(q;q)_n}.$$

For $n = 0$ we have $(\alpha)^*_{\lambda,0} = 1$ and $(a;q)_0 = (b;q)_0 = (q;q)_0 = 1$, hence $c_0 = 1$ and therefore $\mathsf{H}(0) = 1$. On the disc of convergence $|u| < \lambda$ (or for terminating series), termwise differentiation is valid, and

$$\mathsf{H}'(u) = \sum_{n\ge1} n\,c_n\,u^{n-1}, \qquad \text{so} \qquad \mathsf{H}'(0) = c_1.$$

Using $(a;q)_1 = 1-a$, $(b;q)_1 = 1-b$, $(q;q)_1 = 1-q$ and $(\alpha)^*_{\lambda,1} = \Gamma_\lambda(\alpha+1)/\Gamma_\lambda(\alpha)$, we find

$$c_1 = (\alpha)^*_{\lambda,1}\,\frac{\prod_{j=1}^r (1-a_j)}{\prod_{k=1}^s (1-b_k)}\,\frac{1}{1-q},$$

which is exactly the claimed formula. □

Remark 16. Let $\mathbf{b} = (b_1, \ldots, b_{s-1}, b_s)$ and $\mathbf{b}' = (b_1, \ldots, b_{s-1})$. If the last denominator parameter tends to 0, i.e. $b_s \to 0$, then $(b_s;q)_n \to 1$ for each fixed n. Consequently, in the nonterminating case (for $|u| < \lambda$) and in the terminating case (for all u), the coefficients of $\mathsf{H}(\mathbf{a};\mathbf{b};u)$ depend analytically on b_s and we have coefficientwise (hence locally uniform) convergence

$$\lim_{b_s\to 0} \mathsf{H}(\mathbf{a};\mathbf{b};u) = \mathsf{H}(\mathbf{a};\mathbf{b}';u).$$

In other words, the parameter b_s simply drops out. The same conclusion is immediate from the integral representation in Theorem 4.

Example 1. Let $r = 1$, $s = 0$, and $a_1 = q^{-N}$ with $N \in \mathbb{N}$. Assume also $0 < \mathrm{Re}(\alpha + n) < 1/\lambda$ for $0 \le n \le N$, so that the moment hypotheses of Theorem 4 hold. Then the basic part of the coefficient reduces by the q-binomial theorem (see [2]) to a degree-N polynomial, and the theorem yields the closed form

$$\mathsf{H}\left[\begin{matrix} q^{-N} \\ - \end{matrix}; q,\ \lambda,\ \alpha;\ u\right] = \int_0^\infty (utq^{-N};q)_N\, d\mu_{\lambda,\alpha}(t).$$

For $N = 1$ this becomes the explicit linear identity

$$\mathsf{H}\left[\begin{matrix} q^{-1} \\ - \end{matrix}; q,\ \lambda,\ \alpha;\ u\right] = 1 - \frac{\alpha}{q\,[1-\lambda(\alpha+1)]}\, u,$$

which agrees with the defining series (1).

9. Limit transitions

A consistency check for the hybrid construction is that familiar coefficient arrays reappear under controlled limits. We record two coefficientwise transitions and state explicitly what they do, and do not, recover.

Theorem 8. *Fix $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then*

$$\lim_{\lambda\to 0^+} \Gamma_\lambda(z) = \Gamma(z).$$

Moreover, fix $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Assume that for all sufficiently small $\lambda > 0$ one has $1 - \lambda(\alpha + m + 1) \neq 0$ for $0 \le m \le n-1$. Then

$$\lim_{\lambda\to 0^+} (\alpha)^*_{\lambda,n} = (\alpha)_n.$$

Consequently, at every fixed coefficient index for which the denominator factors are nonsingular, the formal coefficientwise limit is

$$\mathsf{H}\left[\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}; q,\ \lambda,\ \alpha;\ u\right] \longrightarrow \sum_{n=0}^{\infty} (\alpha)_n \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{k=1}^{s}(b_k;q)_n} \frac{u^n}{(q;q)_n}.$$

Proof. By Lemma 1, for all sufficiently small $\lambda > 0$ one has

$$\Gamma_\lambda(z) = \lambda^{-z}\Gamma(z)\frac{\Gamma(\frac{1}{\lambda} - z)}{\Gamma(\frac{1}{\lambda})}.$$

As $\lambda \to 0^+$, the Gamma-quotient asymptotic $\Gamma(w - z)/\Gamma(w) = w^{-z}(1 + O(w^{-1}))$, with $w = 1/\lambda$, gives $\Gamma_\lambda(z) \to \Gamma(z)$ for $\mathrm{Re}(z) > 0$.

For the degenerate rising factorial, use the finite product formula

$$(\alpha)^*_{\lambda,n} = \prod_{m=0}^{n-1} \frac{\alpha + m}{1 - \lambda(\alpha + m + 1)}.$$

Letting $\lambda \to 0^+$ in this finite product yields $(\alpha)^*_{\lambda,n} \to \prod_{m=0}^{n-1}(\alpha + m) = (\alpha)_n$, and the coefficientwise limit follows. □

Remark 17. The limit in Theorem 8 is a mixed ordinary–basic series: it contains the ordinary rising factorial $(\alpha)_n$ multiplied by q-Pochhammer coefficients. It should not be identified automatically with a standard basic hypergeometric series. The associated operator limit replaces $\mathcal{R}_{\lambda,\alpha}(\Theta)$ coefficientwise by $\alpha+\Theta$, giving the formal identity

$$\Big[(1-T_q)\prod_{k=1}^{s}(1-b_k q^{-1}T_q)-u(\alpha+\Theta)\prod_{j=1}^{r}(1-a_jT_q)\Big]F(u)=0,$$

for the limiting series F, in terminating or uniformly convergent regimes.

For the $q\to 1^-$ limit, the standard coefficientwise asymptotics of q-shifted factorials [2,5] require that the parameterisation be fixed. Let

$$q=e^{-\varepsilon},\qquad a_j=q^{A_j},\qquad b_k=q^{B_k},\qquad u=(1-q)^{1+s-r}z,$$

where A_j, B_k and z are fixed and the denominator parameters avoid the usual exceptional values. Then, for each fixed n,

$$\frac{(q^{A_j};q)_n}{(1-q)^n}\to(A_j)_n,\qquad \frac{(q^{B_k};q)_n}{(1-q)^n}\to(B_k)_n,\qquad \frac{(q;q)_n}{(1-q)^n}\to n!.$$

Therefore the scaled coefficient limit of (1) is

$$\mathsf{H}\left[\begin{matrix}q^{A_1},\dots,q^{A_r}\\ q^{B_1},\dots,q^{B_s}\end{matrix};q,\lambda,\alpha;(1-q)^{1+s-r}z\right]\longrightarrow\sum_{n=0}^{\infty}(\alpha)^*_{\lambda,n}\frac{\prod_{j=1}^{r}(A_j)_n}{\prod_{k=1}^{s}(B_k)_n}\frac{z^n}{n!},\tag{31}$$

coefficientwise. If one also lets $\lambda\to 0^+$ in a regime that justifies interchange of limits, the factor $(\alpha)^*_{\lambda,n}$ further tends to $(\alpha)_n$, producing the ordinary hypergeometric coefficient array with an additional numerator parameter α.

The arrow in (31) is, in the nonterminating case, only coefficientwise. A concrete locally uniform regime is the fixed terminating case: if some $A_j=-N$ with $N\in\mathbb{N}_0$, then $a_j=q^{-N}$ for every q and both sides contain only $N+1$ terms, so the convergence is locally uniform in z on every compact subset of $\mathbb{C}$. More generally, local uniform convergence on a compact set K would follow from a q-uniform summable majorant for the scaled summands; no such majorant is assumed here.

A differential-equation limit of (15) requires the same scaling and an expansion such as $T_qf(u)=f(e^{-\varepsilon}u)=f(u)-\varepsilon uf'(u)+O(\varepsilon^2)$ in a specified function space. We do not supply the needed function-space estimates, and therefore make no nonterminating claim of locally uniform function convergence or convergence of the operator equation.

10. A short computational check (terminating case)

This computation is retained only as a reproducible illustration of the already proved coefficient identity in Theorem 2; it is not evidence for any nonterminating analytic assertion. We use the exact rational parameters

$$q=\frac{3}{5},\qquad \lambda=\frac{2}{5},\qquad \alpha=\frac{1}{5},\qquad a=q^{-5}=\frac{3125}{243},\qquad b=\frac{3}{10}.$$

The degree-five polynomial obtained from (1) has the exact coefficients

$$\begin{aligned}
c_0&=1,\\
c_1&=-\frac{360250}{22113},\\
c_2&=\frac{153106250000}{73437273},\\
c_3&=\frac{263151367187500000}{3095160745131},\\
c_4&=\frac{48373413085937500000000}{553591165391895267},
\end{aligned}$$

$$c_5 = \frac{327825546264648437500000000000}{128266282176780144942 09}.$$

Substitution of $P(u) = \sum_{n=0}^{5} c_n u^n$ into the scalar operator

$$(1 - T_q)(1 - bq^{-1}T_q) - u\mathcal{R}_{\lambda,\alpha}(\Theta)(1 - aT_q),$$

and simplification in exact rational arithmetic gives the zero polynomial identically. As a separate rounding check, the coefficients were regenerated from the exact recurrence and evaluated with 100 decimal digits of working precision. Table 1 reports one such run; the small residuals measure only floating-point rounding.

Table 1. Rounding residuals for the terminating degree-five check evaluated with 100-digit working precision

Test point u	|LHS| at 100-digit precision
-0.4	3.6×10^{-97}
0.2	0 to the displayed 100 digits
0.7	7.5×10^{-96}
1.1	1.2×10^{-94}

The exact symbolic residual is identically zero; the displayed numbers are rounding residuals from 100-digit arithmetic.

The test points, working precision, exact parameters, and exact polynomial coefficients are therefore all specified. No conclusion about boundary convergence, nonterminating integrals, or analytic continuation is drawn from this finite computation.

11. Conclusion and further directions

We introduced a degenerate–q hybrid normalised coefficient series by multiplying the normalised basic-hypergeometric coefficient array by the degenerate rising factorial obtained from Γ_λ. The paper's central mechanism is the Euler transport of the rational ratio $(\alpha)^*_{\lambda,n+1} / (\alpha)^*_{\lambda,n}$, which yields the hybrid Heine-type equation (15). This identity is coefficientwise formal in general, analytic inside the disk $|u| < \lambda$ in the nonterminating case, and purely finite in the terminating case.

The convergence theory shows that the nonterminating disc of absolute convergence is $|u| < \lambda$ and that the boundary requires a phase-sensitive analysis. The contiguity theory separates numerator shifts from the corrected nonsymmetric denominator-shift relation. In the scalar case we constructed two Frobenius-type series after an explicit conjugation calculation, with the second denominator parameter corrected to $q^{\rho+1}$. The resulting connection coefficients are valid within the nonresonant Frobenius span rather than for an unspecified class of all local solutions.

For terminating series we obtained genuine degenerate-Gamma moment representations and kernel-averaged transformation formulas. For nonterminating series, the same coefficient prescriptions are interpreted through meromorphic regularisation of Beta/Gamma quotients. The symmetric Beta-kernel formula was stated with separate complex-convergence and real-positivity hypotheses. Finally, the limit transitions were formulated with the parameter scaling needed to distinguish mixed ordinary–basic limits from standard basic or ordinary hypergeometric limits.

Several further directions remain natural.

(O1) *Analytic continuation.* Develop analytic continuation beyond $|u| < \lambda$ using the hybrid Heine equation and determine how the phase-dependent boundary behaviour influences singularity structure.

(O2) *Scalar connection theory.* Define a precise functional solution class for the scalar equation and derive connection matrices between local Frobenius-type bases in that class.

(O3) *Terminating hybrid polynomials.* Study the polynomial family

$$P_N(u) := \mathsf{H}\left[\begin{matrix} q^{-N}, b \\ c \end{matrix}; q, \lambda, \alpha; u\right],$$

and determine whether it admits orthogonality or biorthogonality relations derived from the degenerate Beta kernel.

(O4) *Transformations and summations.* Investigate which classical basic-hypergeometric transformations survive under the transmutation operator $\mathcal{J}_{\lambda,\alpha}$ and which require additional regularisation.

(O5) *Coupled limits.* Study scaling regimes in which $q \to 1^-$ and $\lambda = \lambda(q)$ jointly tend to their classical limits, with the operator equation controlled in an appropriate function space.

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