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A generalized fractional framework for Ostrowski-type inequalities with preinvex mappings

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Abstract: In this work, we present the generalized framework of preinvex mapping namely the n -fractional polynomial s -like m -preinvex mapping. Several fundamental propositions are suggested to clarify the structural features of this newly proposed concept, followed by a detailed study of its rich algebraic properties. Additionally, we obtain a new lemma using a k -fractional integral operator that is essential to the analysis that follows. We derive refinements of Ostrowski type inequalities in the context of fractional calculus based on the lemma. The finding presented herein offer meaningful refinements and novel insights, significantly extending and enriching the existing literature on mathematical inequalities and fractional convex analysis.

Keywords: convex function, n -fractional polynomial s -like m -preinvex mapping, ostrowski type inequality, k -fractional integral operator

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1. Introduction

Convex theory provides solid concepts and approaches to deal with a wide range of applied science problems. Convex optimization has provided effective and scalable methods of solutions in communication systems and signal processing [1]. The convex duality theory is a useful tool in stochastic optimization and financial mathematics research enterprise [2]. Economic modeling and risk assessment are other applications of convex analysis especially via its uses in economics and the financial markets [3,4]. In order to address the practical issues that arise in nonlinear programming, statistics, control theory, optimization, etc., many mathematicians and researchers have been working to apply novel concepts of convexity theory. Additionally, a broad class of inequalities is established by the theory of convexity. Integral inequalities arise from the theory of inequalities within the framework of fractional operators. The literature has developed Ostrowski-type inequalities for a variety of convex function types. In 1938, Ostrowski arrived at the following astounding integral inequality.

Theorem 1. [5] Let $\mathcal{G} : I \rightarrow \mathbb{R}$ be a differentiable function in the interior I° of I , and let $\tau_c, \tau_d \in I^\circ$ with $\tau_c < \tau_d$. If $|\mathcal{G}'(x)| \leq M$ for all $x \in [\tau_c, \tau_d]$, then

$$\left| \mathcal{G}(x) - \frac{1}{\tau_d - \tau_c} \int_{\tau_c}^{\tau_d} \mathcal{G}(t) dt \right| \leq M(\tau_d - \tau_c) \left[\frac{1}{4} + \frac{\left(x - \frac{\tau_c + \tau_d}{2}\right)^2}{(\tau_d - \tau_c)^2} \right],$$

$\forall x \in [\tau_c, \tau_d]$. In the above famous integral inequality the constant value $\frac{1}{4}$ is an amazing choice in the aspect that it cannot be substituted by a smaller one.

Wang et al. [6] pioneered Ostrowski-type fractional integral inequalities via Hadamard fractional integrals under the s - σ condition, establishing a foundational link between logarithmic kernels and generalized convexity. Thatsatian et al. [7] obtained sharp Ostrowski-type fractional integral inequalities for p -convex functions using generalized fractional integrals, unifying several classical and fractional results. Meftah and Merad [8] developed new Ostrowski-type fractional integral inequalities for differentiable harmonically convex functions via Riemann–Liouville fractional integrals (RLFIO). Basci and Baleanu [9] introduced Ostrowski-type fractional integral inequalities involving the ψ -Hilfer fractional integral operator, encompassing various existing fractional operators. Budak et al. [10] established Ostrowski-type fractional integral inequalities for fractional integral operators with exponential kernels. Budak and Pehlivan [11] formulated weighted Ostrowski, trapezoid, and midpoint type inequalities for Riemann–Liouville fractional integrals, yielding refined weighted bounds. Nasir et al. [12] derived Ostrowski-type fractional integral inequalities for functions whose second derivatives are preinvex via fractional integral operators. Hassan and Khan [13] established fractional Ostrowski-type inequalities using (s, r) -convex functions, providing sharper bounds under generalized convexity assumptions. Kashuri et al. [14] presented fractional weighted Ostrowski-type integral inequalities along with applications, unifying several known fractional inequality results. Ahmad et al. [15] refined Ostrowski-type integral inequalities involving the Atangana–Baleanu fractional integral operator with non-singular kernel. Ali et al. [16] generalized Ostrowski-type integral inequalities within the framework of quantum calculus using q -integral operators.

Tariq et al. [17] explored the new version of Ostrowski type inequalities via conformable fractional operators. Sahoo et al. [18,19] and Tariq et al. [20] investigated the new kinds of Ostrowski-type fractional inequalities involving exponential convexity and polynomial exponential s -convexity with applications respectively. Tariq and Saad [21] derived the some new variants of Ostrowski type inequalities via generalized harmonic convex functions. Tariq et al. [22] presents the new sorts of Ostrowski inequalities involving s -type p -convexity via hypergeometric functions.

This research introduces a new class of functions, namely n -fractional polynomial s -like convex involving Raina's functions, which generalize existing convexity concepts in a fractional framework. It presents a new Ostrowski-type inequality derived via the k -fractional operator, offering a fresh perspective on classical inequalities. Several related corollaries and remarks are establish by linking the obtained results with Riemann–Liouville fractional integrals involving the Mittag–Leffler function. Overall, the work contributes unique theoretical developments and practical extensions, distinguishing it from earlier studies in the field.

This work, which elaborates on the topic and draws inspiration from the extensive literature in this field, is organized as follows. In §2, we present several foundational theorems, definitions, remarks, and corollaries that will be essential in subsequent sections. §3 introduces the new concept of n -fractional polynomial s -like m -preinvexity and presents its algebraic properties. §4 explores a novel modification of the Ostrowski-type inequalities using the k -fractional operator. Finally, §5 concludes the study with remarks on the results and potential directions for future research.

2. Preliminaries

This section establishes the analytical framework necessary for subsequent developments by reviewing fundamental concepts from fractional calculus, convex analysis, and special functions. We begin with a concise examination of classical convexity, s -like convexity, and n -fractional polynomial convexity. Furthermore, we present a unified framework for generalized convex sets and introduce both Riemann–Liouville and k -fractional integral operators. For completeness, we review several Ostrowski-type inequalities via the k -fractional operator. These preliminary considerations establish the requisite mathematical context and notation for formulating and proving the principal results of this paper.

2.1. Classical and generalized: Convexity and preinvexity

We commence by recalling the classical definition of convexity, which serves as the foundation for various generalizations discussed throughout this work.

Definition 1. [23] A function $\mathcal{G} : I \rightarrow \mathbb{R}$ is said to be *convex* if

$$\mathcal{G}(\nu\tau_c + (1-\nu)\tau_d) \leq \nu\mathcal{G}(\tau_c) + (1-\nu)\mathcal{G}(\tau_d), \quad (1)$$

holds for all $\tau_c, \tau_d \in I$ and $\nu \in [0, 1]$.

The concept of convexity has been generalized in numerous directions to accommodate broader classes of functions arising in optimization, analysis, and applied mathematics. One such generalization is the notion of *s-like convexity*.

Definition 2. [24,25] Let $s \in [0, 1]$. A function $\mathcal{G} : X^\circ \rightarrow \mathbb{R}$ is called *s-like convex* on X° if

$$\mathcal{G}(\nu\tau_c + (1-\nu)\tau_d) \leq (1-s(1-\nu))\mathcal{G}(\tau_c) + (1-s\nu)\mathcal{G}(\tau_d), \quad (2)$$

holds for all $\tau_c, \tau_d \in X^\circ$ and $\nu \in [0, 1]$.

Another important extension involves fractional polynomial weights, leading to the following definition.

Definition 3. [26] Let $n \in \mathbb{N}$. A function $\mathcal{G} : X^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ is called *n-fractional polynomial convex* if

$$\mathcal{G}(\nu\tau_c + (1-\nu)\tau_d) \leq \frac{1}{n} \sum_{j=1}^n \nu^{1/j} \mathcal{G}(\tau_c) + \frac{1}{n} \sum_{j=1}^n (1-\nu)^{1/j} \mathcal{G}(\tau_d), \quad (3)$$

holds for all $\tau_c, \tau_d \in X^\circ$ and $\nu \in [0, 1]$.

Definition 4. [27] Let $A \subseteq \mathbb{R}^n$ and let $\Delta : A \times A \times (0, 1] \rightarrow \mathbb{R}^n$ be a given mapping. The set A is said to be *m-invex* with respect to Δ if

$$m\tau_c + \nu\Delta(\tau_d, \tau_c, m) \in A,$$

for all $\tau_c, \tau_d \in A$, $m \in (0, 1]$, and $\nu \in [0, 1]$.

Example 1. [27] Let $m = \frac{1}{4}$, and define the domain $A = \left(-\frac{\Delta}{4}, \frac{\Delta}{4}\right)$, $\mathbb{X} = (-1, 1)$, and define the mapping

$$\Delta(\tau_d, \tau_c, m) = \begin{cases} m \tan(\tau_d - \tau_c), & \text{if } \tau_c, \tau_d \in \left(0, \frac{\Delta}{4}\right); \\ -m \tan(\tau_d - \tau_c), & \text{if } \tau_c, \tau_d \in \left(-\frac{\Delta}{4}, 0\right); \\ m \tan(\tau_c), & \text{if } \tau_c \in \left(0, \frac{\Delta}{4}\right), \tau_d \in \left(-\frac{\Delta}{4}, 0\right); \\ -m \tan(\tau_c), & \text{if } \tau_c \in \left(-\frac{\Delta}{4}, 0\right), \tau_d \in \left(0, \frac{\Delta}{4}\right). \end{cases}$$

Then, the set $A = \left(-\frac{\Delta}{4}, \frac{\Delta}{4}\right)$ is *m-invex* with respect to Δ , but not convex.

Example 2. [28] Let $\Psi = [0, 2] \cup [4, 6]$ and define

$$\Gamma(\tau_c, \tau_d, m) = \begin{cases} \tau_c + m(\tau_d - 4) & \text{if } 4 \leq \tau_d \leq 6, \ 0 \leq \tau_c \leq 2, \\ \tau_c + m(\tau_d - 2) & \text{if } 0 \leq \tau_d \leq 2, \ 0 \leq \tau_c \leq 2, \\ 4 + m(\tau_d - \tau_c) & \text{if } 0 \leq \tau_d \leq 2, \ 4 \leq \tau_c \leq 6, \\ 2 + m(\tau_d - \tau_c) & \text{if } 4 \leq \tau_d \leq 6, \ 4 \leq \tau_c \leq 6. \end{cases}$$

Then, the set Ψ is an *m-invex* set with respect to Γ , but it is not convex.

Weir and Mond [29] were the first to introduce the concept of invex sets and elaborate on the notion of preinvex functions in 1988.

Definition 5. [29] Let $A \subseteq \mathbb{R}^n$ and let $\Delta : A \times A \rightarrow \mathbb{R}^n$ be a mapping. A function $\mathcal{G} : A \rightarrow \mathbb{R}$ is said to be *preinvex* with respect to Δ if

$$\mathcal{G}(\tau_c + \nu \Delta(\tau_d, \tau_c)) \leq (1 - \nu) \mathcal{G}(\tau_c) + \nu \mathcal{G}(\tau_d),$$

for all $\tau_c, \tau_d \in A$ and $\nu \in [0, 1]$.

For foundational and relevant developments on this topic, see the contributions in [30–32]. Over the past decade, numerous researchers have extended and refined the notion of preinvexity in various directions. In particular, Kalsoom [28] introduced and studied the concept of *generalized m-preinvexity*, defined as follows:

Definition 6. Let $A \subseteq \mathbb{R}^n$ and let $\Delta : A \times A \times (0, 1] \rightarrow \mathbb{R}^n$ be a mapping. A function $\mathcal{G} : A \rightarrow \mathbb{R}$ is said to be *generalized m-preinvex* with respect to Δ if

$$\mathcal{G}(m\tau_c + \nu \Delta(\tau_d, \tau_c, m)) \leq m(1 - \nu) \mathcal{G}(\tau_c) + \nu \mathcal{G}(\tau_d), \quad (4)$$

for all $\tau_c, \tau_d \in A$, $m \in (0, 1]$, and $\nu \in [0, 1]$.

An important generalization of Condition C, tailored for *m*-preinvex functions, was introduced and studied by Du in [33]. This version extends the classical structure by incorporating an additional parameter $m \in (0, 1]$ into the mapping.

Definition 7. [33] Let $A \subset \mathbb{R}$ be an open invex set with respect to a mapping

$$\Delta : A \times A \times (0, 1] \rightarrow \mathbb{R}.$$

Then, for any $\tau_c, \tau_d \in A$, $\nu \in [0, 1]$, and $m \in (0, 1]$, the following conditions are satisfied:

$$\begin{aligned} \Delta(\tau_c, m\tau_c + \nu \Delta(\tau_d, \tau_c, m), m) &= -\nu \Delta(\tau_d, \tau_c, m), \\ \Delta(\tau_d, m\tau_c + \nu \Delta(\tau_d, \tau_c, m), m) &= (1 - \nu) \Delta(\tau_d, \tau_c, m), \\ \Delta(\tau_d, \tau_c, m) &= -\Delta(\tau_c, \tau_d, m). \end{aligned} \quad (5)$$

2.2. Fractional integral operators

Fractional integral operators constitute fundamental tools in fractional calculus, enabling the extension of classical integration theory to non-integer orders. We present both the classical Riemann–Liouville operators and their *k*-fractional generalizations.

2.2.1. Riemann–Liouville fractional integrals

Definition 8. [34] Let $\mathcal{G} \in L_1[v_a, v_b]$ and $\gamma > 0$. The *left-sided Riemann–Liouville fractional integral* of order γ is defined by

$$\mathcal{I}_{v_a+}^{\gamma} \mathcal{G}(x) = \frac{1}{\Gamma(\gamma)} \int_{v_a}^x (x - u)^{\gamma-1} \mathcal{G}(u) du, \quad x > v_a, \quad (6)$$

and the *right-sided Riemann–Liouville fractional integral* of order γ is defined by

$$\mathcal{I}_{v_b-}^{\gamma} \mathcal{G}(x) = \frac{1}{\Gamma(\gamma)} \int_x^{v_b} (u - x)^{\gamma-1} \mathcal{G}(u) du, \quad x < v_b, \quad (7)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

2.2.2. *k*-fractional integrals

A natural generalization of the Riemann–Liouville fractional integrals leads to the concept of *k*-fractional integrals, which introduce an additional parameter *k* providing greater flexibility in modeling.

Definition 9. [35] Let $\gamma > 0$ and $k > 0$. The *left-sided k -fractional integral* is defined by

$$\zeta_{\tau_c^-}^{\gamma,k} \mathcal{G}(x) = \frac{1}{k\Gamma_k(\gamma)} \int_{\tau_c}^x (x - \tau)^{\gamma/k-1} \mathcal{G}(\tau) d\tau, \quad x > \tau_c, \quad (8)$$

and the *right-sided k -fractional integral* is defined by

$$\zeta_{\tau_d^+}^{\gamma,k} \mathcal{G}(x) = \frac{1}{k\Gamma_k(\gamma)} \int_x^{\tau_d} (\tau - x)^{\gamma/k-1} \mathcal{G}(\tau) d\tau, \quad x < \tau_d, \quad (9)$$

where $\Gamma_k(\cdot)$ denotes the k -Gamma function defined by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t^{k/k}} dt, \quad \alpha > 0, \quad k > 0. \quad (10)$$

2.3. Ostrowski-type inequalities via k -Fractional operators

Ostrowski-type inequalities provide error estimates for integral approximations and play a significant role in numerical analysis and approximation theory. We present several such inequalities involving k -fractional integrals for various classes of convex functions.

Theorem 2. [36] Let $\mathcal{G} : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ be a differentiable function on (τ_1, τ_2) with $\tau_1 < \tau_2$ and $\mathcal{G}' \in L[\tau_1, \tau_2]$. Suppose that $|\mathcal{G}'|$ is an s -Godunova-Levin function of the second kind on $[\tau_1, \tau_2]$ and that $|\mathcal{G}'(x)| \leq M$ for all $x \in [\tau_1, \tau_2]$. Then, for all $x \in [\tau_1, \tau_2]$, the following inequality holds:

$$\left| \left(\frac{(x - \tau_1)^{\gamma/k} + (\tau_2 - x)^{\gamma/k}}{\tau_2 - \tau_1} \right) \mathcal{G}(x) - \frac{\Gamma_k(\gamma + k)}{\tau_2 - \tau_1} \left[\zeta_{x^-}^{\gamma,k} \mathcal{G}(\tau_1) + \zeta_{x^+}^{\gamma,k} \mathcal{G}(\tau_2) \right] \right| \leq M \left[\frac{(x - \tau_1)^{\gamma/k+1} + (\tau_2 - x)^{\gamma/k+1}}{\tau_2 - \tau_1} \right] \left[\frac{1}{\gamma/k + 1 - s} + \frac{\Gamma_k(\gamma + k)\Gamma_k(k - sk)}{\Gamma_k(\gamma + 2k - sk)} \right]. \quad (11)$$

Theorem 3. [37] Let $\mathcal{G} : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (τ_1, τ_2) . Suppose that $\mathcal{G}$ is an exponentially convex function satisfying $\mathcal{G}'(x) \geq 1$ and $|e^{\mathcal{G}(\nu)}| \leq M$ for all $x, \nu \in [\tau_1, \tau_2]$. Then the following inequality holds:

$$\left| \left((x - \tau_1)^{\gamma/k} + (\tau_2 - x)^{\gamma/k} \right) e^{\mathcal{G}(x)} - \left(\Gamma_k(\gamma + k) \zeta_{\tau_1^+}^{\gamma,k} e^{\mathcal{G}(x)} + \Gamma_k(\gamma + k) \zeta_{\tau_2^-}^{\gamma,k} e^{\mathcal{G}(x)} \right) \right| \leq \frac{M\rho}{\rho + k} \left[(x - \tau_1)^{\gamma/k+1} + (\tau_2 - x)^{\gamma/k+1} \right], \quad (12)$$

where $\rho > 0$ is a parameter associated with the exponential convexity.

Theorem 4. [38] Let $\mathcal{G} : [\tau_1, \tau_2] \subset (0, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function with $\mathcal{G}' \in L[\tau_1, \tau_2]$. Suppose that $|\mathcal{G}'|$ is an (s, r) -convex function on $[\tau_1, \tau_2]$ and that $|\mathcal{G}'(x)| \leq M$ for all $x \in [\tau_1, \tau_2]$. Then, for $\gamma, k > 0$ and $x \in [\tau_1, \tau_2]$, we have

$$\left| \left(\frac{(x - \tau_1)^{\gamma/k} + (\tau_2 - x)^{\gamma/k}}{\tau_2 - \tau_1} \right) \mathcal{G}(x) - \frac{k\Gamma(\gamma + 1)}{\tau_2 - \tau_1} \left[\zeta_{x^-}^{\gamma,k} \mathcal{G}(\tau_1) + \zeta_{x^+}^{\gamma,k} \mathcal{G}(\tau_2) \right] \right| \leq M \left[\int_0^1 t^{\gamma/k} t^{rs} dt + \int_0^1 t^{\gamma/k} (1 - t^r)^s dt \right] \left[\frac{(x - \tau_1)^{\gamma/k+1} + (\tau_2 - x)^{\gamma/k+1}}{\tau_2 - \tau_1} \right]. \quad (13)$$

Theorem 5. [39] Let $\mathcal{G} : [\tau_1, \tau_2] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (τ_1, τ_2) with $\tau_1 < \tau_2$ and $\mathcal{G}' \in L[\tau_1, \tau_2]$. Suppose that $|\mathcal{G}'|$ is an MT-convex function on $[\tau_1, \tau_2]$ and that $|\mathcal{G}'(x)| \leq M$ for all $x \in [\tau_1, \tau_2]$. Then, for $\gamma > 0$ and $x \in [\tau_1, \tau_2]$, the following inequality holds:

$$\left| \left(\frac{(x - \tau_1)^{\gamma/k} + (\tau_2 - x)^{\gamma/k}}{\tau_2 - \tau_1} \right) \mathcal{G}(x) - \frac{\Gamma_k(\gamma + k)}{\tau_2 - \tau_1} \left[\zeta_{x^-}^{\gamma,k} \mathcal{G}(\tau_1) + \zeta_{x^+}^{\gamma,k} \mathcal{G}(\tau_2) \right] \right|$$

$$\leq M \frac{\Gamma(\gamma + k/2) \Gamma_k(k/2)}{2\Gamma_k(\gamma + k)} \cdot \frac{(x - \tau_1)^{\gamma/k+1} + (\tau_2 - x)^{\gamma/k+1}}{\tau_2 - \tau_1}. \quad (14)$$

3. n -fractional polynomial s -like m -preinvex mapping and algebraic properties

In this section, we introduce a new and highly flexible class of generalized convex functions, termed n -fractional polynomial s -like m -preinvex mapping. This notion unifies several existing convexity structures and extends them into a fractional and nonlocal setting. We investigate its fundamental algebraic properties, establish relationships with known convexity concepts, and demonstrate its richness through illustrative examples. A series of structural results covering stability, closure properties, composition, and ordering are rigorously derived, highlighting the analytical strength and versatility of the proposed framework.

Definition 10. Let $n \in \mathbb{N}$, $\mu_j \geq 0$ ($j = \overline{1, n}$) such that

$$\sum_{j=1}^n \mu_j > 0,$$

let $X^0 \subset \mathbb{R}$ be an interval, and let

$$\Delta : X^0 \times X^0 \times (0, 1] \rightarrow \mathbb{R},$$

be a mapping such that X^0 is m -invex with respect to Δ , i.e.

$$m\tau_c + \nu\Delta(\tau_d, \tau_c, m) \in X^0 \quad \text{for all } \tau_c, \tau_d \in X^0, m \in (0, 1], \nu \in [0, 1].$$

Let $\mathcal{G} : X^0 \rightarrow \mathbb{R}$ be a non-negative mapping. Fix $s \in [0, 1]$ and $m \in (0, 1]$. Then $\mathcal{G}$ is said to be an n -fractional s -like m -preinvex mapping (with respect to Δ , at the pair (s, m)) if

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \frac{m \sum_{j=1}^n \mu_j (1 - s\nu)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d), \quad (15)$$

for all $\nu \in [0, 1]$ and $\tau_c, \tau_d \in X^0$.

Remark 1. • If $n = m = 1$ and $\Delta(\tau_d, \tau_c) = \tau_d - \tau_c$, then Definition 10 reduces to the fractional s -like convexity introduced in [25].

- The choice of preinvex mapping allows the model to capture nonlinear and nonlocal behaviors that do not arise in classical convexity.

We now present example to demonstrate the applicability of Definition 10 and to highlight its distinction from classical convexity.

Example 3. Let $n = 1$, $\mu_1 = 1$, $m = 1$, $s = \frac{1}{2}$, $X^0 = [0, 1]$, and let

$$\Delta(\tau_d, \tau_c) = \tau_d - \tau_c,$$

so that X^0 is m -invex (in fact convex) with respect to Δ . Define

$$\mathcal{G}(x) = \sqrt{x}, \quad x \in [0, 1].$$

Since $\mathcal{G}''(x) = -\frac{1}{4}x^{-3/2} < 0$ for all $x \in (0, 1]$, the mapping $\mathcal{G}$ is strictly concave and hence *not* convex on $[0, 1]$; equivalently, since $\Delta(\tau_d, \tau_c) = \tau_d - \tau_c$ reduces preinvexity (Definition 5) to ordinary convexity, $\mathcal{G}$ is also not preinvex with respect to this Δ . Nevertheless, for all $\tau_c, \tau_d \in [0, 1]$ and $\nu \in [0, 1]$,

$$\mathcal{G}((1 - \nu)\tau_c + \nu\tau_d) \leq \left(1 - \frac{1}{2}\nu\right) \mathcal{G}(\tau_c) + \left(1 - \frac{1}{2}(1 - \nu)\right) \mathcal{G}(\tau_d),$$

which is precisely inequality (15) of Definition 10 with $n = m = 1$, $s = \frac{1}{2}$. Hence $\mathcal{G}$ is n -fractional s -like m -preinvex without being convex, showing the class introduced in Definition 10 is strictly larger than the classical convex (and preinvex) class. Moreover, the value $s = \frac{1}{2}$ is sharp: letting $\tau_c \rightarrow 0$, $\tau_d = 1$, and $\nu \rightarrow 1^-$ forces $s \leq \frac{1}{2}$, so no larger value of s makes the inequality hold on all of $[0, 1]$.

Lemma 1. *The following inequalities hold for all $\nu \in [0, 1]$, $j \in \mathbb{N}$, and $m \in (0, 1]$:*

$$\begin{aligned} \nu &\leq \frac{1}{n} \sum_{j=1}^n \left(1 - (1 - \nu)^j\right), \\ m(1 - \nu) &\leq \frac{m}{n} \sum_{j=1}^n \left(1 - \nu^j\right). \end{aligned}$$

Proof. The proof follows directly from the convexity of the power function and Jensen's inequality. For the first inequality, observe that

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \left(1 - (1 - \nu)^j\right) &= 1 - \frac{1}{n} \sum_{j=1}^n (1 - \nu)^j \\ &\geq 1 - (1 - \nu) = \nu, \end{aligned}$$

where the inequality follows from $(1 - \nu)^j \leq (1 - \nu)$ for $j \geq 1$ and $\nu \in [0, 1]$. The second inequality follows analogously. $\square$

Proposition 1. *Every non-negative generalized m -preinvex mapping is n -fractional polynomial s -like m -preinvex mapping.*

Proof. Let $\mathcal{G} : X^0 \rightarrow \mathbb{R}$ be a non-negative generalized m -preinvex mapping. Then for all $\tau_c, \tau_d \in X^0$ and $\nu \in [0, 1]$,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq m(1 - \nu)\mathcal{G}(\tau_c) + \nu\mathcal{G}(\tau_d). \quad (16)$$

Claim. For every $x \in [0, 1]$ and every integer $j \geq 1$,

$$x^{1/j} \geq x.$$

Proof of Claim. The statement is immediate at $x = 0$ and $x = 1$. For $x \in (0, 1)$, since $1/j \in (0, 1]$ we have $\frac{1}{j} - 1 \leq 0$, and therefore

$$\frac{x^{1/j}}{x} = x^{\frac{1}{j}-1} \geq 1,$$

because $x \leq 1$ raised to a non-positive exponent is ≥ 1 . Hence $x^{1/j} \geq x$. $\square$

Since $s \in [0, 1]$ and $\nu \in [0, 1]$, we have $s\nu \leq \nu$, so

$$1 - s\nu \geq 1 - \nu \geq 0,$$

and $1 - s\nu \in [0, 1]$. Applying the Claim with $x = 1 - s\nu$ gives, for every $j \geq 1$,

$$(1 - s\nu)^{1/j} \geq 1 - s\nu \geq 1 - \nu.$$

Since $\mu_j \geq 0$ for each j and $\sum_{j=1}^n \mu_j > 0$, taking the μ_j -weighted average of this termwise inequality preserves the direction:

$$\frac{\sum_{j=1}^n \mu_j (1 - s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \geq 1 - \nu. \quad (17)$$

By the same argument applied with ν replaced by $1 - \nu$, since $s(1 - \nu) \leq 1 - \nu$,

$$1 - s(1 - \nu) \geq 1 - (1 - \nu) = \nu,$$

and hence, for every $j \geq 1$,

$$(1 - s(1 - \nu))^{1/j} \geq \nu,$$

so that

$$\frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{1/j}}{\sum_{j=1}^n \mu_j} \geq \nu. \quad (18)$$

Since $\mathcal{G} \geq 0$ by hypothesis, multiplying (17) by $m \mathcal{G}(\tau_c) \geq 0$ and (18) by $\mathcal{G}(\tau_d) \geq 0$ does not reverse either inequality, giving

$$m(1 - \nu)\mathcal{G}(\tau_c) \leq \frac{m \sum_{j=1}^n \mu_j (1 - s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c), \quad \nu \mathcal{G}(\tau_d) \leq \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d).$$

Adding these two inequalities to inequality (16) yields

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \frac{m \sum_{j=1}^n \mu_j (1 - s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d),$$

which is precisely the defining inequality (15) of Definition 10. Hence $\mathcal{G}$ is an n -fractional polynomial s -like m -preinvex mapping. $\square$

Proposition 2. Every n -fractional polynomial s -like m -preinvex mapping is a generalized (h, m) -preinvex mapping with

$$h(\nu) = \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{1/j}}{\sum_{j=1}^n \mu_j}.$$

Proof. By Definition 10, for any n -fractional polynomial s -like m -preinvex $\mathcal{G}$, we have

$$\begin{aligned} \mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) &\leq \frac{m \sum_{j=1}^n \mu_j (1 - s\nu)^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{\frac{1}{j}}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d) \\ &\leq m \cdot h(1 - \nu) \mathcal{G}(\tau_c) + h(\nu) \mathcal{G}(\tau_d). \end{aligned}$$

This establishes that $\mathcal{G}$ is generalized (h, m) -preinvex mapping. $\square$

Theorem 6. Let $\mathcal{G}_1, \mathcal{G}_2 : X^0 \times X^0 \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -fractional s -like m -preinvex mapping as in Definition 10. Then:

1. The sum of mapping

$$(\mathcal{G}_1 + \mathcal{G}_2)(\tau_c, \tau_d) := \mathcal{G}_1(\tau_c, \tau_d) + \mathcal{G}_2(\tau_c, \tau_d),$$

is also n -fractional s -like m -preinvex mapping.

2. For any scalar multiplication $\gamma \geq 0$, the mapping

$$(\gamma \mathcal{G}_1)(\tau_c, \tau_d) := \gamma \cdot \mathcal{G}_1(\tau_c, \tau_d),$$

is n -fractional s -like m -preinvex mapping.

Proof. Let $\tau_c, \tau_d \in X^o$ and $\nu, s \in [0, 1]$ be arbitrary.

By Definition 10, for each point in X^o , we have

$$\begin{aligned}\mathcal{G}_1\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) &\leq \frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_1(\tau_d) + \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_1(\tau_c), \\ \mathcal{G}_2\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) &\leq \frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_2(\tau_d) + \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_2(\tau_c).\end{aligned}$$

Adding these two inequalities yields

$$(\mathcal{G}_1 + \mathcal{G}_2)\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) \leq \frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} (\mathcal{G}_1 + \mathcal{G}_2)(\tau_d) + \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} (\mathcal{G}_1 + \mathcal{G}_2)(\tau_c),$$

showing that $\mathcal{G}_1 + \mathcal{G}_2$ satisfies the n -fractional s -like m -preinvex mapping condition.

Let $\gamma \geq 0$. Then

$$(\gamma\mathcal{G}_1)\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) = \gamma\mathcal{G}_1\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right).$$

Using the n -fractional s -like preinvex of $\mathcal{G}_1$, we have

$$\gamma\mathcal{G}_1\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) \leq \gamma \left[\frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_1(\tau_d) + \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_1(\tau_c) \right].$$

Since $\gamma \geq 0$, multiplying preserves the inequality:

$$(\gamma\mathcal{G}_1)\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) \leq \frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} (\gamma\mathcal{G}_1)(\tau_d) + \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} (\gamma\mathcal{G}_1)(\tau_c).$$

Thus, $\gamma\mathcal{G}_1$ is also n -fractional s -like m -preinvex mapping. $\square$

Theorem 7. If $\mathcal{G}_z : \mathbb{R}^n \rightarrow \mathbb{R}$ is an n -fractional polynomial s -like m -preinvex mapping, then

$$M = \{x \in \mathbb{R}^n : \mathcal{G}_z(x) \leq 0, z = 1, 2, 3, \dots, n\}, \quad (19)$$

is a m -invex set.

Proof. Since $\mathcal{G}_z(x)$ ($z = 1, 2, 3, \dots, n$) is n -fractional polynomial s -like m -preinvex mapping for $m \in (0, 1]$ and $\nu \in [0, 1]$, then for all $\tau_c, \tau_d \in \mathbb{R}^n$, we have

$$\mathcal{G}_z\left(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)\right) \leq \frac{m \sum_{j=1}^n \mu_j(1-s\nu)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_z(\tau_c) + \frac{\sum_{j=1}^n \mu_j(1-s(1-\nu))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_z(\tau_d).$$

When $\tau_c, \tau_d \in M$, we know $\mathcal{G}_z(\tau_c) \leq 0$ and $\mathcal{G}_z(\tau_d) \leq 0$. Thus, the above inequality implies

$$\mathcal{G}_z(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq 0, \quad z = 1, 2, 3, \dots, n. \quad (20)$$

Therefore, $m\tau_c + \nu\Delta(\tau_d, \tau_c, m) \in M$, which establishes that M is a m -invex set. $\square$

Theorem 8. Let G be an n -fractional polynomial s -like m -preinvex mapping on a generalized m -invex set A with respect to Δ , and suppose in addition that

$$A(\nu) + B(\nu) \leq 1 \quad \text{for all } \nu \in [0, 1],$$

where

$$A(\nu) = \frac{m \sum_{j=1}^n \mu_j (1 - s\nu)^{1/j}}{\sum_{j=1}^n \mu_j}, \quad B(\nu) = \frac{\sum_{j=1}^n \mu_j (1 - s(1 - \nu))^{1/j}}{\sum_{j=1}^n \mu_j}.$$

Then G is also quasi n -fractional polynomial s -like m -preinvex; that is, for all $\tau_c, \tau_d \in X$ and $\nu \in [0, 1]$,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \max\{\mathcal{G}(\tau_c), \mathcal{G}(\tau_d)\}.$$

Proof. Without loss of generality, assume $\mathcal{G}(\tau_c) \leq \mathcal{G}(\tau_d)$ for $\tau_c, \tau_d \in X$. By Definition 10,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq A(\nu)\mathcal{G}(\tau_c) + B(\nu)\mathcal{G}(\tau_d).$$

Since $A(\nu) \geq 0$ and $\mathcal{G}(\tau_c) \leq \mathcal{G}(\tau_d)$,

$$A(\nu)\mathcal{G}(\tau_c) + B(\nu)\mathcal{G}(\tau_d) \leq A(\nu)\mathcal{G}(\tau_d) + B(\nu)\mathcal{G}(\tau_d) = [A(\nu) + B(\nu)]\mathcal{G}(\tau_d).$$

By hypothesis, $A(\nu) + B(\nu) \leq 1$, and since $\mathcal{G}(\tau_d) \geq 0$, multiplying preserves the inequality:

$$[A(\nu) + B(\nu)]\mathcal{G}(\tau_d) \leq \mathcal{G}(\tau_d).$$

Combining the three displays,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \mathcal{G}(\tau_d) = \max\{\mathcal{G}(\tau_c), \mathcal{G}(\tau_d)\}.$$

Similarly, if $\mathcal{G}(\tau_d) \leq \mathcal{G}(\tau_c)$, the same argument with the roles of τ_c, τ_d interchanged gives

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \mathcal{G}(\tau_c) = \max\{\mathcal{G}(\tau_c), \mathcal{G}(\tau_d)\}.$$

In either case,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) \leq \max\{\mathcal{G}(\tau_c), \mathcal{G}(\tau_d)\},$$

which completes the proof. $\square$

Theorem 9. Let $\mathcal{G}_z : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ($z = 1, 2, \dots, n$) be n -fractional polynomial s -like m -preinvex mapping. Then the function

$$\mathcal{G} = \sum_{z=1}^n v_z \mathcal{G}_z, \quad v_z \geq 0, \quad z = 1, 2, 3, \dots, n, \quad (21)$$

is also n -fractional polynomial s -like m -preinvex mapping, where v_z are positive constants.

Proof. The proof follows directly from the linearity of the convexity condition and the non-negativity of the coefficients v_z . For any $\tau_c, \tau_d \in A$ and $\nu \in [0, 1]$,

$$\mathcal{G}(m\tau_c + \nu\Delta(\tau_d, \tau_c, m)) = \sum_{z=1}^n v_z \mathcal{G}_z(m\tau_c + \nu\Delta(\tau_d, \tau_c, m))$$

$$\begin{aligned}
&\leq \sum_{z=1}^n v_z \left[\frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_z(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}_z(\tau_d) \right] \\
&= \frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \sum_{z=1}^n v_z \mathcal{G}_z(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \sum_{z=1}^n v_z \mathcal{G}_z(\tau_d) \\
&= \frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d).
\end{aligned}$$

This establishes that $\mathcal{G}$ is n -fractional polynomial s -like m -preinvex mapping. $\square$

Theorem 10. Let $\mathcal{G} : \mathbb{R}_0 \rightarrow \mathbb{R}_0 = [0, \infty)$ be n -fractional polynomial s -like m -preinvex mapping with respect to $\Delta : \mathbb{R}_0 \times \mathbb{R}_0 \times (0, 1] \rightarrow \mathbb{R}_0$ for $m \in (0, 1]$ and $v \in [0, 1]$. Suppose that Δ is monotonically increasing and $\mathcal{G}$ is monotonically decreasing for fixed $\tau_c, \tau_d \in \mathbb{R}_0$ and $m_1 \leq m_2$ ($m_1, m_2 \in (0, 1]$). If $\mathcal{G}$ is n -fractional polynomial s -like m_1 -preinvex mapping on $\mathbb{R}_0$ with respect to Δ , then $\mathcal{G}$ is n -fractional polynomial s -like m_2 -preinvex mapping on $\mathbb{R}_0$ with respect to Δ .

Proof. Given that $\mathcal{G}$ is n -fractional polynomial s -like m_1 -preinvex mapping, for all $\tau_c, \tau_d \in \mathbb{R}_0$, we have

$$\mathcal{G}(m_1 \tau_c + v \Delta(\tau_d, \tau_c, m_1)) \leq \frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d). \quad (22)$$

Combining the monotonicity properties of $\mathcal{G}$ and Δ with respect to m for fixed $\tau_c, \tau_d \in \mathbb{R}_0$ and $m_1 \leq m_2$, we obtain

$$\begin{aligned}
\mathcal{G}(m_2 \tau_c + v \Delta(\tau_d, \tau_c, m_2)) &\leq \mathcal{G}(m_1 \tau_c + v \Delta(\tau_d, \tau_c, m_1)) \\
&\leq \frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d) \\
&\leq \frac{\sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_c) + \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} \mathcal{G}(\tau_d).
\end{aligned}$$

This completes the proof. $\square$

4. Ostrowski-type inequalities via n -Fractional polynomial s -like m -preinvex mapping

Assumption A*: Suppose $\mathcal{G} : [m\tau_c, m\tau_c + \Delta(\tau_d, \tau_c, m)] \rightarrow R$ is a differentiable mapping on X^0 such that $\mathcal{G}' \in \mathcal{L}[m\tau_c, m\tau_c + \Delta(\tau_d, \tau_c, m)]$ with $m\tau_c < m\tau_c + \Delta(\tau_d, \tau_c, m)$, $\gamma > 0$ and $k > 0$. Assume that $|\mathcal{G}'|$ satisfies the Definition 10 with $|\mathcal{G}'| \leq S$.

Lemma 2. Under Assumption A*, the following equality holds:

$$\frac{\Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)}$$

$$\begin{aligned}
& - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))^-}^{\gamma, k} \mathcal{G}(x) \right] \\
& = \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 \left(v^{\gamma/k} - 1 \right) \mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m)) dv \\
& + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 v^{\gamma/k} \mathcal{G}'(x + v\Delta(\tau_d, x, m)) dv,
\end{aligned}$$

for all $x \in [m\tau_c, m\tau_c + \Delta(\tau_d, \tau_c, m)]$.

Proof. Let

$$J_1 = \int_0^1 \left(v^{\gamma/k} - 1 \right) \mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m)) dv, \quad J_2 = \int_0^1 v^{\gamma/k} \mathcal{G}'(x + v\Delta(\tau_d, x, m)) dv.$$

First we find J_1 , so substitute $t = m\tau_c + v\Delta(x, \tau_c, m)$, so t ranges over $[m\tau_c, x]$ as v ranges over $[0, 1]$. Then

$$\Delta^{\gamma/k+1}(x, \tau_c, m) J_1 = \int_{m\tau_c}^x \left[(t - m\tau_c)^{\gamma/k} - \Delta^{\gamma/k}(x, \tau_c, m) \right] \mathcal{G}'(t) dt.$$

Integrating by parts with $u = (t - m\tau_c)^{\gamma/k} - \Delta^{\gamma/k}(x, \tau_c, m)$, $dv = \mathcal{G}'(t) dt$: the term vanishes at $t = x$ (where $u = 0$) and contributes $\Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c)$ at $t = m\tau_c$. Hence

$$\Delta^{\gamma/k+1}(x, \tau_c, m) J_1 = \Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c) - \frac{\gamma}{k} \int_{m\tau_c}^x (t - m\tau_c)^{\gamma/k-1} \mathcal{G}(t) dt.$$

By Definition 9, this integral equals $k \Gamma_k(\gamma) \zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c)$, and using $\gamma \Gamma_k(\gamma) = \Gamma_k(\gamma + k)$,

$$\Delta^{\gamma/k+1}(x, \tau_c, m) J_1 = \Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c) - \Gamma_k(\gamma + k) \zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c). \quad (23)$$

Next we find J_2 , so write $c^* = m\tau_c + \Delta(\tau_d, \tau_c, m)$ and substitute $t = x + v\Delta(\tau_d, x, m)$, so t ranges over $[x, c^*]$. Then

$$\Delta^{\gamma/k+1}(\tau_d, x, m) J_2 = \int_x^{c^*} (t - x)^{\gamma/k} \mathcal{G}'(t) dt.$$

Integrating by parts with $u = (t - x)^{\gamma/k}$, $dv = \mathcal{G}'(t) dt$: the term vanishes at $t = x$ and contributes $\Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(c^*)$ at $t = c^*$. Hence

$$\Delta^{\gamma/k+1}(\tau_d, x, m) J_2 = \Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(c^*) - \frac{\gamma}{k} \int_x^{c^*} (t - x)^{\gamma/k-1} \mathcal{G}(t) dt.$$

By Definition 9, this integral equals $k \Gamma_k(\gamma) \zeta_{(c^*)^-}^{\gamma, k} \mathcal{G}(x)$, so

$$\Delta^{\gamma/k+1}(\tau_d, x, m) J_2 = \Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(c^*) - \Gamma_k(\gamma + k) \zeta_{(c^*)^-}^{\gamma, k} \mathcal{G}(x). \quad (24)$$

Adding (23) and (24) and dividing by $\Delta(\tau_d, \tau_c, m)$ gives the stated equality. $\square$

Theorem 11. Assume that $|\mathcal{G}'|$ satisfies Definition 10 and, under Assumption A^* , the following Ostrowski-type inequality associated with the k -fractional operator holds:

$$\begin{aligned}
& \left| \frac{\Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\
& \left. - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))^-}^{\gamma, k} \mathcal{G}(x) \right] \right|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \cdot \frac{S}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 (1-v^{\gamma/k}) \left[(1-s(1-v))^{1/j} + (1-sv)^{1/j} \right] dv \\ &\quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \cdot \frac{S}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 v^{\gamma/k} \left[(1-s(1-v))^{1/j} + (1-sv)^{1/j} \right] dv. \end{aligned}$$

Proof. By the identity in Lemma 2 and taking absolute values of both sides, we have:

$$\begin{aligned} &\left| \frac{\Delta^{\gamma/k}(x, \tau_c, m)\mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m)\mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\ &\quad \left. - \frac{\Gamma_k(\gamma+k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma,k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))^-}^{\gamma,k} \mathcal{G}(x) \right] \right| \\ &\leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 |v^{\gamma/k} - 1| |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \\ &\quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv. \end{aligned}$$

Since $v \in [0, 1]$ and $\gamma, k > 0$, we have $|v^{\gamma/k} - 1| = 1 - v^{\gamma/k}$. Applying Definition 10, for any $v \in [0, 1]$:

$$|\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| \leq \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} |\mathcal{G}'(x)| + \frac{m \sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} |\mathcal{G}'(\tau_c)|,$$

and

$$|\mathcal{G}'(mx + v\Delta(\tau_d, x, m))| \leq \frac{\sum_{j=1}^n \mu_j (1-s(1-v))^{1/j}}{\sum_{j=1}^n \mu_j} |\mathcal{G}'(x)| + \frac{m \sum_{j=1}^n \mu_j (1-sv)^{1/j}}{\sum_{j=1}^n \mu_j} |\mathcal{G}'(\tau_d)|.$$

Under Assumption A^* , $|\mathcal{G}'(x)|, |\mathcal{G}'(\tau_c)|, |\mathcal{G}'(\tau_d)| \leq S$. Substituting these upper bounds into the integrals gives:

$$\begin{aligned} &\left| \frac{\Delta^{\gamma/k}(x, \tau_c, m)\mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m)\mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\ &\quad \left. - \frac{\Gamma_k(\gamma+k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma,k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))^-}^{\gamma,k} \mathcal{G}(x) \right] \right| \\ &\leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 (1-v^{\gamma/k}) \left\{ \frac{S}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \left[(1-s(1-v))^{1/j} + (1-sv)^{1/j} \right] \right\} dv \\ &\quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 v^{\gamma/k} \left\{ \frac{S}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \left[(1-s(1-v))^{1/j} + (1-sv)^{1/j} \right] \right\} dv. \end{aligned}$$

Factoring out the constant terms yields the stated inequality. $\square$

Theorem 12. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $|\mathcal{G}'|^q$ satisfies Definition 10 and, under Assumption A^* , the following Ostrowski-type inequality associated with the k -fractional operator hol

$$\left| \frac{\Delta^{\gamma/k}(x, \tau_c, m)\mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m)\mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right|$$

$$\begin{aligned}
& - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))}^{\gamma, k} \mathcal{G}(x) \right] \Big| \\
& \leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \left(\int_0^1 (1 - v^{\gamma/k})^p dv \right)^{1/p} \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j}] dv \right)^{1/q} \\
& + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \cdot \left(\frac{k}{p\gamma + k} \right)^{1/p} \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j}] dv \right)^{1/q}.
\end{aligned}$$

Proof. By the identity in Lemma 2 and applying the triangle inequality, we have:

$$\begin{aligned}
& \left| \frac{\Delta^{\gamma/k}(x, \tau_c, m) \mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m) \mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\
& \quad \left. - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x^-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))}^{\gamma, k} \mathcal{G}(x) \right] \right| \\
& \leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 |v^{\gamma/k} - 1| |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \\
& \quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv. \tag{25}
\end{aligned}$$

Since $v \in [0, 1]$ and $\gamma, k > 0$, we have $|v^{\gamma/k} - 1| = 1 - v^{\gamma/k}$. Applying Hölder's inequality to each integral gives:

$$\int_0^1 (1 - v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \leq \left(\int_0^1 (1 - v^{\gamma/k})^p dv \right)^{1/p} \left(\int_0^1 |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))|^q dv \right)^{1/q}, \tag{26}$$

and

$$\int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv \leq \left(\int_0^1 (v^{\gamma/k})^p dv \right)^{1/p} \left(\int_0^1 |\mathcal{G}'(x + v\Delta(\tau_d, x, m))|^q dv \right)^{1/q}. \tag{27}$$

Notice that $\int_0^1 v^{p\gamma/k} dv = \frac{k}{p\gamma + k}$. Applying Definition 10 to $|\mathcal{G}'|^q$ under Assumption A^* , we have:

$$|\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))|^q \leq \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j [(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j}], \tag{28}$$

and similarly:

$$|\mathcal{G}'(x + v\Delta(\tau_d, x, m))|^q \leq \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j [(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j}]. \tag{29}$$

Integrating Eqs. (28) and (29) over $v \in [0, 1]$ and substituting them along with the evaluated integral $\left(\frac{k}{p\gamma + k}\right)^{1/p}$ into inequalities (26) and (27) yields:

$$\begin{aligned}
& \int_0^1 (1 - v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \leq \left(\int_0^1 (1 - v^{\gamma/k})^p dv \right)^{1/p} \\
& \quad \times \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j}] dv \right)^{1/q}, \tag{30}
\end{aligned}$$

and

$$\int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv \leq \left(\frac{k}{p\gamma + k} \right)^{1/p} \times \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1-s(1-v))^{1/j} + (1-sv)^{1/j}] dv \right)^{1/q}. \quad (31)$$

Finally, substituting (30) and (31) back into inequality (25) completes the proof. $\square$

Theorem 13. Let $q \geq 1$. Assume that $|\mathcal{G}'|^q$ satisfies Definition 10 and, under Assumption A^* , the following Ostrowski-type inequality associated with the k -fractional operator holds:

$$\begin{aligned} & \left| \frac{\Delta^{\gamma/k}(x, \tau_c, m)\mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m)\mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\ & \quad \left. - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))}^{\gamma, k} \mathcal{G}(x) \right] \right| \\ & \leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \left(\frac{\gamma}{\gamma + k} \right)^{1-1/q} \left(\frac{\gamma}{\gamma + k} \cdot \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1-s(1-v))^{1/j} + (1-sv)^{1/j}] dv \right)^{1/q} \\ & \quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \left(\frac{k}{\gamma + k} \right)^{1-1/q} \left(\frac{k}{\gamma + k} \cdot \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 [(1-s(1-v))^{1/j} + (1-sv)^{1/j}] dv \right)^{1/q}. \end{aligned}$$

Proof. By the identity in Lemma 2 and applying the triangle inequality, we have:

$$\begin{aligned} & \left| \frac{\Delta^{\gamma/k}(x, \tau_c, m)\mathcal{G}(m\tau_c) + \Delta^{\gamma/k}(\tau_d, x, m)\mathcal{G}(m\tau_c + \Delta(\tau_d, \tau_c, m))}{\Delta(\tau_d, \tau_c, m)} \right. \\ & \quad \left. - \frac{\Gamma_k(\gamma + k)}{\Delta(\tau_d, \tau_c, m)} \left[\zeta_{x-}^{\gamma, k} \mathcal{G}(m\tau_c) + \zeta_{(m\tau_c + \Delta(\tau_d, \tau_c, m))}^{\gamma, k} \mathcal{G}(x) \right] \right| \\ & \leq \frac{\Delta^{\gamma/k+1}(x, \tau_c, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 (1-v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \\ & \quad + \frac{\Delta^{\gamma/k+1}(\tau_d, x, m)}{\Delta(\tau_d, \tau_c, m)} \int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv. \end{aligned} \quad (32)$$

Applying the Power Mean inequality for $q \geq 1$ to each integral gives:

$$\begin{aligned} & \int_0^1 (1-v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \leq \left(\int_0^1 (1-v^{\gamma/k}) dv \right)^{1-1/q} \\ & \quad \times \left(\int_0^1 (1-v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))|^q dv \right)^{1/q}, \end{aligned} \quad (33)$$

and

$$\begin{aligned} & \int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv \leq \left(\int_0^1 v^{\gamma/k} dv \right)^{1-1/q} \\ & \quad \times \left(\int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))|^q dv \right)^{1/q}. \end{aligned} \quad (34)$$

Evaluating the elementary integrals directly yields:

$$\int_0^1 (1 - v^{\gamma/k}) dv = 1 - \frac{k}{\gamma + k} = \frac{\gamma}{\gamma + k}, \quad (35)$$

and

$$\int_0^1 v^{\gamma/k} dv = \frac{k}{\gamma + k}. \quad (36)$$

Applying Definition 10 to $|\mathcal{G}'|^q$ under Assumption A^* , we bound the derivative terms:

$$|\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))|^q \leq \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \left[(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j} \right], \quad (37)$$

and similarly:

$$|\mathcal{G}'(x + v\Delta(\tau_d, x, m))|^q \leq \frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \left[(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j} \right]. \quad (38)$$

Substituting Eqs. (35) and (37) into inequality (33) gives:

$$\begin{aligned} & \int_0^1 (1 - v^{\gamma/k}) |\mathcal{G}'(m\tau_c + v\Delta(x, \tau_c, m))| dv \\ & \leq \left(\frac{\gamma}{\gamma + k} \right)^{1-1/q} \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 (1 - v^{\gamma/k}) \left[(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j} \right] dv \right)^{1/q}. \end{aligned} \quad (39)$$

Similarly, substituting Eqs. (36) and (38) into inequality (34) gives:

$$\begin{aligned} & \int_0^1 v^{\gamma/k} |\mathcal{G}'(x + v\Delta(\tau_d, x, m))| dv \\ & \leq \left(\frac{k}{\gamma + k} \right)^{1-1/q} \left(\frac{S^q}{\sum_{j=1}^n \mu_j} \sum_{j=1}^n \mu_j \int_0^1 v^{\gamma/k} \left[(1 - s(1 - v))^{1/j} + (1 - sv)^{1/j} \right] dv \right)^{1/q}. \end{aligned} \quad (40)$$

Finally, substituting (39) and (40) back into inequality (32) completes the proof. $\square$

5. Conclusion

In this paper, we have developed a new framework within fractional convex analysis by introducing the new definition namely n -fractional polynomial s -like m -preinvex mappings. The fundamental properties and structural behavior of this new notion were rigorously investigated through several propositions, revealing its rich algebraic and analytical features.

A key contribution of the paper is the establishment of a novel lemma involving a k -fractional integral operator, which serves as an essential tool for deriving refined integral inequalities. On the basis of this lemma, we obtained several Ostrowski-type inequalities. Overall, the findings presented in this paper significantly extend the current literature by unifying various known results and introducing new generalizations within the theory of fractional integral inequalities.

Future research directions may include the investigation of analogous inequalities for other classes of fractional operators, extensions to multivariate settings, and potential applications in fractional differential equations and optimization problems.

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