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A new Iyengar-type inequality via a refined integral identity

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Abstract: In this paper, we present new sharp forms of Iyengar-type integral inequalities for differentiable and twice-differentiable functions. Using exact integral identities and the Cauchy-Schwarz inequality, we derive optimal L^2-bounds for the deviation between the midpoint value and the integral mean of a function. These results refine and extend several classical Hermite-Hadamard and Iyengar inequalities.

Keywords: Convex function, Hermite-Hadamard inequalities, trapezoid , midpoint inequalities, Iyengar inequality

MSC: 26A51, 26D15, 26D10.

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1. Introduction

The concept of convexity plays a central role in many areas of mathematical analysis, such as optimization theory, approximation theory, functional analysis, and the study of integral inequalities. A function $f : [a, b] \to \mathbb{R}$ is called *convex* if, for all $x, y \in [a, b]$ and $t \in [0, 1]$, the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds. Convex functions possess rich geometric and analytical properties that naturally lead to various fundamental inequalities.

One of the most celebrated results associated with convexity is the *Hermite–Hadamard inequality*, which provides a sharp relationship between the midpoint value of a convex function and its integral mean. If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities are reversed when f is concave. This classical result serves as the foundation for a large body of work exploring refined, generalized, and fractional versions of (1).

Over the past decades, numerous authors have contributed to the development of inequalities related to (1) and its variants. Agarwal and Dragomir [1] applied Hayashi's inequality to obtain new bounds for differentiable functions, thereby linking derivative-based estimates with Hermite–Hadamard-type results. Cerone and Dragomir [2] introduced a weighted generalization of Iyengar-type inequalities involving bounded first derivatives, providing a more flexible framework for the estimation of integral means. Mitrinovic et. al [3] presented a comprehensive monograph on inequalities concerning functions, their integrals, and derivatives, which has become a foundational reference for subsequent research in this area.

The study of the deviation between the arithmetic and integral means dates back to the pioneering work of Iyengar [4], who established the following result: if $f : [a, b] \to \mathbb{R}$ is differentiable and $|f'(x)| \leq M$ for all $x \in [a, b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}.$$

Subsequent studies have refined, generalized, and extended Iyengar's result in multiple directions. Cheng [5] established an improved version of Iyengar's inequality, deriving sharper estimates based on differential bounds. Franjic et. al [6] later presented further refinements, emphasizing tightness of the constants involved. Huy and Ngo [7] extended the inequality to quadrature formulas involving n-knots, thus connecting the classical Iyengar result with numerical integration techniques. Milovanovic and Pecaric [8] studied related applications in approximation theory and integral inequalities, offering additional insights into the practical aspects of Iyengar-type bounds. Dragomir and Wang [9] explored applications of these inequalities to estimate error bounds for the trapezoidal quadrature rule, demonstrating their utility in numerical analysis.

More recent research has incorporated fractional calculus and generalized convexity concepts to establish new versions of Iyengar-type inequalities. Sarikaya [10] introduced generalized Iyengar-type inequalities in the setting of fractional differential calculus, whereas Sarikaya, Yaldiz, and Budak [11] obtained weighted Iyengar-type inequalities for conformable fractional integrals, broadening the classical framework to fractional and weighted contexts.

Motivated by these advances, the present paper develops *new Iyengar-type inequalities* that directly involve the midpoint value

$$f\left(\frac{a+b}{2}\right),$$

rather than the traditional arithmetic mean

$$\frac{f(a)+f(b)}{2},$$

ensuring clarity and consistency with the main functional $E_m(f)$ introduced below. By employing precise integral identities and sharp L^2-norm estimates based on the Cauchy–Schwarz inequality, we derive refined inequalities with optimal constants. These results not only generalize and strengthen previously known Hermite–Hadamard and Iyengar-type inequalities but also provide a unified analytical framework applicable to a broad class of differentiable and convex-type functions.

For clarity and consistency throughout the paper, we introduce the interval length

$$\ell := b-a, \qquad m := \frac{a+b}{2},$$

and the midpoint deviation functional

$$E_m(f) := \frac{1}{\ell}\int_a^b f(x)\,dx - f(m).$$

2. Main Result

In this section, we establish our main results, which present new and sharp forms of Iyengar-type inequalities expressed in terms of the midpoint value of a differentiable function. The proofs are based on exact integral identities and the application of the Cauchy–Schwarz inequality in the L^2-space framework. By carefully analyzing the kernel structure arising from these identities, we derive optimal constants and provide complete equality characterizations. These results not only refine the classical Iyengar inequality but also extend the Hermite-Hadamard framework to settings involving first and second derivatives, thereby offering tighter analytical bounds for differentiable and twice-differentiable functions.

Theorem 1. *Let $f \in C^1([a,b])$, and set*

$$\Delta := f(b) - f(a).$$

Then

$$\left| f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx \right| \le \frac{\ell^{1/2}}{2\sqrt{3}} \sqrt{\int_a^b [f'(x)]^2\,dx \; - \; \frac{\Delta^2}{\ell}}.$$

The constant $\dfrac{\ell^{1/2}}{2\sqrt{3}}$ is optimal.

Moreover, equality in the Cauchy-Schwarz step formally requires

$$f'(x) = \frac{\Delta}{\ell} + \lambda\, K(x) \quad \text{a.e. on } [a,b],$$

for some $\lambda \in \mathbb{R}$, *where*

$$K(x) = \begin{cases} x-a, & x \in [a,m], \\ x-b, & x \in (m,b]. \end{cases}$$

However, for $\lambda \neq 0$, *this profile does not belong to* $C^1([a,b])$. *Consequently, under the hypothesis* $f \in C^1([a,b])$, *equality in the stated inequality occurs only in the trivial affine case* ($\lambda = 0$). *Thus, the constant is best possible, but non-attained in* $C^1([a,b])$ *except for affine functions.*

Proof. Define K as above. Writing

$$f(x) = f(a) + \int_a^x f'(t)\,dt,$$

and averaging over $[a,b]$ yields the exact identity

$$\frac{1}{\ell}\int_a^b f(x)\,dx - f(m) = \frac{1}{\ell}\int_a^b K(x)\,f'(x)\,dx.$$

Set $g := f'$ and define

$$\Delta_m := f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx = -\frac{1}{\ell}\int_a^b K(x)g(x)\,dx.$$

Decompose g into its mean and zero-mean parts,

$$g(x) = c + h(x), \qquad c := \frac{1}{\ell}\int_a^b g(x)\,dx = \frac{\Delta}{\ell}, \qquad \int_a^b h(x)\,dx = 0.$$

Since $\int_a^b K(x)\,dx = 0$, we obtain

$$\int_a^b K(x)g(x)\,dx = \int_a^b K(x)h(x)\,dx.$$

Applying the Cauchy-Schwarz inequality in $L^2(a,b)$ gives

$$\left|\int_a^b K(x)h(x)\,dx\right| \le \|K\|_2\,\|h\|_2.$$

A direct computation yields

$$\|K\|_2^2 = \int_a^b K(x)^2\,dx = 2\int_0^{\ell/2} u^2\,du = \frac{\ell^3}{12}, \qquad \|K\|_2 = \frac{\ell^{3/2}}{2\sqrt{3}}.$$

Moreover,

$$\|h\|_2^2 = \int_a^b h(x)^2\,dx = \int_a^b g(x)^2\,dx - \ell c^2 = \int_a^b [f'(x)]^2\,dx - \frac{\Delta^2}{\ell}.$$

Combining these relations gives

$$|\Delta_m| \le \frac{1}{\ell}\|K\|_2\,\|h\|_2 = \frac{\ell^{1/2}}{2\sqrt{3}}\sqrt{\int_a^b [f'(x)]^2\,dx - \frac{\Delta^2}{\ell}}.$$

The constant cannot be improved within this framework, since equality in Cauchy-Schwarz is formally obtained when h is proportional to K. However, such extremal profiles fail to lie in $C^1([a,b])$ unless $\lambda = 0$, which completes the proof. □

Remark 1. Assume in addition that $|f'(x)| \leq M$ a.e. on $[a,b]$. Since

$$\int_a^b [f'(x)]^2\,dx \leq M^2(b-a),$$

the main estimate yields the following corollary bound:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{b-a}{2\sqrt{3}}\sqrt{M^2 - \frac{\Delta^2}{(b-a)^2}}\,.$$

Although this inequality is weaker than the sharp L^2-based form, it is often useful in situations where only a uniform bound M for the derivative is available.

Theorem 2. *Let $f \in C^2([a,b])$. Then the following midpoint-type estimate holds:*

$$\left| f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx \right| \leq \frac{\ell^{3/2}}{8\sqrt{5}}\,\|f''\|_{L^2(a,b)}.$$

Moreover, equality in the above inequality (in the L^2 sense) holds if and only if

$$f''(x) = \lambda\,\mathcal{L}(x) \quad \text{a.e. on } [a,b],$$

for some $\lambda \in \mathbb{R}$, where

$$\mathcal{L}(x) = \begin{cases} -\frac{1}{2}(x-a)^2, & a \leq x \leq m, \\ -\frac{1}{2}(x-b)^2, & m < x \leq b. \end{cases}$$

Since $\mathcal{L}$ is not continuously differentiable at $x = m$, any nontrivial choice $\lambda \neq 0$ would contradict the assumption $f \in C^2([a,b])$. Hence, under this regularity hypothesis, equality occurs only in the trivial case $f'' \equiv 0$, that is, precisely when f is affine on $[a,b]$.

Proof. Define

$$K(x) = \begin{cases} x-a, & a \leq x \leq m, \\ x-b, & m < x \leq b. \end{cases}$$

A direct computation yields the identity

$$\frac{1}{\ell}\int_a^b f(x)\,dx - f(m) = \frac{1}{\ell}\int_a^b K(x)\,f'(x)\,dx.$$

Since $\int_a^b K(x)\,dx = 0$, writing $f'(x) = f'(m) + \int_m^x f''(t)\,dt$ eliminates the constant term and gives

$$\int_a^b K(x) f'(x)\,dx = \int_a^b K(x)\int_m^x f''(t)\,dt\,dx.$$

Interchanging the order of integration, we obtain

$$f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx = -\frac{1}{\ell}\int_a^b \mathcal{L}(t)\,f''(t)\,dt,$$

where $\mathcal{L}$ is defined above.

Applying the Cauchy–Schwarz inequality in $L^2(a,b)$ gives

$$\left| f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx \right| \leq \frac{1}{\ell}\|\mathcal{L}\|_{L^2(a,b)}\,\|f''\|_{L^2(a,b)}.$$

A straightforward calculation shows that

$$\|\mathcal{L}\|_{L^2(a,b)}^2 = 2\int_0^{\ell/2} \left(\frac{u^2}{2}\right)^2 du = \frac{\ell^5}{320},$$

and hence

$$\|\mathcal{L}\|_{L^2(a,b)} = \frac{\ell^{5/2}}{8\sqrt{5}}.$$

Substituting this into the previous estimate yields the desired inequality.

The equality characterization follows from the equality case of Cauchy-Schwarz. □

Corollary 1. *Under the assumptions of Theorem 2, suppose in addition that $|f''(x)| \le N$ a.e. on $[a,b]$. Then*

$$\left| f(m) - \frac{1}{\ell}\int_a^b f(x)\,dx \right| \le \frac{\ell^2}{8\sqrt{5}}\,N,$$

where $\ell = b - a$.

Remark 2. The constant $\frac{\ell^{3/2}}{8\sqrt{5}}$ in Theorem 2 is optimal within the Cauchy–Schwarz approach, since it is determined by $\|\mathcal{L}\|_{L^2(a,b)}$.

Formally, equality in the estimate of Theorem 2 occurs when f'' is a scalar multiple of $\mathcal{L}$. However, under the standing assumption $f \in C^2([a,b])$, this can only happen in the trivial case $f'' \equiv 0$.

3. Applications and illustrative examples

Throughout this section, we write

$$\ell := b - a, \qquad m := \frac{a+b}{2}, \qquad E_m(f) := \frac{1}{\ell}\int_a^b f(x)\,dx - f(m).$$

The following examples illustrate Theorems 1 and 2. They are intended to clarify the size and behaviour of the bounds in concrete cases, without introducing new inequalities.

Lemma 1 (Comparison of the C^1- and C^2-based bounds). *Let $f \in C^2([a,b])$. Denote*

$$B_1 := \frac{\ell^{1/2}}{2\sqrt{3}}\sqrt{\int_a^b [f'(x)]^2 dx - \frac{(f(b)-f(a))^2}{\ell}}, \qquad B_2 := \frac{\ell^{3/2}}{8\sqrt{5}}\,\|f''\|_{L^2(a,b)}.$$

Then

$$|E_m(f)| \le B_1, \qquad |E_m(f)| \le B_2,$$

and moreover,

$$B_1 \le B_2 \quad \Longleftrightarrow \quad \int_a^b [f'(x)]^2 dx - \frac{(f(b)-f(a))^2}{\ell} \le \frac{3\ell^2}{80}\,\|f''\|_{L^2(a,b)}^2.$$

Proof. The bounds follow directly from Theorems 1 and 2. Squaring $B_1 \le B_2$ gives

$$\frac{\ell}{12}\left(\int_a^b [f'(x)]^2 dx - \frac{(f(b)-f(a))^2}{\ell}\right) \le \frac{\ell^3}{320}\|f''\|_{L^2(a,b)}^2,$$

and multiplying both sides by $12/\ell$ yields the stated equivalence. □

Remark 3. The first bound B_1 is sharper whenever the oscillation of f' around its mean is sufficiently small compared to $\ell\|f''\|_{L^2}$. In contrast, B_2 becomes preferable for functions with small L^2-curvature.

Example 1 (A cubic polynomial)**.** Let $f(x) = x^3$ on $[0,1]$. Then $\ell = 1$, $m = \frac{1}{2}$, and $\Delta = f(1) - f(0) = 1$. A direct computation gives

$$f\left(\tfrac{1}{2}\right) = \frac{1}{8}, \qquad \int_0^1 f(x)\,dx = \frac{1}{4}, \qquad E_m(f) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

(i) From Theorem 1,

$$|E_m(f)| \leq \frac{1}{2\sqrt{3}}\sqrt{\int_0^1 [f'(x)]^2 dx - \Delta^2}.$$

Since $f'(x) = 3x^2$, one obtains

$$\int_0^1 [f'(x)]^2 dx = 9\int_0^1 x^4 dx = \frac{9}{5}.$$

Hence

$$|E_m(f)| \leq \frac{1}{2\sqrt{3}}\sqrt{\frac{9}{5} - 1}.$$

(ii) From Theorem 2,

$$|E_m(f)| \leq \frac{1}{8\sqrt{5}}\,\|f''\|_{L^2(0,1)}.$$

Since $f''(x) = 6x$,

$$\|f''\|^2_{L^2(0,1)} = 36\int_0^1 x^2 dx = 12,$$

and therefore

$$|E_m(f)| \leq \frac{\sqrt{12}}{8\sqrt{5}}.$$

This example allows a direct numerical comparison between the first- and second-derivative bounds.

Example 2 (A trigonometric function)**.** Let $f(x) = \sin x$ on $[0,\pi]$. Then $\ell = \pi$, $m = \pi/2$, and $\Delta = 0$. Since

$$\int_0^\pi \sin x\,dx = 2, \qquad f(m) = 1,$$

we have

$$E_m(f) = \frac{2}{\pi} - 1.$$

Theorem 1 gives

$$|E_m(f)| \leq \frac{\pi^{1/2}}{2\sqrt{3}}\sqrt{\int_0^\pi \cos^2 x\,dx},$$

while Theorem 2 gives

$$|E_m(f)| \leq \frac{\pi^{3/2}}{8\sqrt{5}}\,\|\sin x\|_{L^2(0,\pi)}.$$

Again, the exact error is known, so the inequalities serve to illustrate the size of the constants in a smooth non-polynomial case.

Example 3 (Polynomial class of degree ≤ 3)**.** If f is a polynomial of degree at most 3 on $[a,b]$, then $f' \in L^2$ and $f'' \in L^2$ automatically. The midpoint deviation satisfies separately

$$|E_m(f)| \leq \frac{\ell^{1/2}}{2\sqrt{3}}\sqrt{\int_a^b [f'(x)]^2 dx - \frac{(f(b) - f(a))^2}{\ell}},$$

and

$$|E_m(f)| \leq \frac{\ell^{3/2}}{8\sqrt{5}}\|f''\|_{L^2(a,b)}.$$

No general ordering between these two bounds is asserted here; each estimate follows independently from its respective theorem.

Example 4 (Convex functions)**.** If $f \in C^2([a,b])$ is convex, then $f''(x) \geq 0$ on $[a,b]$. In this case Theorem 2 yields

$$|E_m(f)| \leq \frac{\ell^{3/2}}{8\sqrt{5}} \|f''\|_{L^2(a,b)}.$$

No claim is made that monotonicity alone implies any sign condition on f''; convexity (or concavity) is the appropriate assumption when such structure is needed.

Example 5 (Midpoint rule error estimate)**.** For $f \in C^2([a,b])$, the midpoint quadrature rule reads

$$\int_a^b f(x)\,dx \approx \ell\, f(m).$$

In normalized form,

$$|E_m(f)| = \left| \frac{1}{\ell} \int_a^b f(x)\,dx - f(m) \right|.$$

Theorem 2 provides the rigorous bound

$$|E_m(f)| \leq \frac{\ell^{3/2}}{8\sqrt{5}} \|f''\|_{L^2(a,b)},$$

which yields an L^2-based error estimate expressed solely in terms of the second derivative.

4. Conclusion

In this work, we have presented two complementary bounds for the midpoint deviation of smooth functions. The first bound applies to differentiable functions and refines the classical estimate by incorporating the deviation of the derivative from its mean. The second bound, based on the second derivative, provides a sharp L^2-type estimate with the constant $\frac{\ell^{3/2}}{8\sqrt{5}}$, and equality can only occur for affine functions under the C^2 hypothesis.

These results clarify the behavior of midpoint errors and give practical L^2-based tools for analyzing integral inequalities of Hermite-Hadamard and Iyengar type.

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References

[1] Agarwal, R. P., & Dragomir, S. S. (1996). An application of Hayashi's inequality for differentiable functions. *Computers & Mathematics With Applications*, *32*(6), 95-99.

[2] Cerone, P., & Dragomir, S. S. (2012). On a weighted generalization of Iyengar type inequalities involving bounded first derivative. *RGMIA Research Report Collection*, *2*(2).

[3] Mitrinovic, D. S., Pecaric, J., & Fink, A. M. (1994). *Inequalities for Functions and Their Integrals and Derivatives*. Kluwer Academic Publishers, Dordrecht, 1994.

[4] Iyengar, K. S. K. (1938). Note on an inequality, *Math. Student*, *6*, 75-76.

[5] Cheng, X. L. (2001). The Iyengar-type inequality. *Applied Mathematics Letters*, *14*(8), 975-978.

[6] Franjić, I., Pečarić, J., & Perić, I. (2006). Note on an Iyengar type inequality. *Applied Mathematics Letters*, *19*(7), 657-660.

[7] Huy, V. N. (2010). On an Iyengar-type inequality involving quadratures in n knots. *Applied Mathematics and Computation*, *217*(1), 289-294.

[8] Milovanović, G. V., & Pečarić, J. E. (1976). Some considerations on Iyengar's inequality and some related applications. Publikacije Elektrotehničkog fakulteta. *Serija Matematika I Fizika*, (544/576), 166-170.
[9] Dragomir, S., & Wang, S. (1998). Applications of Iyengar's type inequalities to the estimation of error bounds for the trapezoidal quadrature rule. *Tamkang Journal of Mathematics, 29*(1), 55-58.
[10] Sarikaya, M. Z. (2023). On the generalized Iyengar type inequalities. *Fractional Differential Calculus, 13*(2), 133-139.
[11] Sarikaya, M. Z., Yaldiz, H., & Budak, H. (2017). On weighted Iyengar-type inequalities for conformable fractional integrals. *Mathematical Sciences, 11*(4), 327-331.