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# A new operator, based on summation of Fourier series, and its interpolation analogues for filtering of onedimensional signals

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**Abstract:** Summability methods for trigonometric Fourier series play a fundamental role in approximation theory and signal processing. Among them, Fejér means provide a classical regularization tool ensuring uniform convergence for continuous functions. In this paper, we investigate an operator constructed as the arithmetic mean of the first  $n$  Fejér means. This approach leads to an additional averaging procedure and naturally strengthens the smoothing effect compared to a single Fejér mean of the same order. The operator is studied both in the time and frequency domains. In the time domain, it is represented as a convolution operator with a positive summability kernel. Its normalization and structural properties are established, including preservation of constants and removability of the singularity at the origin. In the frequency domain, the operator is described via its Fourier multipliers, obtained as averages of the corresponding Fejér multipliers. Their monotonic decay with respect to the harmonic index is analyzed, which provides insight into the enhanced attenuation of high-frequency components. A discrete (interpolation-type) analogue defined on a uniform grid is also introduced and interpreted as a quadrature approximation of the continuous convolution representation. Explicit representations of the operator and its kernel are derived. The smoothing character of the method is justified theoretically and confirmed numerically for periodic signals with additive noise. The experiments demonstrate improved suppression of high harmonics compared to classical Fejér summation of the same order. The proposed operator can be regarded as a strengthened low-pass Fourier multiplier method and may be effectively applied to smoothing and filtering of one-dimensional periodic signals.

**Keywords:** Fejér means, Fourier multipliers, summability methods, trigonometric approximation, smoothing operators

**MSC:** 42A24, 41A25, 42A10.

## 1. Introduction

The usage of different operators, generated by linear methods of summation of Fourier series, is a classical way to filter onedimensional signals [1] and in other methods of its processing [1–4]. As classical example to do it we may use partial Fourier sums [1]

$$S_n(t) = S_n(f, t) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(kt) + b_k \sin(kt)), \quad (1)$$

where  $f = f(t)$  is a function, what describes studying signal,  $a_k$  and  $b_k$  are Fourier coefficients of this function, Fejér operators [1]

$$\sigma_n(t) = \sigma_n(f, t) = \frac{1}{n} \sum_{k=0}^{n-1} S_k(t), \quad (2)$$

and its interpolation analogues. Usage of these operators guarantee a filtration of signals in frequency domain, but effect of its usage may be not enough strong, especially in case, when we work with signals with large bandwidth. In this work we propose a new operator  $\mu_n(t)$ , described by formula

$$\mu_n(t) = \mu_n(f, t) = \frac{1}{n} \sum_{k=1}^n \sigma_k(t), \quad (3)$$

what gives more strong effect as signal filter.

## 2. Materials and methods

To study (3) as method of signal processing a research is built on two steps. First step is based on representation of (3) as operator and making an interpolation analogue of (3) using this operator. Second step based on filtering a concrete signal with random noise using (2) and (3) and comparison of given results. To do it, Python programming language is used. Two scripts on Python is written. First script is written for generation of a signal with addition of random noise using next code:

Listing 1: Code of script for generation of considered signal

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from numpy import pi
from scipy.special import gamma, digamma

# signal forming
def f(x : np.ndarray) -> np.ndarray:
    return gamma(np.exp(np.sin(x)+np.cos(2*x)/2))*(np.sin(2*x)-np.cos(3*x)/3)
        + digamma(np.exp(np.sin(3*x)-np.cos(4*x)/4))*(np.sin(4*x)+np.cos(5*x)/5)

N = int(1e6+1)
x = np.linspace(0, 2*pi, N)
y = f(x)

# addition of random noise
noise = np.random.normal(0, 1, N)
y = y + noise

# plotting of generated signal
plt.plot(x, y)
plt.xlabel("time_in_seconds ,_t")
plt.ylabel("Noised_signal")
plt.grid()
plt.show()

# saving of generated signal
data = pd.DataFrame()
data["signal"] = y
data.index = x
```

```
data.to_csv("signal.csv")
```

Second Python script realize a filtration of signal, generated by first script, using (2) and (3) for come other values of parameter  $n$  and visualize result of this filtration. This script realizes next code:

Listing 2: Code of script for signal filtering and visualizing of results

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from numpy import pi

# needed functions
def fejer_sum(x : np.ndarray, f : np.ndarray, n : int) -> np.ndarray:
    def m_sigma(n : int, k : int) -> float:
        return 1.0 - k/n

    if n<0:
        return zeros(len(x))

    res = np.ones(len(x)) * 0.5 * np.mean(f)
    for k in range(1, n):
        res = res + m_sigma(n, k)*( np.mean(f*np.cos(k*x))
        *np.cos(k*x)+np.mean(f*np.sin(k*x))*np.sin(k*x) )
    return res/n

def new_sum(x : np.ndarray, f : np.ndarray, n : int) -> np.ndarray:
    def l_mu(n : int, k : int) -> float:
        def H(N : int):
            return sum([1.0/m for m in range(1, N+1)])

        return k * (H(n)-H(k)) / n

    if n<0:
        return np.zeros(len(x))

    res = np.ones(len(x)) * 0.5 * np.mean(f)
    for k in range(1, n+1):
        res = res + l_mu(n, k)*( np.mean
        *np.cos(k*x)+np.mean(f*np.sin(k*x))*np.sin(k*x) )
    return res

# signal reading and filtering
data = pd.read_csv("signal.csv")
data.columns = ["time", "signal"]

t = data["time"].to_numpy()
f = data["signal"].to_numpy()
ns = [5, 10, 15]

fig, axs = plt.subplots(3, 2)
for i in range(3):
```

```

axs[i, 0].plot(t, fejer_sum(t, f, ns[i]), label="Operator_(2),_n="+str(ns[i]))
axs[i, 0].grid()
axs[i, 0].legend()

axs[i, 1].plot(t, new_sum(t, f, ns[i]), label="Operator_(3),_n="+str(ns[i]))
axs[i, 1].grid()
axs[i, 1].legend()
plt.show()

```

After use of these two scripts a comparison of (2) and (3) as filters of signals is made.

### 3. Results

#### 3.1. Some properties of operators (3)

Let describe some propositions about operator (3).

**Proposition 1.** *The operator (3) admits the Fourier multiplier representation*

$$\mu_n(t) = \mu_n(f, t) = \frac{1}{n} \frac{a_0}{2} + \sum_{k=1}^{n-1} \lambda_{n,k} (a_k \cos kt + b_k \sin kt), \quad (4)$$

where  $\lambda_{n,k} = \frac{k}{n} (H_n - H_k)$ , and  $H_n = \sum_{k=1}^n \frac{1}{k}$ .

**Proof.** Let denote

$$A_k(t) = a_k \cos kt + b_k \sin kt.$$

Now let rewrite a formula (3).

$$\begin{aligned} \mu_n(t) = \mu_n(f, t) &= \frac{1}{n} \sum_{k=1}^n \sigma_k(t) \\ &= \frac{1}{n} \cdot \left( \frac{a_0}{2} + \frac{a_0}{2} + \frac{1}{2} A_1 + \frac{a_0}{2} + \frac{1}{3} A_1 + \frac{2}{3} A_2 + \frac{a_0}{2} + \frac{1}{4} A_1 + \frac{2}{4} A_2 + \frac{3}{4} A_3 + \dots \right. \\ &\quad \left. + \frac{a_0}{2} + \frac{1}{n} A_1 + \frac{2}{n} A_2 + \dots + \frac{n-1}{n} A_{n-1} \right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) A_k \\ &= \frac{a_0}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) (a_k \cos kt + b_k \sin kt). \end{aligned}$$

Proposition proofed.  $\square$

**Proposition 2.** *The operator (3) may be represented by integral*

$$\mu_n(t) = \mu_n(f, t) = \frac{1}{n} \int_0^{2\pi} f(u) \frac{n \sin \frac{u-t}{2} - \sin \frac{n(u-t)}{2} \cos \frac{(n+1)(u-t)}{2}}{\sin \frac{u-t}{2} (1 - \cos(u-t))} du. \quad (5)$$

**Proof.** To prove this proposition, let use a fact, that Fejér operator (2) may be represented by integral

$$\sigma_n(t) = \frac{1}{2n\pi} \int_0^{2\pi} f(u) \frac{1 - \cos(n(u-t))}{1 - \cos(u-t)} du.$$

If we replace this into (3) and doing some manipulations, we will have next

$$\mu_n(t) = \mu_n(f, t) = \frac{1}{n} \int_0^{2\pi} f(u) \frac{n - \sum_{k=1}^n \cos(k(u-t))}{1 - \cos(u-t)} du.$$

Using the equality

$$\sum_{k=1}^n \cos kx = \frac{\sin \frac{nx}{2} \cos \frac{(n+1)x}{2}}{\sin \frac{x}{2}},$$

and doing some additional manipulations, we will have, that

$$\mu_n(t) = \mu_n(f, t) = \frac{1}{n} \int_0^{2\pi} f(u) \frac{n \sin \frac{u-t}{2} - \sin \frac{n(u-t)}{2} \cos \frac{(n+1)(u-t)}{2}}{\sin \frac{u-t}{2} (1 - \cos(u-t))} du.$$

Proposition proofed.  $\square$

**Proposition 3.** Let consider, that  $K_n(t)$  is a kernel of (5). So,

$$\frac{1}{\pi} \int_0^{2\pi} K_n(t) dt = 1.$$

for all natural  $n$ .

**Proof.** After doing some operations with  $K_n(t)$ , we will see, that

$$K_n = \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) (\cos kt + \sin kt),$$

where

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

Using it, we will have next.

$$\begin{aligned} \int_0^{2\pi} K_n(t) dt &= \int_0^{2\pi} \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) (\cos kt + \sin kt) \\ &= \frac{1}{2} \int_0^{2\pi} dt + \frac{k}{n} (H_n - H_k) \left( \int_0^{2\pi} \cos kt dt + \int_0^{2\pi} \sin kt dt \right) \\ &= \pi. \end{aligned}$$

As the result, we will have, that

$$\frac{1}{\pi} \int_0^{2\pi} K_n(t) dt = \frac{\pi}{\pi} = 1.$$

Proposition proofed.  $\square$

**Proposition 4.** Let consider, that  $K_n(t)$  is a kernel of (5). So, limit

$$\lim_{t \rightarrow 0} K_n(t) dt,$$

equals

$$\frac{1}{2} + \frac{1}{n} \int_0^1 \frac{\sum_{k=1}^{n-1} kx^k}{1-x} dx - \frac{n-1}{2} \int_0^1 \frac{x^n}{1-x} dx,$$

and is finite for all finite  $n$ .

**Proof.** As in proof of proposition 3, let use representation of  $K_n(t)$  in a form

$$K_n = \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) (\cos kt + \sin kt),$$

where

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

Using it, let find  $\lim_{t \rightarrow 0} K_n(t) dt$ .

$$\lim_{t \rightarrow 0} K_n(t) dt = \lim_{t \rightarrow 0} \left( \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k) (\cos kt + \sin kt) \right) = \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} (H_n - H_k).$$

Using an integral representation of harmonic numbers  $H_n$  in form

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x} dx,$$

we will have next

$$\begin{aligned} \lim_{t \rightarrow 0} K_n(t) dt &= \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k}{n} \int_0^1 \frac{x^k - x^n}{1 - x} dx \\ &= \frac{1}{2} + \frac{1}{n} \int_0^1 \frac{1}{1 - x} \left( \sum_{k=1}^{n-1} kx^k \right) dx - \frac{1}{n} \left( \sum_{k=1}^{n-1} k \right) \int_0^1 \frac{x^n}{1 - x} dx \\ &= \frac{1}{2} + \frac{1}{n} \int_0^1 \frac{\sum_{k=1}^{n-1} kx^k}{1 - x} dx - \frac{n-1}{2} \int_0^1 \frac{x^n}{1 - x} dx. \end{aligned}$$

It's easy to see, that for finite  $n$  an expression

$$\frac{1}{2} + \frac{1}{n} \int_0^1 \frac{\sum_{k=1}^{n-1} kx^k}{1 - x} dx - \frac{n-1}{2} \int_0^1 \frac{x^n}{1 - x} dx,$$

is finite too. It may be shown by making one integral from second and third terms in this expression. It may be shown using Bézout's theorem, that given integral is finite for every finite  $n$ .

Proposition proved.  $\square$

Let analyze described propositions. Proposition 1 shows, that operator (3) is a triangular linear summation method of Fourier series with multipliers  $\lambda_{n,k}$ , which may be calculated by formula

$$\lambda_{n,k} = \frac{k}{n} (H_n - H_k) = \frac{k}{n} \sum_{m=k}^n \left( 1 - \frac{k}{m+1} \right),$$

what is different from the previously described ones. For example, multipliers  $\lambda_{n,k}$  for a Fejér operator may be calculated by formula

$$\lambda_{n,k} = 1 - \frac{k}{n},$$

and for Abel-Poisson partial sums [5] may be calculated by formula

$$\lambda_{n,k} = e^{-\frac{k}{n}}.$$

In addition, proposition 1 shows, that operator (3) satisfies the classical conditions of regular linear summation methods: bounded multipliers, convergence of  $\lambda_{n,k}$  to 1 for fixed  $k$  when  $n \rightarrow \infty$ , and preservation of constant functions.

Second proposition 2 allows to write an discrete variant of (5), what in some literature, related with Fourier analysis (for example, in [2] or [5]), is called as interpolation analogue of operator and may be written by formula

$$\tilde{\mu}_n(t) = \tilde{\mu}_n(f, t) = \frac{1}{mn} \sum_{k=1}^m f(t_k) \frac{n \sin \frac{t_k-t}{2} - \sin \frac{n(t_k-t)}{2} \cos \frac{(n+1)(t_k-t)}{2}}{\sin \frac{t_k-t}{2} (1 - \cos(t_k - t))}, \tag{6}$$

where sequence of argument values  $\{t_k\}_{k=1}^m$  gives an uniform grid in interval  $[0; 2\pi]$ .

Third and fourth Propositions 3 and 4 are about properties of a kernel of integral representation of operator (3). These propositions say, that integral of this kernel is finite, what may be useful in approximation theory and signal processing, especially, in approximation of digital signals using discrete variant (6).

### 3.2. Comparative analysis of (2) and (3) as digital filters

For demonstration difference of signal filtering using (2) and (3) firstly let compare behavior of Fourier coefficient multipliers of these operators. To do it, let denote Fourier coefficient multipliers for Fejér operators (2) as

$$\lambda_{\sigma_n}(n, k) = 1 - \frac{k}{n},$$

and Fourier coefficient multipliers for operator (3) as

$$\lambda_{\mu_n}(n, k) = \frac{k}{n} (H_n - H_k).$$

Now let visualize  $\lambda_{\sigma_n}(n, k)$  and  $\lambda_{\mu_n}(n, k)$  for some different values of parameter  $n$ .

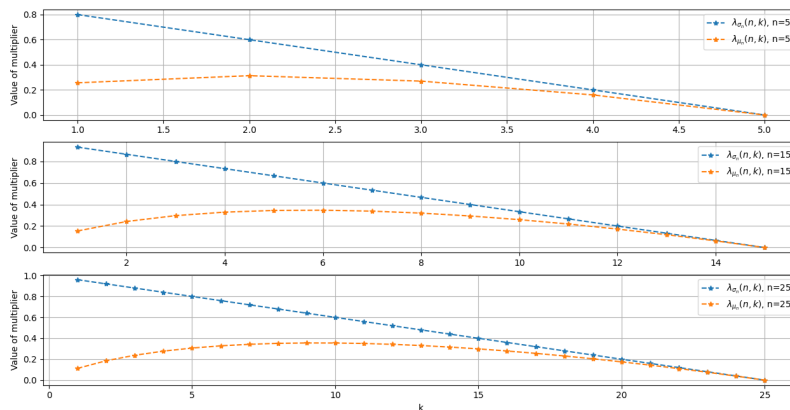


Figure 1. A behavior of  $\lambda_{\sigma_n}(n, k)$  and  $\lambda_{\mu_n}(n, k)$  for different values of  $n$

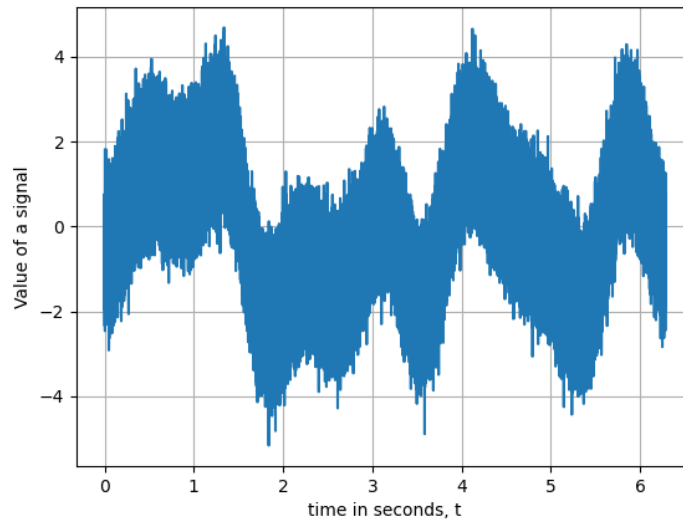
As we may see on Figure 1, the introduced operator (3) may give really stronger frequency suppression than Fejér operator (2), but this suppression has other behavior: unlike Fejér operator  $\sigma_n(t)$  operator  $\mu_n(t)$  will suppress all harmonics, but the further harmonics is from  $\lfloor \frac{n}{3} \rfloor + 1$ -th, a more suppression of this harmonics will be expected. This factor may be used for signal filtering, especially in case of localization needed frequency.

To demonstrate described effect let consider a digital signal, continuous equivalent of which may be described by function

$$f(t) = \Gamma \left( e^{\sin t + \frac{\cos 2t}{2}} \right) \left( \sin 2t - \frac{\cos 3t}{3} \right) + \psi \left( e^{\sin 3t - \frac{\cos 4t}{4}} \right) \left( \sin 4t + \frac{\cos 5t}{5} \right), t \in [0, 2\pi].$$

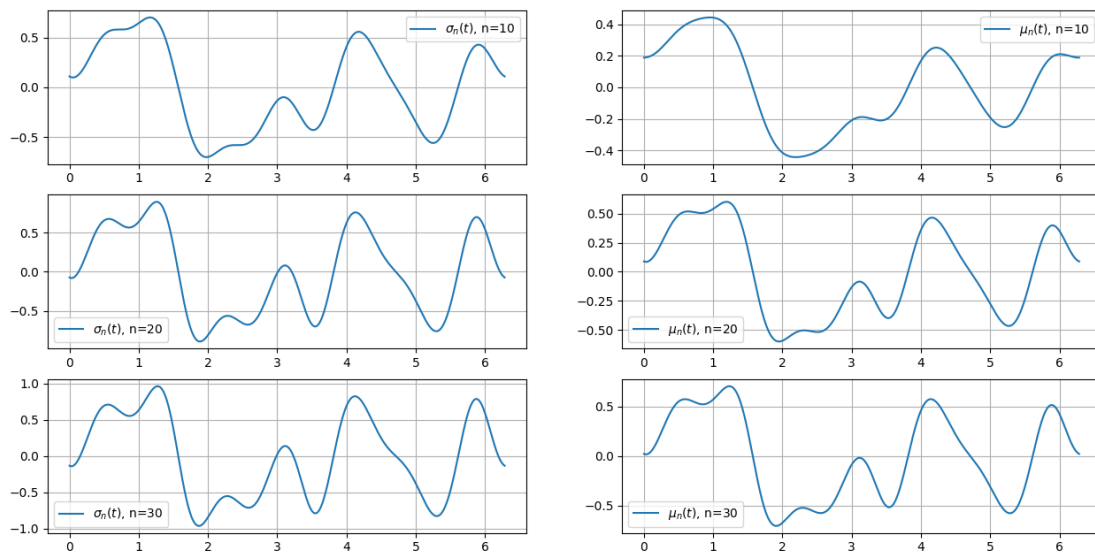
and let add to every  $i$ -th sample of signal and additive white Gaussian noise  $v_i \sim N(0, 1)$ .

Firstly, let visualize noised signal.



**Figure 2.** A signal, generated for demonstration of difference between filtering using (2) and (3)

Now, let visualize a result of firtering of signal, shown on Figure 2, using (2) and (3) with different values of parameter  $n$ .



**Figure 3.** A result of filtering of the signal using (2) and (3)

We see, that visually interpolation analogues of (3) may be used as signal filter and as approximator of digital signals like interpolation analogues of Fejér operators (2), but in case of small values of parameter  $n$  looks worth. But, as we may see on Figure 3, it may be solved by using of higher values of  $n$ .

To make more detailed research of an effect of approximation of the signal using interpolation analogues of (2) and (3) let visualize a dependence between mean squared error

$$MSE = \frac{1}{N} \sum_{k=1}^N (f(t_k) - \tilde{f}(t_k))^2,$$

(between real digital signal  $f(t_k)$  and result of usage of interpolation analoque  $\tilde{f}(t_k)$  of (2) and (3)) and value of parameter  $n$ .

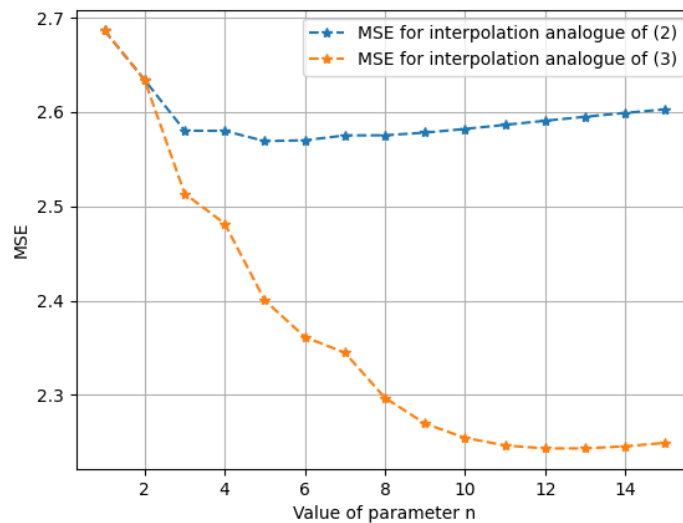


Figure 4. Dependence between MSE and value of parameter  $n$

On Figure 4 we may see, that in case of  $n > 2$  interpolation analogues of (3) may give less MSE, what means, that for same values of parameter  $n$  and sample rate interpolation analogues of operator  $\mu_n(t)$  gives better approximation of digital signal, than interpolation analogues of Fejér operator  $\sigma_n(t)$ .

#### 4. Conclusions

This research is considering operator  $\mu_n(f, t)$ , defined by formula (3). As a research shows, this operator has an integral representation (5). Discrete form (6) of this integral representation, what sometimes is called interpolation analogue of an operator, may be used for approximation of one-dimensional digital signal in time domain, what may be seen on Figure 2 and Figure 3, but this approximation will be better, that approximation using interpolation analogue of Fejér operator in case of same value of parameter  $n$  and same sampling rate, what we may see on Figure 4. In addition, research shows, that operator (3) admits the Fourier multiplier representation, what was demonstrated and proofed as one of propositions.

These aspects may have practical value in DSP. For example, in future research approximation properties of (3) may be studied and compared with approximation properties of other, widely used operators like Fejér operator (2) [1], Abel-Poisson operators [2] and its triangular analogue [5] or other. As other perspective of future research interpolation analogue (6) may be studied and compared with other interpolation methods [7-9], especially, in case of DSP [10,11]. In addition, future research may be based on studying of (6) as a numerical method [12]. As existing research shows, (6) may be considered in context of modeling [13,14,14], fundamental research of approximation and filtering methods [1,16] and many other aspects of DSP.

#### References

- [1] Hamming, R. W. (1998). *Digital Filters*. Courier Corporation.
- [2] Makarchuk, A., Kal'Chuk, I., Kharkevych, Y., & Kharkevych, G. (2022, December). Application of trigonometric interpolation polynomials to signal processing. In *P2022 IEEE 4th International Conference on Advanced Trends in Information Theory (ATIT)* (pp. 156-159). IEEE.
- [3] Chihara, N., Takata, T., Fujiwara, Y., Noda, K., Toyoda, K., Higuchi, K., & Onizuka, M. (2023). Effective detection of variable celestial objects using machine learning-based periodic analysis. *Astronomy and Computing*, 45, 100765.
- [4] Bergmann, R., & Merkert, D. (2020). FFT-based homogenization on periodic anisotropic translation invariant spaces. *Applied and Computational Harmonic Analysis*, 48(1), 266-292.
- [5] Barabash, O., Musienko, A., Makarchuk, A., Ivanna, S., Obidin, D., & Ilin, O. (2024, May). Abel-Poisson partial sums in signal theory. In *2024 International Congress on Human-Computer Interaction, Optimization and Robotic Applications (HORA)* (pp. 1-4). IEEE.

- [6] Koval, O., Barabash, O., Kharkevych, Y., Makarchuk, A., Kal'chuk, I., & Kharkevych, G. (2024, November). Some Interpolation Analogues as a Tool for Signals Approximation in Network Technologies. In *2024 IEEE 5th International Conference on Advanced Trends in Information Theory (ATIT)* (pp. 185-189). IEEE.
- [7] Zimmermann, R., & Bergmann, R. (2024). Multivariate Hermite interpolation on Riemannian manifolds. *SIAM Journal on Scientific Computing*, 46(2), A1276-A1297.
- [8] De Marchi, S., Dell'Accio, F., & Nudo, F. (2024). A mixed interpolation-regression approximation operator on the triangle. *Dolomites Research Notes on Approximation*, 17(3), 33-44.
- [9] Campagna, R., De Marchi, S., Perracchione, E., & Santin, G. (2022). Stable interpolation with exponential-polynomial splines and node selection via greedy algorithms. *Advances in Computational Mathematics*, 48(6), 69.
- [10] Vaseghi, S. V. (2008). *Advanced Digital Signal Processing and Noise Reduction*. John Wiley & Sons.
- [11] Lyons, R. G. (2004). *Understanding Digital Signal Processing*. 2nd ed. Eds.; Prentice-Hall, International, New Jersey, USA.
- [12] Hamming, R. (2012). *Numerical Methods for Scientists and Engineers*. Courier Corporation.
- [13] Aziz, S., & Chiba, H. (2025). Numerical Analysis of Approximate Solutions and Linear Growth in a Glial Cell Dynamics Model. *Journal of Applied Mathematics and Physics*, 13(12), 4506–4547.
- [14] Oko, T. O., Eze, E. O., & Adeyeye, A. C. (2025). Approximation of symmetric periodic solutions of the Sitnikov problem. *Open Journal of Mathematical Sciences*, 9, 32-44.
- [15] Bhatia, R. (2026). Numerical solution of Burgers' equation using collocation of uniform hyperbolic polynomial B-splines. *Journal of Applied Mathematics and Computing*, 72(2), 98.
- [16] Youssef, A. M., Rageh, T. M., Fareed, A. F., & Abdelwahed, M. S. (2025). Approximations for Fractional Derivatives and Fractional Integrals Using Padé Approximation. *Journal of Applied Mathematics*, 2025(1), 6678349.



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