Open Journal of
Mathematical Sciences

PSR Press

Article

A note on complete minimal submanifolds of Riemannian manifolds

Sergey Stepanov[1,2,*]

[1] Department of Scientific Information on Fundamental and Applied Mathematics, Russian Institute for Scientific and Technical Information, Russian Academy of Sciences, Moscow, Russia; s.e.stepanov@mail.ru

[2] Department of Mathematics and Data Analysis, Finance University, Moscow, Russia

Received: 23 Apr 2026; Revised: 20 Jun 2026; Accepted: 25 Jun 2026; Published: 19 Jul 2026

Abstract: In this paper, we study complete minimal submanifolds of Riemannian manifolds by employing a generalized Bochner technique. First, we provide a generalization of the classical theorem of Chern, Kobayashi, and do Carmo on compact minimal submanifolds to the case of complete parabolic minimal submanifolds. Second, we investigate the rigidity of complete stable minimal parabolic hypersurfaces in Riemannian manifolds, showing that under certain curvature conditions such hypersurfaces must be totally geodesic.

Keywords: minimal parabolic submanifold, stable minimal parabolic hypersurface

MSC: 53C42, 53C21, 31C12.

---

1. Introduction

Minimal submanifolds constitute a central topic in the calculus of variations and geometric analysis. In this paper, we study complete minimal submanifolds of Riemannian manifolds by means of a generalized Bochner technique, which provides effective tools for deriving rigidity results under curvature conditions (see, e.g., [1–3]).

The classical Bochner method establishes deep relations between curvature and global properties of Riemannian manifolds, particularly in the compact setting (see [4]). Its extensions to noncompact manifolds require additional analytic input, typically in the form of maximum principles or Liouville-type theorems (see. for example, [5–8]).

A natural framework for such extensions is provided by *parabolic manifolds*, i.e., complete Riemannian manifolds that do not admit a positive Green function (see [2,9]). A basic sufficient condition for parabolicity is finite volume, although this condition is not necessary. More generally, parabolicity depends on the volume growth and the potential-theoretic structure of the manifold. In particular, parabolic manifolds form a broad class that includes many geometrically significant examples.

The main goal of this paper is to extend classical rigidity results for minimal submanifolds to the setting of *complete parabolic manifolds*.

First, we prove a parabolic analogue of the classical pinching theorem of Shiing-Shen Chern, Manfredo do Carmo, and Shoshichi Kobayashi. In the compact case, this theorem asserts that a minimal submanifold in a sphere satisfying a suitable pinching condition on the second fundamental form must be parallel. In Theorem 2, we extend this result to *complete parabolic minimal submanifolds*, replacing compactness by a Liouville-type argument for bounded subharmonic functions. We emphasize that the pinching condition is nontrivial only in the case of *positive ambient curvature*, so the result is most relevant for space forms with $C > 0$.

Second, we investigate rigidity properties of *complete stable minimal hypersurfaces*. In Theorem 3, we show that if such a hypersurface is parabolic and admits a positive supersolution of the Jacobi equation, then it must be totally geodesic. The proof relies on the Liouville property for positive superharmonic functions on parabolic manifolds. While related rigidity results are known in the literature, the formulation given here isolates a simple set of sufficient conditions combining stability, parabolicity, and a maximum principle argument.

As a consequence, we obtain a simplified rigidity statement (Corollary 2) under the assumption that the Jacobi operator admits a positive solution. This allows one to reduce the geometric problem to a purely analytic condition.

The paper is organized as follows. In §2, we establish a parabolic extension of the Chern–do Carmo–Kobayashi theorem for minimal submanifolds in space forms of positive curvature (see [10]). In §3, we apply Liouville-type arguments to derive rigidity results for complete stable minimal hypersurfaces.

2. Parabolic minimal submanifolds in Riemannian manifolds

Let (M, g) be an n-dimensional complete Riemannian manifold isometrically immersed into an $(n+k)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature C, where $k \geq 1$. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on M and $\overline{M}$, respectively (see details in [11]).

For vector fields $X, Y \in C^\infty(TM)$, the Gauss formula is

$$\overline{\nabla}_X Y = \nabla_X Y + \varphi(X, Y),$$

where φ is the second fundamental form, a symmetric tensor field with values in the normal bundle (see details in [11]).

Definition 1. The *mean curvature vector* is defined by

$$H = \frac{1}{n} \sum_{i=1}^{n} \varphi(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame. The submanifold is called *minimal* if $H \equiv 0$.

Let $\widetilde{g}$ denote the natural metric on the bundle $TM \oplus T^\perp M$. The squared norm of the second fundamental form is given by

$$\|\varphi\|^2 = \sum_{i,j=1}^{n} \overline{g}\big(\varphi(e_i, e_j), \varphi(e_i, e_j)\big). \tag{1}$$

Definition 2. A submanifold (M, g) is called *totally geodesic* if $\varphi \equiv 0$, and *totally umbilical* if $\varphi = Hg$.

We recall the following classical result due to Chern, do Carmo (see [10]), and Kobayashi on complete minimal submanifolds.

Theorem 1. *Let (M, g) be an n-dimensional compact minimal submanifold of an $(n+k)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ of constant curvature C. If (M, g) is not totally geodesic and*

$$\|\varphi\|^2 \leq \frac{kn}{2k-1} C,$$

then (M, g) is a parallel submanifold and equality holds.

Remark 1. A submanifold is called parallel if its second fundamental form is parallel with respect to the *Van der Waerden–Bortolotti connection* on $TM \oplus T^\perp M$ (see details in [11]),

Next, we consider complete parabolic minimal submanifolds of Riemannian manifolds of constant sectional curvature C. This class is broad enough to include a variety of geometrically significant examples, and its richness depends on both the ambient geometry and the volume growth behavior. In particular, every complete Riemannian manifold of finite volume is parabolic, providing a natural source of examples, although this condition is not necessary.

We emphasize that the pinching condition in Theorem 2 is nontrivial only in the case $C > 0$. Therefore, the most relevant examples arise in positively curved ambient spaces, such as spheres. In this setting, parabolicity provides a natural extension of compactness, allowing one to replace integration arguments by Liouville-type principles for bounded subharmonic functions.

Theorem 2. *Let (M, g) be an n-dimensional complete parabolic minimal submanifold (in particular, of finite volume) of an m-dimensional Riemannian manifold of constant sectional curvature $C > 0$. Suppose that (M, g) is not totally geodesic and that*

$$\|\varphi\|^2 \leq \frac{kn}{2k-1} C.$$

Then (M, g) is a parallel submanifold. Moreover, equality holds in the above inequality.

Proof. Assume $C > 0$. Then the pinching condition in the statement of Theorem 2 is nontrivial.

We use the Simons-type Bochner formula for minimal submanifolds in a space form (see, e.g., Chern–do Carmo–Kobayashi [10]):

$$\frac{1}{2}\Delta\|\varphi\|^2 = \|\nabla\varphi\|^2 + \left(nC - \left(2 - \frac{1}{k}\right)\|\varphi\|^2\right)\|\varphi\|^2.$$

Here $\Delta = \operatorname{div} \nabla$ is the nonnegative Laplace operator, and φ denotes the second fundamental form with the usual normalization

$$\|\varphi\|^2 = \sum_{i,j} \left\langle \varphi(e_i, e_j), \varphi(e_i, e_j) \right\rangle.$$

The coefficient $2 - \frac{1}{k}$ is the codimension-dependent constant appearing in the classical Chern–do Carmo–Kobayashi estimate.

By the pinching assumption

$$\|\varphi\|^2 \leq \frac{kn}{2k-1}C = \frac{nC}{2 - \frac{1}{k}},$$

we have

$$nC - \left(2 - \frac{1}{k}\right)\|\varphi\|^2 \geq 0.$$

Therefore

$$\Delta\|\varphi\|^2 \geq 0.$$

Moreover,

$$0 \leq \|\varphi\|^2 \leq \frac{kn}{2k-1}C,$$

so $f = \|\varphi\|^2$ is a bounded nonnegative subharmonic function on the complete parabolic manifold (M, g).

By the Liouville property of parabolic manifolds, every bounded nonnegative subharmonic function on (M, g) is constant. Hence $\|\varphi\|^2$ is constant. Substituting this into the Bochner formula gives

$$0 = \|\nabla\varphi\|^2 + \left(nC - \left(2 - \frac{1}{k}\right)\|\varphi\|^2\right)\|\varphi\|^2.$$

Both terms on the right-hand side are nonnegative. Hence, they both vanish. In particular,

$$\|\nabla\varphi\|^2 = 0,$$

and therefore φ is parallel.

Since (M, g) is assumed not to be totally geodesic, $\|\varphi\|^2 \not\equiv 0$. Consequently,

$$nC - \left(2 - \frac{1}{k}\right)\|\varphi\|^2 = 0,$$

and hence

$$\|\varphi\|^2 = \frac{nC}{2 - \frac{1}{k}} = \frac{kn}{2k-1}C.$$

Thus, equality holds in the pinching condition, and (M, g) is a parallel submanifold. □

Remark 2. The pinching condition in Theorem 2 is meaningful only in the case $C > 0$. Indeed:

If $C = 0$, then the inequality reduces to $\|\varphi\|^2 \leq 0$, hence $\varphi \equiv 0$, which contradicts the assumption that the submanifold is not totally geodesic.

If $C < 0$, the right-hand side is negative, while $\|\varphi\|^2 \geq 0$, so the inequality cannot be satisfied. Therefore, in both cases $C \leq 0$, the assumptions of Theorem 2 are either inconsistent or reduce to the trivial totally geodesic case.

Corollary 1. *Let (M, g) be an n-dimensional complete parabolic minimal hypersurface in a Riemannian manifold of constant sectional curvature $C > 0$. Assume that (M, g) is not totally geodesic and that*

$$\|\varphi\|^2 \leq nC.$$

Then φ is parallel and equality holds:

$$\|\varphi\|^2 = nC.$$

Consequently, around each point of (M, g) there exists a local Riemannian product decomposition

$$(M, g) = (M_1, g_1) \times \cdots \times (M_r, g_r),$$

such that $g = g_1 + \cdots + g_r$, and the second fundamental form has the form

$$\varphi = \lambda_1 g_1 + \cdots + \lambda_r g_r,$$

where the constants $\lambda_1, \ldots, \lambda_r$ satisfy $n_1\lambda_1 + \cdots + n_r\lambda_r = 0$, and

$$n_1\lambda_1^2 + \cdots + n_r\lambda_r^2 = nC.$$

Here $n_\alpha = \dim M_\alpha$, $\alpha = 1, \ldots, r$, and $n_1 + \cdots + n_r = n$.

Proof. The conclusion follows directly from Theorem 2. Under the stated assumptions, the second fundamental form φ is parallel and satisfies $\|\varphi\|^2 = nC$. The local product decomposition and the representation of φ follow from the classical theorem of Eisenhart (see [12]) on parallel symmetric 2-tensors. □

In the hypersurface case ($k = 1$), the pinching condition reduces to $\|\varphi\|^2 \leq nC$, and the conclusion of Corollary 1 remains valid without modification.

3. Complete stable minimal parabolic submanifolds

Definition 3. We recall that a minimal hypersurface is called stable if the associated quadratic form

$$Q(\eta, \eta) = \int_M \left(|\nabla \eta|^2 - \left(\|\varphi\|^2 + \mathrm{Ric}(N, N) \right) \eta^2 \right) d\mu,$$

is nonnegative for all $\eta \in C_0^\infty(M)$. The corresponding Jacobi operator is (see [13])

$$L = \Delta + \|\varphi\|^2 + \mathrm{Ric}(N, N),$$

where $\Delta = \operatorname{div} \nabla$.

We emphasize that stability does not imply a differential inequality of the form $Lu \leq 0$ for arbitrary smooth functions u. In the following theorem, the existence of a positive function satisfying such an inequality is therefore assumed explicitly.

Theorem 3. *Let (M, g) be a complete two-sided minimal hypersurface in an $(n+1)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$. Let N be a globally defined unit normal vector field along M. Assume that*

$$\overline{\mathrm{Ric}}(N, N) \geq 0,$$

at every point of M, and that (M, g) is parabolic. Suppose that there exists a function $u \in C^2(M)$ such that $u > 0$ and $Lu \leq 0$, where

$$L = \Delta + \|\varphi\|^2 + \overline{\mathrm{Ric}}(N, N),$$

is the Jacobi operator. Then (M, g) is totally geodesic. Moreover,

$$\overline{\mathrm{Ric}}(N, N) = 0,$$

at every point of M.

Proof. Since $u > 0$ and $Lu \leq 0$, we have

$$\Delta u \leq -qu \leq 0,$$

where $q = \|\varphi\|^2 + \mathrm{Ric}(N, N) \geq 0$. Hence u is a positive superharmonic function on the complete parabolic manifold M.

By the Liouville property of parabolic manifolds, every positive superharmonic function on (M, g) is constant (see [7]). Hence u is constant and consequently $\Delta u = 0$. Substituting this into $\Delta u + qu \leq 0$ gives $qu \leq 0$. Since $u > 0$ and $q \geq 0$, it follows that $q = 0$. Therefore

$$\|\varphi\|^2 + \overline{\mathrm{Ric}}(N, N) = 0.$$

Since both terms are nonnegative, we obtain

$$\|\varphi\|^2 = 0, \qquad \overline{\mathrm{Ric}}(N, N) = 0.$$

Thus M is totally geodesic, and the ambient Ricci curvature in the normal direction vanishes identically. □

It is known that on a complete, non-compact stable minimal hypersurface there exists a globally defined positive function u satisfying $Lu = 0$, where L is the Jacobi operator. If, in addition, $\overline{\mathrm{Ric}}(N, N) \geq 0$, then

$$\Delta u = -\left(\|\varphi\|^2 + \overline{\mathrm{Ric}}(N, N)\right) u \leq 0,$$

so that u is a positive superharmonic function. On a parabolic manifold, every positive superharmonic function is constant (see [7]). This observation allows us to formulate the following corollary.

Corollary 2. *Let (M, g) be a complete two-sided minimal hypersurface in an $(n+1)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$. Assume that*

$$\overline{\mathrm{Ric}}(N, N) \geq 0,$$

at every point of M, and that (M, g) is parabolic. Suppose that there exists a function $u \in C^2(M)$ such that $u > 0$, $Lu = 0$, where

$$L = \Delta + \|\varphi\|^2 + \overline{\mathrm{Ric}}(N, N).$$

Then (M, g) is totally geodesic and

$$\overline{\mathrm{Ric}}(N, N) = 0.$$

Proof. Since $Lu = 0$, we have

$$\Delta u = -qu \leq 0, \qquad q = \|\varphi\|^2 + \overline{\mathrm{Ric}}(N, N) \geq 0.$$

Thus u is a positive superharmonic function (see [8]). By parabolicity, u is constant, and the conclusion follows as in Theorem 3. $\square$

Data Availability: No datasets were generated or analyzed during the current study.

Funding Information: The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

References

[1] Mikeš, J., & Stepanov, S. E. (2022). What is the Bochner technique and where is it applied. *Lobachevskii Journal of Mathematics*, *43*(4), 709.

[2] Li, P. (2012). *Geometric Analysis* (Vol. 134). Cambridge University Press.

[3] Jost, J. (2005). *Riemannian Geometry and Geometric Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg.

[4] Petersen, P. (2016). Symmetric Spaces and Holonomy. In *Riemannian Geometry* (pp. 365-394). Cham: Springer International Publishing.

[5] Rovenski, V., Stepanov, S., & Tsyganok, I. (2023, May). Lichnerowicz-type Laplacians in the Bochner technique. In *International Workshop on Differential Geometry* (pp. 167-207). Cham: Springer Nature Switzerland.

[6] Mikeš, J., Rovenski, V., & Stepanov, S. E. (2020). An example of Lichnerowicz-type Laplacian. *Annals of Global Analysis and Geometry*, *58*(1), 19-34.

[7] Cheng, S. Y., & Yau, S. T. (1975). Differential equations on Riemannian manifolds and their geometric applications. *Communications on Pure and Applied Mathematics*, *28*(3), 333-354.

[8] Yau, S. T. (1976). Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana University Mathematics Journal*, *25*(7), 659-670.

[9] Grigor'Yan, A. (1999). Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bulletin of the American Mathematical Society*, *36*(2), 135-249.

[10] Chern, S. S., Carmo, M. D., & Kobayashi, S. (2012). Minimal submanifolds of a sphere with second fundamental form of constant length. In *Manfredo P. do Carmo–Selected Papers* (pp. 47-63). Berlin, Heidelberg: Springer Berlin Heidelberg.

[11] Chen, B. Y. (2000). Riemannian submanifolds. In *Handbook of Differential Geometry* (Vol. 1, pp. 187-418). North-Holland.

[12] Eisenhart, L. P. (1923). Symmetric tensors of the second order whose first covariant derivatives are zero. *Transactions of the American Mathematical Society*, *25*(2), 297-306.

[13] Catino, G., Mastrolia, P., & Roncoroni, A. (2024). Two rigidity results for stable minimal hypersurfaces. *Geometric and Functional Analysis*, *34*(1), 1-18.