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Auxiliary principle approach for mixed general bivariational inequalities

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Abstract: We introduce a class of mixed general bivariational inequalities in a real Hilbert space and show that several known models, including mixed variational inequalities, bivariational inequalities, general variational inequalities, and complementarity problems, arise as special cases. An auxiliary-principle framework is then used to derive predictor–corrector type iterative schemes. A basic descent estimate is established under a g-partially relaxed monotonicity assumption, and a convergence theorem is obtained under natural continuity and uniqueness hypotheses. The presentation has been streamlined to make the algorithmic steps explicit, and a scalar example is included to illustrate the resolvent formulation.

Keywords: bivariational inequalities, auxiliary principle technique, iterative methods, convergence

MSC: 47A12, 49J40, 65K15, 90C33

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1. Introduction

Variational inequality theory provides a unified framework for equilibrium, obstacle, optimization, and complementarity problems. Since the foundational works of Stampacchia [1] and Lions and Stampacchia [2], the subject has been extended in several directions to accommodate nonlinear operators, nonsmooth terms, and generalized constraints. For broad background on theory, applications, and numerical methods, we refer to [1–17].

A useful extension of the classical variational inequality is the mixed variational inequality, where a nondifferentiable term is incorporated into the model. Such problems arise naturally in equilibrium theory, optimization, mechanics, and engineering applications. Because of the nonsmooth term, standard projection- and descent-type arguments are not always directly applicable. One fruitful alternative is the auxiliary principle technique introduced by Trémolieres, Lions, & Glowinski [6]; see also Noor [8–12] and Zhu and Marcotte [17]. This technique has repeatedly proved useful for deriving iterative schemes with transparent convergence mechanisms.

In this paper we study a mixed general bivariational inequality involving a bifunction $N(\cdot,\cdot)$, operators A, T, and g, and a lower-semicontinuous function φ. The formulation unifies several classes of variational inequalities and complementarity problems. We derive auxiliary-principle-based predictor–corrector methods and establish a Fejér-type descent estimate under a g-partially relaxed monotonicity condition. To keep the convergence analysis rigorous, the main convergence statement is formulated under explicit continuity, cluster-point, and uniqueness assumptions. We also include a scalar example that clarifies the resolvent interpretation of the proposed algorithm.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H, and let $\varphi : H \to \mathbb{R}\cup\{+\infty\}$ be a proper function.

For a bifunction $N : H \times H \to H$ and continuous operators $A, T, g : H \to H$, we consider the problem of finding $u \in H$ such that

$$\langle N(A(u), T(u)), g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \forall v \in H. \tag{1}$$

We call (1) the *mixed general bivariational inequality*. When φ is proper, convex, and lower semicontinuous, (1) is equivalent to the inclusion

$$0 \in N(A(u), T(u)) + \partial\varphi(g(u)), \tag{2}$$

where $\partial\varphi$ denotes the subdifferential of φ.

Special cases

1. If $A = I$ and $T = I$, then (1) becomes

$$\langle N(u,u), g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \forall v \in H, \tag{3}$$

which is a mixed general variational inequality [11].

2. If $g = I$, then (1) reduces to

$$\langle N(A(u), T(u)), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in H, \tag{4}$$

which is a mixed bivariational inequality.

3. If $\varphi = I_K$ is the indicator function of a closed convex set K, then (1) is equivalent to

$$\langle N(A(u), T(u)), g(v) - g(u)\rangle \geq 0, \qquad \forall v \in K, \tag{5}$$

which is a general bivariational inequality.

4. If, in addition, $N(A(u), T(u)) \equiv T(u)$, then (5) becomes

$$\langle T(u), g(v) - g(u)\rangle \geq 0, \qquad \forall v \in K, \tag{6}$$

which is the general variational inequality introduced in [7].

5. If K is a convex cone and $K^* = \{w \in H : \langle w, v\rangle \geq 0 \text{ for all } v \in K\}$ is its polar cone, then (5) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \qquad N(A(u), T(u)) \in K^*, \qquad \langle N(A(u), T(u)), g(u)\rangle = 0, \tag{7}$$

which is a general bicomplementarity problem [13].

6. If $g = I$ in (7), then we obtain the bicomplementarity problem: find $u \in K$ such that

$$u \in K, \qquad N(A(u), T(u)) \in K^*, \qquad \langle N(A(u), T(u)), u\rangle = 0. \tag{8}$$

7. If $g = I$ in (5), then we recover the bivariational inequality

$$\langle N(A(u), T(u)), v - u\rangle \geq 0, \qquad \forall v \in K. \tag{9}$$

8. If $N(A(u), T(u)) \equiv T(u)$ in (9), then we obtain the classical variational inequality

$$\langle T(u), v - u\rangle \geq 0, \qquad \forall v \in K, \tag{10}$$

introduced by Lions and Stampacchia [2].

Thus, for suitable choices of N, A, T, g, and φ, the mixed general bivariational inequality (1) provides a unifying model for a broad family of problems.

We also recall a simple Hilbert space identity.

Lemma 1. *For all $u, v \in H$,*

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{11}$$

Definition 1. For all $u, v, z \in H$, the bifunction $N(\cdot,\cdot) : H \times H \to H$ involving the operators A and T with respect to g is said to be:

(i) *g-partially relaxed strongly monotone* if there exists $\alpha > 0$ such that

$$\langle N(A(u),T(u)) - N(A(v),T(v)), g(z) - g(v)\rangle \geq -\alpha\|g(u) - g(z)\|^2;$$

(ii) *g-partially relaxed monotone* if

$$\langle N(A(u),T(u)) - N(A(v),T(v)), g(z) - g(v)\rangle \geq 0;$$

(iii) *Lipschitz continuous* if there exists $\beta > 0$ such that

$$\|N(A(u),T(u)) - N(A(v),T(v))\| \leq \beta\|u - v\|.$$

When $z = u$, Definition 1(i) reduces to the usual g-monotonicity of the composite mapping $u \mapsto N(A(u),T(u))$. In particular, the partially relaxed formulation is weaker than the coercive assumptions commonly used in proximal or descent analyses; compare [9,11].

Definition 2 ([4])**.** If B is a maximal monotone operator on H, then its resolvent is defined by

$$J_B(u) = (I + \lambda B)^{-1}(u), \qquad \forall u \in H,$$

where $\lambda > 0$ is fixed.

It is well known that the resolvent of a maximal monotone operator is single-valued and nonexpansive. Since $\partial\varphi$ is maximal monotone whenever φ is proper, convex, and lower semicontinuous, we write

$$J_\varphi(u) := (I + \lambda\partial\varphi)^{-1}(u). \tag{12}$$

Lemma 2. *Let φ be proper, convex, and lower semicontinuous. For given $u, z \in H$, the following are equivalent:*

(a)

$$\langle u - z, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in H; \tag{13}$$

(b) $u = J_\varphi(z)$.

3. Auxiliary principle and iterative methods

Let $u \in H$ satisfy (1). Consider the auxiliary problem of finding $w \in H$ such that

$$\langle \rho N(A(u),T(u)) + g(w) - g(u), g(v) - g(w)\rangle + \rho\varphi(g(v)) - \rho\varphi(g(w)) \geq 0, \qquad \forall v \in H, \tag{14}$$

where $\rho > 0$ is fixed. If $w = u$, then (14) reduces to (1). This observation leads to the following predictor–corrector scheme.

Algorithm 1. Given $u_0 \in H$, generate sequences $\{u_n\}$ and $\{w_n\}$ by solving

$$\langle \rho N(A(w_n),T(w_n)) + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle + \rho\varphi(g(v)) - \rho\varphi(g(u_{n+1})) \geq 0, \quad \forall v \in H, \tag{15}$$

and

$$\langle \beta N(A(u_n),T(u_n)) + g(w_n) - g(u_n), g(v) - g(w_n)\rangle + \beta\varphi(g(v)) - \beta\varphi(g(w_n)) \geq 0, \quad \forall v \in H, \tag{16}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Remark 1. If $g = I$, then Algorithm 1 takes the form

$$\langle \rho N(A(w_n),T(w_n)) + u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\varphi(v) - \rho\varphi(u_{n+1}) \geq 0,$$

$$\langle \beta N(A(u_n),T(u_n)) + w_n - u_n, v - w_n\rangle + \beta\varphi(v) - \beta\varphi(w_n) \geq 0,$$

for all $v \in H$.

Remark 2. If φ is proper, convex, and lower semicontinuous, then Lemma 2 yields the equivalent resolvent formulation

$$g(u_{n+1}) = J_\varphi[g(w_n) - \rho N(A(w_n), T(w_n))], \qquad g(w_n) = J_\varphi[g(u_n) - \beta N(A(u_n), T(u_n))]. \tag{17}$$

If, moreover, $\varphi = I_K$, then Algorithm 1 reduces to the corresponding projection-type method on K.

The first result gives the basic descent inequalities needed in the convergence analysis.

Theorem 1. *Let $\bar{u} \in H$ be a solution of (1), and let $\{u_n\}$ and $\{w_n\}$ be generated by Algorithm 1. If $N(\cdot,\cdot)$ is g-partially relaxed monotone, then for every $n \geq 0$,*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \|g(u_{n+1}) - g(u_n)\|^2, \tag{18}$$
$$\|g(w_n) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \|g(w_n) - g(u_n)\|^2. \tag{19}$$

Proof. Because $\bar{u}$ solves (1), we have

$$\langle \rho N(A(\bar{u}), T(\bar{u})), g(v) - g(\bar{u})\rangle + \rho\varphi(g(v)) - \rho\varphi(g(\bar{u})) \geq 0, \qquad \forall v \in H, \tag{20}$$
$$\langle \beta N(A(\bar{u}), T(\bar{u})), g(v) - g(\bar{u})\rangle + \beta\varphi(g(v)) - \beta\varphi(g(\bar{u})) \geq 0, \qquad \forall v \in H. \tag{21}$$

Taking $v = u_{n+1}$ in (20) and $v = \bar{u}$ in (15), and then adding the resulting inequalities, we obtain

$$\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle \geq \rho \left\langle N(A(w_n), T(w_n)) - N(A(\bar{u}), T(\bar{u})), g(u_{n+1}) - g(\bar{u})\right\rangle. \tag{22}$$

By g-partial relaxed monotonicity, the right-hand side is nonnegative, and therefore

$$\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle \geq 0. \tag{23}$$

Applying Lemma 1 with

$$\xi = g(\bar{u}) - g(u_{n+1}), \qquad \eta = g(u_{n+1}) - g(u_n),$$

yields

$$2\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle = \|g(\bar{u}) - g(u_n)\|^2 - \|g(\bar{u}) - g(u_{n+1})\|^2 - \|g(u_{n+1}) - g(u_n)\|^2.$$

Combining this identity with (23) gives (18).
Next, take $v = \bar{u}$ in (16) and $v = w_n$ in (21). Adding the two relations, we obtain

$$\langle g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle \geq \beta \left\langle N(A(u_n), T(u_n)) - N(A(\bar{u}), T(\bar{u})), g(w_n) - g(\bar{u})\right\rangle. \tag{24}$$

Again, the right-hand side is nonnegative by Definition 1(ii), so

$$\langle g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle \geq 0. \tag{25}$$

Applying Lemma 1 with

$$\xi = g(\bar{u}) - g(w_n), \qquad \eta = g(w_n) - g(u_n),$$

we obtain (19). □

Corollary 1. *Under the assumptions of Theorem 1, the sequence $\{\|g(u_n) - g(\bar{u})\|\}$ is nonincreasing and*

$$\sum_{n=0}^{\infty} \|g(u_{n+1}) - g(u_n)\|^2 < \infty. \tag{26}$$

Consequently,

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0. \tag{27}$$

Moreover, (19) *implies*

$$\|g(w_n) - g(u_n)\|^2 \le \|g(u_n) - g(\bar{u})\|^2 - \|g(w_n) - g(\bar{u})\|^2, \tag{28}$$

so in particular $\|g(w_n) - g(u_n)\| \to 0$ *whenever* $\|g(u_n) - g(\bar{u})\|$ *and* $\|g(w_n) - g(\bar{u})\|$ *have the same limit.*

The next theorem gives a rigorous convergence statement. It is formulated with an explicit cluster-point assumption, which is automatic, for example, when H is finite dimensional and $\{u_n\}$ is bounded.

Theorem 2. *Assume that the solution of* (1) *is unique and denote it by* $\bar{u}$*. Suppose that the mapping*

$$F(u) := N(A(u), T(u)),$$

is continuous, that $g : H \to H$ *is invertible with continuous inverse, and that the sequence* $\{u_n\}$ *generated by Algorithm 1 has a cluster point. Then*

$$u_n \to \bar{u} \qquad \text{and} \qquad w_n \to \bar{u}.$$

Proof. Let $\hat{u}$ be a cluster point of $\{u_n\}$, and let $u_{n_j} \to \hat{u}$ along a subsequence. By (27) and continuity of g^{-1}, we also have

$$u_{n_j+1} \to \hat{u}.$$

From (19), the nonnegative quantity

$$\|g(w_n) - g(u_n)\|^2 \le \|g(u_n) - g(\bar{u})\|^2 - \|g(w_n) - g(\bar{u})\|^2,$$

vanishes along the same subsequence, hence $g(w_{n_j}) - g(u_{n_j}) \to 0$. Since g^{-1} is continuous,

$$w_{n_j} \to \hat{u}.$$

Fix $v \in H$. From (15),

$$\rho\varphi(g(u_{n_j+1})) \le \rho\varphi(g(v)) + \left\langle \rho F(w_{n_j}) + g(u_{n_j+1}) - g(u_{n_j}), g(v) - g(u_{n_j+1}) \right\rangle.$$

Passing to the limit superior and using the continuity of F, together with $u_{n_j+1} \to \hat{u}$, $w_{n_j} \to \hat{u}$, and $g(u_{n_j+1}) - g(u_{n_j}) \to 0$, we obtain

$$\limsup_{j\to\infty} \varphi(g(u_{n_j+1})) \le \varphi(g(v)) + \langle F(\hat{u}), g(v) - g(\hat{u})\rangle. \tag{29}$$

Since φ is lower semicontinuous,

$$\varphi(g(\hat{u})) \le \liminf_{j\to\infty} \varphi(g(u_{n_j+1})).$$

Combining this with (29) gives

$$\langle F(\hat{u}), g(v) - g(\hat{u})\rangle + \varphi(g(v)) - \varphi(g(\hat{u})) \ge 0, \qquad \forall v \in H.$$

Thus $\hat{u}$ solves (1). By uniqueness, $\hat{u} = \bar{u}$.

Now Theorem 1 shows that the sequence $d_n := \|g(u_n) - g(\bar{u})\|$ is nonincreasing. Because $u_{n_j} \to \bar{u}$, we have $d_{n_j} \to 0$, hence $d_n \to 0$. Since g^{-1} is continuous, it follows that $u_n \to \bar{u}$. Finally, (19) yields $\|g(w_n) - g(u_n)\| \to 0$, and therefore $w_n \to \bar{u}$ as well. □

Remark 3. Theorem 2 is intentionally stated under explicit continuity and cluster-point hypotheses. These assumptions make the limit passage transparent and avoid hidden compactness assumptions in infinite-dimensional Hilbert spaces.

4. Illustrative example

We include a simple scalar example to make the resolvent formulation concrete.

Example 1. Let $H = \mathbb{R}$, let $A = T = g = I$, define

$$N(A(u), T(u)) \equiv 0, \qquad \varphi(x) = \mu|x|, \quad \mu > 0.$$

Then (1) becomes

$$\mu|v| - \mu|u| \geq 0, \qquad \forall v \in \mathbb{R}. \tag{30}$$

Hence the unique solution is $u = 0$. In this case the composite mapping $u \mapsto N(A(u), T(u))$ is constant, so the g-partial relaxed monotonicity condition is satisfied trivially.

Since φ is proper, convex, and lower semicontinuous, Algorithm 1 reduces to the resolvent iteration

$$w_n = J_\varphi(u_n), \qquad u_{n+1} = J_\varphi(w_n).$$

For $\varphi(x) = \mu|x|$, the resolvent J_φ is the soft-thresholding operator

$$S_{\lambda\mu}(x) = \text{sgn}(x) \max\{|x| - \lambda\mu, 0\}.$$

Therefore,

$$w_n = S_{\lambda\mu}(u_n), \qquad u_{n+1} = S_{\lambda\mu}(w_n).$$

Starting from any $u_0 \in \mathbb{R}$, the iterates move monotonically toward 0 and reach the exact solution in finitely many steps whenever the accumulated threshold dominates $|u_0|$. This example illustrates the proximal structure behind the general algorithm.

5. Conclusion

We have studied a mixed general bivariational inequality that unifies several well-known models in variational inequality and complementarity theory. The auxiliary principle technique leads naturally to a two-step predictor–corrector method and to its resolvent representation. The main analytical outcome is a pair of descent inequalities under a g-partially relaxed monotonicity condition. These estimates yield asymptotic regularity of the iterates and, under continuity, cluster-point, and uniqueness hypotheses, convergence to the exact solution.

The analysis in this paper is theoretical. Further work may address existence theory under problem-specific hypotheses, sharper convergence rates, and computational studies for concrete application classes.

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