Open Journal of Mathematical Sciences

PSR Press

Article

Characterization of the boundednes of the discrete Hardy-Littlewood maximal operator and its higher–order commutator on generalized Morrey sequence spaces

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Received: 06 May 2026; Revised: 01 Jul 2026; Accepted: 04 Jul 2026; Published: 17 Jul 2026

Abstract: In this paper, we introduce the function class $G_p^\eta(\mathbb{Z})$ alongside a more general class $G_p(\mathbb{Z})$ to serve as parameter functions for generalized Morrey sequence spaces and generalized mixed Morrey double-sequence spaces. Using these classes, we establish necessary and sufficient conditions for the boundedness of the discrete Hardy–Littlewood maximal operator and its higher-order commutator on generalized Morrey sequence spaces and generalized mixed Morrey double-sequence spaces over $\mathbb{Z}$. The boundedness of the maximal operator is characterized in terms of relations between the parameter functions. To obtain the sufficiency and necessity conditions, we utilize the properties of the $G_p(\mathbb{Z})$ class. For the necessity conditions, we compute the norms of the characteristic functions of a discrete interval in generalized Morrey sequence spaces and of a rectangle in $\mathbb{Z} \times \mathbb{Z}$ in generalized mixed Morrey double-sequence spaces. Furthermore, we characterize the boundedness of the higher-order commutator via the BMO space. Specifically, the proof of sufficiency relies on the Fefferman–Stein inequality within these spaces, whereas the necessity proof adapts the techniques developed for the maximal operator itself.

Keywords: Hardy-Littlewood maximal operator, higher-order commutator, Morrey spaces, double-mixed Morrey spaces, discrete spaces

MSC: 42B35, 46B45, 46A45, 47B38.

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1. Introduction and preliminaries

Morrey spaces originated from the work of Charles B. Morrey in 1938, when studying the regularity properties of solutions to certain elliptic partial differential equations [1]. Since then, various studies have been conducted on these spaces, particularly concerning their structural and geometric properties (see [2,3]), as well as the behavior of specific operators on them (see [4–11]).

For $1 \leq p \leq q < \infty$, the Morrey space $M_q^p(\mathbb{R}^d)$ consists of all measurable functions f defined on $\mathbb{R}^d$ that satisfy the condition

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^d)} = \sup_{(a,r)\in\mathbb{R}^d\times(0,\infty)} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)} |f(y)|^p dy\right)^{1/p} < \infty.$$

When $p = q$, it follows that $M_q^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. Then, Morrey spaces can be viewed as a natural generalization of Lebesgue spaces. Building upon this concept, Nakai and Mizuhara [6,7] introduced the generalized Morrey space $\mathcal{M}_\psi^p(\mathbb{R}^d)$, defined as the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}_\psi^p(\mathbb{R}^d)} = \sup_{(a,r)\in\mathbb{R}^d\times(0,\infty)} \frac{1}{\psi(r)} \left(\frac{1}{r^d}\int_{B(a,r)} |f(y)|^p dy\right)^{1/p} < \infty.$$

Recently, Gunawan et al. [12] introduced another variant of Morrey spaces, known as discrete Morrey spaces. The boundedness of the Hardy-Littlewood operator M on these spaces was subsequently investigated in [13]. Building on this, our previous work [14] examined the boundedness of the higher-order commutators of the Riesz potential I_α and the fractional maximal operator M_α within the discrete Morrey space framework.

These spaces were generalized by Haroske and Skrzypczak [15] into generalized Morrey sequence spaces, alongside a study of their structural properties. The boundedness of M on these generalized spaces was further explored in [16]. Concurrently, Gunawan et al. [17] introduced and analyzed the structural properties of mixed Morrey double-sequence spaces, which serve as another extension of the discrete setting. For a broader overview of discrete settings in this context, we refer the reader to [18–27].

In this paper, our aim is to study the boundedness of M and its higher-order commutator on both generalized Morrey sequence spaces and generalized mixed Morrey double-sequence spaces. Specifically, we establish the necessary and sufficient conditions for the boundedness of the operator M and its higher-order commutators on these spaces.

Before we present our main results, we introduce some foundational definitions and notations. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_0 = [0, \infty)$, and $\mathbb{R}^+ = (0, \infty)$. For $j \in \mathbb{Z}$ and $R \in \mathbb{R}_0$, the discrete interval centered at j with radius R is defined by $Q(j, R) := \{k \in \mathbb{Z} : |j - k| \le R\}$. We also define $k\,Q(j, R) := Q(j, kR)$ for $k > 0$. The cardinality of a subset $Z \subseteq \mathbb{Z}$ is denoted by $\#Z$. Since j and k are integers, the distance $|j - k|$ is inherently an integer. Consequently, by setting $N = \lfloor R \rfloor \in \mathbb{N}_0$, we observe that $Q(j, R) = Q(j, N)$ and $\#Q(j, R) = \#Q(j, N) = 2N + 1$. This equivalence allows us to restrict our attention to integer radii $N \in \mathbb{N}_0$ without any loss of generality.

For a subset Q of $\mathbb{Z}$ and $1 \le p < \infty$, we denote by $\ell^p(Q)$ the Lebesgue sequence space with the norm

$$\|a\|_{\ell^p(Q)} = \left(\sum_{j \in Q} |a_j|^p \right)^{\frac{1}{p}}.$$

Moreover, the weak Lebesgue sequence spaces $w\ell^p(Q)$ is defined as the set of all sequences $a : \mathbb{Z} \to \mathbb{C}$ such that

$$\|a\|_{w\ell^p(Q)} = \sup_{\gamma > 0} \gamma \# \{ j \in Q : |a_j| > \gamma \}^{1/p} < \infty.$$

It can be seen that $\|a\|_{w\ell^p(Q)} \le \|a\|_{\ell^p(Q)}$.

Next, we define the generalized Morrey sequence space and the generalized mixed Morrey double-sequence space. In transitioning from the continuous to the discrete framework, the Lebesgue measure of a ball $B(a, r)$ in $\mathbb{R}^d$ is replaced by the cardinality of the discrete set $Q(j, N)$. Furthermore, while the parameter function ψ is typically defined on $\mathbb{R}^+$ in the continuous setting, it is defined on $\mathbb{N}_0$ in this discrete environment. This structural shift poses unique challenges in the study of the discrete framework.

Let $1 \le p < \infty$ and $\psi : \mathbb{N}_0 \to \mathbb{R}^+$. The generalized Morrey sequence space $\mathcal{M}^p_\psi(\mathbb{Z})$ is defined to be the set of all sequences $a : \mathbb{Z} \to \mathbb{C}$ such that

$$\|a\|_{\mathcal{M}^p_\psi(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \frac{1}{\psi(N)} \frac{1}{\#Q(j, N)^{1/p}} \|a\|_{\ell^p(Q(j,N))} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \frac{1}{\psi(N)} \frac{1}{(2N+1)^{1/p}} \|a\|_{\ell^p(Q(j,N))} < \infty.$$

In addition, we define the generalized weak Morrey sequence space $W\mathcal{M}^p_\psi(\mathbb{Z})$ by the set of any sequence $a : \mathbb{Z} \to \mathbb{C}$ such that

$$\|a\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \frac{1}{\psi(N)} \frac{1}{(2N+1)^{1/p}} \|a\|_{w\ell^p(Q(j,N))} < \infty.$$

Since $\|a\|_{w\ell^p(Q)} \le \|a\|_{\ell^p(Q)}$ for any $Q \subseteq \mathbb{Z}$, we have that

$$\|a\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} \le \|a\|_{\mathcal{M}^p_\psi(\mathbb{Z})}. \tag{1}$$

We turn to the operators used in this paper. The discrete Hardy-Littlewood maximal operator M is defined by

$$(M(a))_j = \sup_{N \in \mathbb{N}_0} \frac{1}{\#Q(j, N)} \sum_{n \in Q(j,N)} |a_n|, \quad j \in \mathbb{Z},$$

for any suitable sequence a on $\mathbb{Z}$. For $b : \mathbb{Z} \to \mathbb{C}$, we define the higher-order commutator M_b^m of the operator M by the formula

$$(M_b^m(a))_j = \sup_{N\in\mathbb{N}_0} \frac{1}{\#Q(j,N)} \sum_{n\in Q(j,N)} |b_n - b_j|^m |a_n|, \quad j \in \mathbb{Z}.$$

Related to M_b^m, the function b on $\mathbb{Z}$ belongs to BMO space if $\|b\|_* < \infty$ where

$$\|b\|_* = \sup_Q \frac{1}{\#Q} \sum_{k\in Q} |b_k - b_Q|,$$

and the supremum is taken over all intervals Q on $\mathbb{Z}$ and we denote $b_Q = \frac{1}{\#Q}\sum\limits_{k\in Q} b_k$ for any interval Q. Moreover, if we define

$$(M^{\#}(b))_j = \sup_{Q\ni j} \frac{1}{\#Q} \sum_{k\in Q} |b_k - b_Q|, \quad j \in \mathbb{Z},$$

then $b \in BMO$ if and only if $\sup_{j\in\mathbb{Z}}(M^{\#}b)_j < \infty$.

We now consider the generalized mixed Morrey double-sequence setting. Let $1 \leq p,q < \infty$ and $\psi_1, \psi_2 : \mathbb{N}_0 \to \mathbb{R}^+$. The generalized mixed Morrey double-sequence space $(\mathcal{M}^q_{\psi_2})(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ is defined to be the set of all sequences $a : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ such that

$$\|a\|_{(\mathcal{M}^q_{\psi_2})(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} = \left\| \|a_{(j,k)}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^q_{\psi_2}(\mathbb{Z}),k} = \sup_{l\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{\psi_2(N)} \frac{1}{(2N+1)^{1/q}} \left(\sum_{k\in Q(l,N)} \|a_{(j,k)}\|^q_{\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} \right)^{1/q},$$

is finite. Here, the notation $\|\{\cdot\}\|_{\mathcal{M}^p_{\psi}(\mathbb{Z}),i}$ indicates that the norm is taken with respect to the index i.

We can see that the norm is evaluated through an iterated mixed-norm structure. Specifically, we first compute the inner norm of the sequence $a_{(j,k)}$ with respect to its first variable j in the generalized Morrey sequence space $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ for a fixed index k. The outer norm is then obtained by taking the $\mathcal{M}^q_{\psi_2}(\mathbb{Z})$-norm of the resulting sequence with respect to k. Explicitly, this requires taking the supremum over all discrete intervals $Q(l,N)$ centered at $l \in \mathbb{Z}$ with radius $N \in \mathbb{N}_0$. For each interval, the ℓ^q-norm of the inner norms is computed, normalized by the interval length $(2N+1)^{1/q}$, and further scaled by the weight $\frac{1}{\psi_2(N)}$. The parameter function $\psi_2(N)$ plays a crucial role here, as it dictates the specific growth or decay conditions that characterize the generalized Morrey sequence space.

Next, the generalized mixed weak Morrey double-sequence space $(\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ and $(W\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ are defined to be the sets of all sequences $a : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ such that

$$\begin{aligned}\|a\|_{(\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} &= \left\| \|a_{(j,k)}\|_{W\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^q_{\psi_2}(\mathbb{Z}),k} \\ &= \sup_{l\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{\psi_2(N)} \frac{1}{(2N+1)^{1/q}} \left(\sum_{k\in Q(l,N)} \|a_{(j,k)}\|^q_{W\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} \right)^{1/q} < \infty,\end{aligned}$$

and

$$\begin{aligned}\|a\|_{(W\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} &= \left\| \|a_{(j,k)}\|_{W\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} \right\|_{W\mathcal{M}^q_{\psi_2}(\mathbb{Z}),k} \\ &= \sup_{l\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{\psi_2(N)} \frac{1}{(2N+1)^{1/q}} \sup_{\gamma>0} \gamma \left\{ k \in Q(l,N) : \|a_{(j,k)}\|_{W\mathcal{M}^p_{\psi_1}(\mathbb{Z}),j} > \gamma \right\}^{1/q} < \infty,\end{aligned}$$

respectively. The inequality (1) and the definitions implies

$$\|a\|_{(W\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \leq \|a\|_{(\mathcal{M}^q_{\psi_2})(W\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \leq \|a\|_{(\mathcal{M}^q_{\psi_2})(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})}. \tag{2}$$

Related to the double-sequence setting, we define

$$(M(a))_{(j,k)} = \sup_{N\in\mathbb{N}_0} \frac{1}{\#Q(j,N)} \sum_{l\in Q(j,N)} |a_{(l,k)}|, \quad (j,k)\in\mathbb{Z}\times\mathbb{Z},$$

for any suitable sequence $a:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$. For $b:\mathbb{Z}\to\mathbb{C}$, we also define

$$(M_b^m(a))_{(j,k)} = \sup_{N\in\mathbb{N}_0} \frac{1}{\#Q(j,N)} \sum_{l\in Q(j,N)} |b_l - b_j|^m |a_{(l,k)}|, \quad (j,k)\in\mathbb{Z}\times\mathbb{Z},$$

for any suitable sequence $a:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$. In addition, for any suitable sequence $a:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$, we write

$$(M^{\#}(a))_{(j,k)} = \sup_{Q\ni j} \frac{1}{\#Q} \sum_{n\in Q} |a_{(n,k)} - a_{Q,k}|, \quad (j,k)\in\mathbb{Z}\times\mathbb{Z},$$

where $a_{Q,k} = \frac{1}{\#Q}\sum\limits_{n\in Q} a_{(n,k)}$ for any interval Q and $k\in\mathbb{Z}$.

We next define the class of parameter functions ψ in the generalized Morrey sequence spaces and generalized mixed Morrey double-sequence spaces which is our interest in this paper. For $\psi:\mathbb{N}_0\to\mathbb{R}^+$ and $1\le p<\frac{1}{\eta}<\infty$, we say that $\psi\in G_p^\eta(\mathbb{Z})$ if and only if

$$\left(\frac{2N+1}{2M+1}\right)^{-\frac{1}{p}} \lesssim \frac{\psi(N)}{\psi(M)} \lesssim \left(\frac{2N+1}{2M+1}\right)^{-\eta}, \quad N,M\in\mathbb{N}, N\ge M. \tag{3}$$

The classical example for $\psi\in G_p^\eta$ is $\psi(N)=(2N+1)^{-1/q}$ where $1\le p<q<\infty$ and $\eta=1/q$.

Condition (3) implies that ψ is almost decreasing (i.e., $\psi(N)\lesssim\psi(M)$ for $N\ge M$) and the map $N\mapsto \psi(N)(2N+1)^{1/p}$ is almost increasing (i.e., $\psi(N)(2N+1)^{1/p}\lesssim\psi(M)(2M+1)^{1/p}$ for $N\le M$). Moreover, for $1\le p<\infty$, a function $\psi:\mathbb{N}_0\to\mathbb{R}^+$ is said to belong to $G_p(\mathbb{Z})$ if ψ is almost decreasing and the map $N\mapsto\psi(N)(2N+1)^{1/p}$ is almost increasing, see [12]. Consequently, we obtain the inclusion $G_p^\eta(\mathbb{Z})\subseteq G_p(\mathbb{Z})$.

The introduction of this restricted class $G_p^\eta(\mathbb{Z})$ is primarily motivated by the fine quantitative control required to establish the boundedness of higher-order commutators. While the broader class $G_p(\mathbb{Z})$ provides a sufficient framework for the discrete Hardy–Littlewood maximal operator, it lacks the strict decay bounds necessary to control the sharper oscillations associated with BMO coefficients. By incorporating the parameter η, the class $G_p^\eta(\mathbb{Z})$ enforces a precise upper scaling bound that prevents the parameter function from degenerating. This controlled decay is a critical technical prerequisite when invoking Fefferman–Stein type inequalities within the discrete mixed-norm setting.

Now, the following theorems are our main results in this paper.

Theorem 1. *Let $\psi_1,\psi_2:\mathbb{N}_0\to\mathbb{R}^+$.*

1. *If $1<p<\infty$ and $\psi_1,\psi_2\in G_p(\mathbb{Z})$, then $\psi_1\lesssim\psi_2$ if and only if M is bounded from $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ to $\mathcal{M}^p_{\psi_2}(\mathbb{Z})$.*
2. *If $\psi_1,\psi_2\in G_1(\mathbb{Z})$, then $\psi_1\lesssim\psi_2$ if and only if M is bounded from $\mathcal{M}^1_{\psi_1}(\mathbb{Z})$ to $W\mathcal{M}^1_{\psi_2}(\mathbb{Z})$.*

Theorem 2. *Let $\psi_1,\psi_2,\psi:\mathbb{N}_0\to\mathbb{R}^+$.*

1. *Suppose that*
$$1<p<\infty, \quad 1\le q<\infty, \quad \text{and } \psi_1,\psi_2\in G_p(\mathbb{Z}).$$
If $\psi_1\lesssim\psi_2$, then M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$. Conversely, if $\psi\in G_q(\mathbb{Z})$ and M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$, then $\psi_1\lesssim\psi_2$.
2. *Suppose that*
$$1\le q<\infty, \quad \text{and } \psi_1,\psi_2\in G_1(\mathbb{Z}).$$
If $\psi_1\lesssim\psi_2$, then M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$. Conversely, if $\psi\in G_q(\mathbb{Z})$ and M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$, then $\psi_1\lesssim\psi_2$.

Theorem 3. *Let $1 < p < 1/\eta < \infty$. Suppose that $\psi \in G_p^\eta(\mathbb{Z})$ and $m \in \mathbb{N}$. Then, $b \in BMO$ if and only if M_b^m is bounded on $\mathcal{M}_\psi^p(\mathbb{Z})$.*

Theorem 4. *Let $1 < p_1 < 1/\eta < \infty, 1 \le p_2 < \infty$, and $\psi_1, \psi_2 : \mathbb{N}_0 \to \mathbb{R}^+$ where $\psi_1 \in G_{p_1}^\eta(\mathbb{Z})$. Suppose that $m \in \mathbb{N}$ and $b : \mathbb{Z} \to \mathbb{C}$.*

(1) *If $b \in BMO$, then M_b^m is bounded on $(\mathcal{M}_{\psi_2}^{p_2})(\mathcal{M}_{\psi_1}^{p_1})(\mathbb{Z} \times \mathbb{Z})$.*
(2) *If M_b^m is bounded on $(\mathcal{M}_{\psi_2}^{p_2})(\mathcal{M}_{\psi_1}^{p_1})(\mathbb{Z} \times \mathbb{Z})$ where $\psi_2 \in G_{p_2}(\mathbb{Z})$, then $b \in BMO$.*

Statement (1) in Theorem 1 characterizes the boundedness of M on generalized Morrey sequence spaces, namely from the sequence space $\mathcal{M}_{\psi_1}^p(\mathbb{Z})$ to the sequence space $\mathcal{M}_{\psi_2}^p(\mathbb{Z})$ via the relation between the functions ψ_1 and ψ_2 under some assumptions. Meanwhile, the statement (2) characterizes the weak type (1,1) boundedness of M on generalized Morrey sequence spaces. This theorem extends the results regarding the boundedness of M on the continuous settings, see [6,7,11]. Moreover, in the discrete setting, the theorem complement our previous results, particularly concerning the necessary condition for the boundedness of M.

Theorem (2) handles the double-mixed setting of generalized Morrey sequence spaces. In the first part of theorem, we characterize the boundedness of M on generalized mixed Morrey double-sequence spaces, namely from the sequence space $(\mathcal{M}_\psi^q)(\mathcal{M}_{\psi_1}^p)(\mathbb{Z} \times \mathbb{Z})$ to the sequence space $(\mathcal{M}_\psi^q)(\mathcal{M}_{\psi_2}^p)(\mathbb{Z} \times \mathbb{Z})$ via the relation between the functions ψ_1 and ψ_2 under some assumptions. On the other hand, the other part characterizes the weak type (1,1) boundedness of M on generalized mixed Morrey double-sequence spaces. Beside the mixed Morrey double-sequence spaces initially introduced by Gunawan *et al.* [17], our theorem extends the boundedness M on continuous setting of mixed Morrey spaces, see [28], to the discrete setting. However, we do not state Theorem 2 as an equivalence, and establishing such an equivalent formulation remains an open problem for future research.

Unlike the previous theorems, we characterize the boundedness of the commutator on generalized Morrey sequence spaces in terms of BMO in Theorem 3 and 4. The boundedness of commutators on generalized Morrey spaces in the continuous setting has been studied in [11,28–30]. The two theorems extend the boundedness of the commutator to the discrete setting, both in the standard and mixed frameworks. Similar to Theorem 2, establishing a complete equivalence for Theorem 4 is left as an open question for future investigations.

2. Proof of Theorem 1 and 2

We need some lemmas in order to prove Theorem 1 and 2. The following first lemma estimates the norm of the characteristic function of the discrete interval Q in generalized Morrey sequence spaces.

Lemma 1. *Let $1 \le p < \infty$ and $\psi \in G_p(\mathbb{Z})$. Then, $\|\chi_{Q(j,N)}\|_{\mathcal{M}_\psi^p(\mathbb{Z})} \sim \psi(N)^{-1}$ and $\|\chi_{Q(j,N)}\|_{W\mathcal{M}_\psi^p(\mathbb{Z})} \sim \psi(N)^{-1}$ for $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$.*

Proof. Let $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. We easily see from definition that

$$\|\chi_{Q(j,N)}\|_{\mathcal{M}_\psi^p(\mathbb{Z})} \ge \frac{1}{\psi(N)} \frac{1}{(2N+1)^{1/p}} \|\chi_{Q(j,N)}\|_{\ell^p(Q(j,N))} = \frac{1}{\psi(N)}. \tag{4}$$

Now, let $k \in \mathbb{Z}$ and $M \in \mathbb{N}_0$. If $M \ge N$, then by the fact the map $N \mapsto \psi(N)(2N+1)^{1/p}$ is almost increasing, we obtain

$$\begin{aligned}
\frac{1}{\psi(M)} \frac{1}{(2M+1)^{1/p}} \|\chi_{Q(j,N)}\|_{\ell^p(Q(k,M))} &= \frac{1}{\psi(M)} \frac{(\#Q(j,N) \cap Q(k,M))^{1/p}}{(2M+1)^{1/p}} \\
&\le \frac{1}{\psi(M)} \frac{\#Q(j,N)^{1/p}}{(2M+1)^{1/p}} \\
&= \frac{1}{\psi(M)} \frac{(2N+1)^{1/p}}{(2M+1)^{1/p}} \lesssim \frac{1}{\psi(N)}.
\end{aligned}$$

Otherwise, if $M < N$, then by the fact that ψ is almost decreasing, we also obtain that

$$\begin{aligned}\frac{1}{\psi(M)}\frac{1}{(2M+1)^{1/p}}\|\chi_{Q(j,N)}\|_{\ell^p(Q(k,M))} &= \frac{1}{\psi(M)}\frac{(\#Q(j,N)\cap Q(k,M))^{1/p}}{(2M+1)^{1/p}} \\ &\le \frac{1}{\psi(M)}\frac{\#Q(k,M)^{1/p}}{(2M+1)^{1/p}} \\ &= \frac{1}{\psi(M)} \lesssim \frac{1}{\psi(N)}.\end{aligned}$$

The two cases allow us to deduce that

$$\|\chi_{Q(j,N)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} = \sup_{k\in\mathbb{Z},M\in\mathbb{N}_0}\frac{1}{\psi(M)}\frac{1}{(2M+1)^{1/p}}\|\chi_{Q(j,N)}\|_{\ell^p(Q(k,M))} \lesssim \frac{1}{\psi(N)}. \tag{5}$$

Hence, (4) and (5) imply $\|\chi_{Q(j,N)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} \sim \psi(N)^{-1}$.
Next, we see that

$$\begin{aligned}\|\chi_{Q(j,N)}\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} &\ge \frac{1}{\psi(N)}\frac{1}{(2N+1)^{1/p}}\|\chi_{Q(j,N)}\|_{w\ell^p(Q(j,N))} \\ &\ge \frac{1}{\psi(N)}\frac{1}{(2N+1)^{1/p}}\sup_{\gamma>0}\gamma\#\{k\in Q(j,N): \chi_{Q(j,N)}(k) > \gamma\}^{1/p} \\ &\ge \frac{1}{\psi(N)}\frac{1}{(2N+1)^{1/p}}\frac{1}{2}\#\left\{k\in Q(j,N): \chi_{Q(j,N)}(k) > \frac{1}{2}\right\}^{1/p} \\ &= \frac{1}{\psi(N)}\frac{1}{(2N+1)^{1/p}}\frac{1}{2}\#Q(j,N)^{1/p} \simeq \frac{1}{\psi(N)}.\end{aligned}$$

On the other hand, by (1) and the estimate $\|\chi_{Q(j,N)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} \sim \frac{1}{\psi(N)}$ we have that

$$\|\chi_{Q(j,N)}\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} \le \|\chi_{Q(j,N)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} \sim \frac{1}{\psi(N)}.$$

Therefore, $\|\chi_{Q(j,N)}\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} \sim \psi(N)^{-1}$ and the desired results are obtained. $\square$

We note that if $k\in\mathbb{C}$ is constant, then from the definition of the norm of $\mathcal{M}^p_\psi(\mathbb{Z})$, it is clear that

$$\|ka\|_{\mathcal{M}^p_\psi(\mathbb{Z})} = |k|\|a\|_{\mathcal{M}^p_\psi(\mathbb{Z})}. \tag{6}$$

On the other hand, related to the weak type norm, for the constant $k\ne 0$ we have that

$$\begin{aligned}\|ka\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} &= \sup_{j\in\mathbb{Z},N\in\mathbb{N}_0}\frac{1}{\psi(N)}\frac{1}{\#Q(j,N)^{1/p}}\sup_{\gamma>0}\gamma\#\{j'\in Q(j,N): |ka_{j'}| > \gamma\}^{1/p} \\ &= \sup_{j\in\mathbb{Z},N\in\mathbb{N}_0}\frac{1}{\psi(N)}\frac{1}{\#Q(j,N)^{1/p}}\sup_{\gamma>0}\gamma\#\{j'\in Q(j,N): |a_{j'}| > \frac{\gamma}{|k|}\}^{1/p} \\ &= |k|\sup_{j\in\mathbb{Z},N\in\mathbb{N}_0}\frac{1}{\psi(N)}\frac{1}{\#Q(j,N)^{1/p}}\sup_{\gamma>0}\frac{\gamma}{|k|}\#\{j'\in Q(j,N): |a_{j'}| > \frac{\gamma}{|k|}\}^{1/p} \\ &= |k|\|a\|_{W\mathcal{M}^p_\psi(\mathbb{Z})}.\end{aligned}$$

If $k=0$, it is clear that $\|ka\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} = k\|a\|_{W\mathcal{M}^p_\psi(\mathbb{Z})}$. Therefore,

$$\|ka\|_{W\mathcal{M}^p_\psi(\mathbb{Z})} = |k|\|a\|_{W\mathcal{M}^p_\psi(\mathbb{Z})}, k\in\mathbb{C}. \tag{7}$$

Next, the following lemma provides an estimate for the norm of a specific sequence in generalized mixed Morrey double-sequence spaces. This sequence is defined as a characteristic function of a rectangle in $\mathbb{Z} \times \mathbb{Z}$. This result generalizes [17, Lemma 2.1].

Lemma 2. *Let $1 \le p_1, p_2 < \infty$. Suppose that $\psi_1 \in G_{p_1}(\mathbb{Z})$ and $\psi_2 \in G_{p_2}(\mathbb{Z})$. Let $j, k \in \mathbb{Z}$ and $M, N \in \mathbb{N}_0$. Define $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ such that $a(l,n) = \chi_{Q(j,N)\times Q(k,M)}(l,n)$. Then, $\|a\|_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim (\psi_1(N)\psi_2(M))^{-1}$ and $\|a\|_{(W\mathcal{M}^{p_2}_{\psi_2})(W\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim (\psi_1(N)\psi_2(M))^{-1}$.*

Proof. It can be seen that $a(l,n) = \chi_{Q(j,N)}(l) \cdot \chi_{Q(k,M)}(n)$. Then, by (6),

$$\|a(l,n)\|_{\mathcal{M}^{p_1}_{\psi_1},l} = \|\chi_{Q(j,N)}(l) \cdot \chi_{Q(k,M)}(n)\|_{\mathcal{M}^{p_1}_{\psi_1},l} = \|\chi_{Q(j,N)}(l)\|_{\mathcal{M}^{p_1}_{\psi_1},l} \cdot \chi_{Q(k,M)}(n).$$

Here, $\chi_{Q(k,M)}(n)$ can be taken outside the norm since it is constant with respect to the index l. Therefore, by the preceding lemma, we obtain that

$$\begin{aligned}
\|a\|_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} &= \left\| \|a(l,n)\|_{\mathcal{M}^{p_1}_{\psi_1},l} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n} \\
&= \left\| \|\chi_{Q(j,N)}(l)\|_{\mathcal{M}^{p_1}_{\psi_1},l} \cdot \chi_{Q(k,M)}(n) \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n} \\
&= \|\chi_{Q(j,N)}(l)\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),l} \, \|\chi_{Q(k,M)}(n)\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n}.
\end{aligned}$$

Note that we have taken $\|\chi_{Q(j,N)}(l)\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),l}$ outside the $\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z})$-norm, since it is constant with respect to the index n.

By the preceding lemma, we thus have that

$$\|a\|_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim \frac{1}{\psi_1(N)\psi_2(M)},$$

as desired. On the other hand, By an analogous argument, paying careful attention to the terms that are constant with respect to the relevant indices, by (7) and the preceding lemma we also have

$$\begin{aligned}
\|a\|_{(W\mathcal{M}^{p_2}_{\psi_2})(W\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} &= \left\| \|a(l,n)\|_{W\mathcal{M}^{p_1}_{\psi_1},l} \right\|_{W\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n} \\
&= \left\| \|\chi_{Q(j,N)}(l)\|_{W\mathcal{M}^{p_1}_{\psi_1},l} \cdot \chi_{Q(k,M)}(n) \right\|_{W\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n} \\
&= \|\chi_{Q(j,N)}(l)\|_{W\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),l} \, \|\chi_{Q(k,M)}(n)\|_{W\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),n} \\
&\sim \frac{1}{\psi_1(N)\psi_2(M)}.
\end{aligned}$$

This completes the proof the lemma. □

The following theorem is our earlier result regarding the sufficient condition for the boundedness of M on generalized Morrey sequence spaces. See [16, Theorem 4] for the general case.

Theorem 5. *Let $1 \le p < \infty$. Suppose that ψ_1 and ψ_2 are positive functions on $\mathbb{N}_0$ such that*

$$\sup_{N\in\mathbb{N}_0} \frac{1}{\psi_2(N)} \sup_{N<n\in\mathbb{N}} \frac{\inf_{n\le m\in\mathbb{N}} (2m+1)^{1/p}\psi_1(m)}{(2n+1)^{1/p}} < \infty. \tag{8}$$

Then, M is bounded from $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ to $\mathcal{M}^p_{\psi_2}(\mathbb{Z})$ for $1 < p < \infty$ and bounded from $\mathcal{M}^1_{\psi_1}(\mathbb{Z})$ to $W\mathcal{M}^1_{\psi_2}(\mathbb{Z})$.

As an immediate consequence of the foundational result, we deduce the following corollary. This result establishes a specialized boundedness properties involving a single function ψ, which will serve as a crucial technical vehicle for proving the necessity part of our main characterization theorem.

Corollary 1. *Let $\psi \in G_p(\mathbb{Z})$ where $1 \le p < \infty$. Then, M is bounded on $\mathcal{M}^p_\psi(\mathbb{Z})$ for $1 < p < \infty$ and bounded from $\mathcal{M}^1_\psi(\mathbb{Z})$ to $W\mathcal{M}^1_\psi(\mathbb{Z})$ for $p = 1$.*

Proof. According to Theorem 5, we only need to prove that the condition $\psi \in G_p(\mathbb{Z})$ implies (8). Indeed, from the assumption $\psi \in G_p(\mathbb{Z})$ we have ψ is almost decreasing and

$$\sup_{N<n\in\mathbb{N}} \psi(n) \lesssim \psi(N), \quad N \in \mathbb{N}_0.$$

Moreover, we have that

$$\sup_{N<n\in\mathbb{N}} \psi(n) = \sup_{N<n\in\mathbb{N}} \frac{(2n+1)^{1/p}\psi(n)}{(2n+1)^{1/p}} \ge \sup_{N<n\in\mathbb{N}} \frac{\inf_{n\le m\in\mathbb{N}}(2m+1)^{1/p}\psi(m)}{(2n+1)^{1/p}}.$$

Hence,

$$\sup_{N<n\in\mathbb{N}} \frac{\inf_{n\le m\in\mathbb{N}}(2m+1)^{1/p}\psi(m)}{(2n+1)^{1/p}} \lesssim \psi(N), \quad N \in \mathbb{N}_0,$$

as desired. □

Now, we prove Theorem 1.

Proof of Theorem 1. First suppose that $\psi_1 \lesssim \psi_2$. By Corollary 1, we have that M is bounded on $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ for $1 < p < \infty$ and bounded from $\mathcal{M}^1_{\psi_1}(\mathbb{Z})$ to $W\mathcal{M}^1_{\psi_1}(\mathbb{Z})$. Since $\psi_1 \lesssim \psi_2$, we then have $\|M(a)\|_{\mathcal{M}^p_{\psi_2}(\mathbb{Z})} \le \|M(a)\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \lesssim \|a\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})}$ for $1 < p < \infty$ and $\|M(a)\|_{W\mathcal{M}^1_{\psi_2}(\mathbb{Z})} \le \|M(a)\|_{W\mathcal{M}^1_{\psi_1}(\mathbb{Z})} \lesssim \|a\|_{\mathcal{M}^1_{\psi_1}(\mathbb{Z})}$. Hence, M is bounded from $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ to $\mathcal{M}^p_{\psi_2}(\mathbb{Z})$ for $1 < p < \infty$ and bounded from $\mathcal{M}^1_{\psi_1}(\mathbb{Z})$ to $W\mathcal{M}^1_{\psi_2}(\mathbb{Z})$.

Next, we assume that M is bounded from $\mathcal{M}^p_{\psi_1}(\mathbb{Z})$ to $\mathcal{M}^p_{\psi_2}(\mathbb{Z})$ for $1 < p < \infty$. Let $j \in \mathbb{Z}$ and $N \in \mathbb{N}$. Since $|a| \le M(|a|)$ on $\mathbb{Z}$ for any $a : \mathbb{Z} \to \mathbb{C}$, then by taking $a := \chi_{Q(j,N)}$ we have from Lemma 1 that

$$\psi_2(N)^{-1} \sim \|a\|_{\mathcal{M}^p_{\psi_2}(\mathbb{Z})} \le \|M(a)\|_{\mathcal{M}^p_{\psi_2}(\mathbb{Z})} \lesssim \|a\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \sim \psi_1(N)^{-1},$$

which implies $\psi_1 \lesssim \psi_2$. Now we assume that M is bounded from $\mathcal{M}^1_{\psi_1}(\mathbb{Z})$ to $W\mathcal{M}^1_{\psi_2}(\mathbb{Z})$. The same choice of a also allows us to get

$$\psi_2(N)^{-1} \sim \|a\|_{W\mathcal{M}^1_{\psi_2}(\mathbb{Z})} \le \|M(a)\|_{W\mathcal{M}^1_{\psi_2}(\mathbb{Z})} \lesssim \|a\|_{\mathcal{M}^1_{\psi_1}(\mathbb{Z})} \sim \psi_1(N)^{-1},$$

which implies $\psi_1 \lesssim \psi_2$. This completes the proof of the theorem. □

While the sufficiency part of Theorem 1 builds upon the boundedness imported from Theorem 5 and the embedding implied by $\psi_1 \lesssim \psi_2$, the chief novelty of this theorem lies in establishing the necessity condition. It is worth emphasizing that this necessity is achieved even though the condition on the function ψ is strengthened. Therefore, Theorem 1 provides a more comprehensive characterization than a straightforward corollary.

Proposition 1. *Let $1 < p < \infty$. Then,*

$$\|M(a)_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})} \lesssim \|a_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})}, \quad k \in \mathbb{Z},$$

for $a_{(\cdot,k)} \in \ell^p(\mathbb{Z})$ for any $k \in \mathbb{Z}$. Moreover,

$$\|M(a)_{(\cdot,k)}\|_{w\ell^1(\mathbb{Z})} \le \|a_{(\cdot,k)}\|_{\ell^1(\mathbb{Z})}, \quad k \in \mathbb{Z},$$

for $a_{(\cdot,k)} \in \ell^1(\mathbb{Z})$ *for any* $k \in \mathbb{Z}$.

Proof. It can be seen that if $a_{(\cdot,k)} \in \ell^\infty(\mathbb{Z})$ for any $k \in \mathbb{Z}$, then

$$(M(a))_{(j,k)} = \sup_{N\in\mathbb{N}_0} \frac{1}{(2N+1)} \sum_{l\in Q(j,N)} |a_{(l,k)}| \le \|a_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})} \sup_{N\in\mathbb{N}_0} \frac{1}{(2N+1)} \sum_{l\in Q(j,N)} 1 = \|a_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})},$$

which implies that

$$\|M(a)_{(\cdot,k)}\|_{\ell^\infty(\mathbb{Z})} \le \|a_{(\cdot,k)}\|_{\ell^\infty(\mathbb{Z})}, \quad k \in \mathbb{Z}.$$

Next, we move to the weak-type estimate (1,1). Motivated by [22], our aim is to prove that there exist $C > 0$ such that

$$\#\{n \in \mathbb{Z} : (M(a))_{(n,k)} > \alpha\} \le \frac{C}{\alpha}\|a_{(\cdot,k)}\|_{\ell^1(\mathbb{Z})}, \quad \alpha > 0 \tag{9}$$

for $a_{(\cdot,k)} \in \ell^1(\mathbb{Z})$ for any $k \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$. For $\alpha > 0$, we set $E_{\alpha,k} = \{n \in \mathbb{Z} : (M(a))_{(n,k)} > \alpha\}$.

If $j \in E_{\alpha,k}$, there exists $N_j \in \mathbb{N}_0$ such that

$$\alpha \le \frac{1}{2N_j+1} \sum_{|n-j|\le N_j} |a_{(n,k)}|.$$

Hence, we have that

$$E_{\alpha,k} \subseteq \cup_{j\in E_{\alpha,k}} Q(j, N_j).$$

Then, by Vitalli covering lemma, there exists a disjoint subcollection $\{Q(j', N_{j'})\}$ such that

$$E_{\alpha,k} \subseteq \cup_{j'} 5Q(j', N_{j'}).$$

Since the subcollection $\{Q(j', N_{j'})\}$ is disjoint, we can see that

$$\alpha \sum_{j'} \#Q(j', N_{j'}) \le \sum_{j'} \sum_{|n-j'|\le N_{j'}} |a_{(n,k)}| \le \sum_{j\in\mathbb{Z}} |a_{(j,k)}| = \|a_{(\cdot,k)}\|_{\ell^1(\mathbb{Z})}.$$

Hence,

$$\#E_{\alpha,k} \le \sum_{j'} \#5Q(j', N_{j'}) \le 5\sum_{j'} \#Q(j', N_{j'}) \le \frac{5}{\alpha}\|a_{(\cdot,k)}\|_{\ell^1(\mathbb{Z})}$$

which proves (9).

Now, for $k \in \mathbb{Z}$, we set $b_k(\cdot) = a_{(\cdot,k)}$. Then, $(M(a))_{(j,k)} = (M(b_k))_j$. Hence, $\|M(b_k)\|_{\ell^\infty} \le \|b_k\|_{\ell^\infty}$ and $\|M(b_k)\|_{w\ell^1} \le 5\|M(b_k)\|_{w\ell^1}$ for $k \in \mathbb{Z}$. By Markinkiewicz interpolation, we have that

$$\|M(b_k)\|_{\ell^p(\mathbb{Z})} \le 2\left(\frac{p}{p-1}\right)^{\frac{1}{p}} 5^{\frac{1}{p}} \|b_k\|_{\ell^p(\mathbb{Z})},$$

which implies

$$\|(M(a))_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})} \le 2\left(\frac{5p}{p-1}\right)^{\frac{1}{p}} \|a_{(\cdot,k)}\|_{\ell^p(\mathbb{Z})}, \quad k \in \mathbb{Z}.$$

This proves the proposition. □

Next, we prove Theorem 2.

Proof of Theorem 2. Suppose that $\psi_1 \lesssim \psi_2$. Let $j, k \in \mathbb{Z}$ and $M, N \in \mathbb{N}_0$. Suppose that $a \in (\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z} \times \mathbb{Z})$. We write $a = a^{(1)} + a^{(2)}$ where

$$a^{(1)}(j', k') = a_{(j',k')} \cdot \chi_{Q(j,2N+1)}(j').$$

It is clear that

$$\|(M(a^{(1)}))_{(\cdot,k')}\|_{\ell^p(Q(j,N))} \leq \|(M(a^{(1)}))_{(\cdot,k')}\|_{\ell^p(\mathbb{Z})}.$$

Proposition 1 then yields

$$\|(M(a))_{(\cdot,k')}\|_{\ell^p(Q(j,N))} \lesssim \|(a^{(1)})_{(\cdot,k')}\|_{\ell^p(\mathbb{Z})} = \|a_{(\cdot,k')}\|_{\ell^p(Q(j,2N+1))}.$$

Next, we shall find the estimate regarding $a^{(2)}$. Let $j' \in Q(j,N)$. For $j'' \in Q(j',n) \cap Q(j,2N+1)^c$, then $N < 2N+1-N < |j''-j| - |j-j'| \leq |j''-j'| \leq n$. Hence, $\sum_{j''\in Q(j',n)\cap Q(j,2N)^c} |a_{j''}| = 0$, for $n \leq N$. Moreover, we also note that $|j''-j| \leq |j''-j'| + |j'-j| \leq n+N < 2n$ for $N < n$ which implies that $Q(j',n) \cap Q(j,2N+1)^c \subseteq Q(j,2n)$. Therefore, by Hölder's inequality and the assumption that $\psi_1 \in G_p^\eta(\mathbb{Z})$, for $k' \in \mathbb{Z}$,

$$\begin{aligned}
\sup_{n\in\mathbb{N}_0} \frac{1}{\#Q(j',n)} \sum_{j''\in Q(j',n)} |a^{(2)}(j'',k')| &= \sup_{N<n\in\mathbb{N}_0} \frac{1}{\#Q(j,n)} \sum_{j''\in Q(j',n)\cap Q(j,2N)^c} |a(j'',k')| \\
&\leq \sup_{N<n\in\mathbb{N}_0} \frac{1}{\#Q(j,n)} \sum_{j''\in Q(j,2n)} |a(j'',k')| \\
&\lesssim \sup_{2N<n\in\mathbb{N}_0} \frac{1}{\#Q(j,n)} \sum_{j''\in Q(j,n)} |a(j'',k')| \\
&\leq \sup_{2N<n\in\mathbb{N}_0} \frac{1}{\#Q(j,n)} \|a_{(\cdot,k')}\|_{\ell^p(Q(j,n))} \|1\|_{\ell^{p'}(Q(j,n))} \\
&\lesssim \sup_{2N<n\in\mathbb{N}_0} \frac{1}{\#Q(j,n)^{\frac{1}{p}}} \|a_{(\cdot,k')}\|_{\ell^p(Q(j,n))} \\
&\leq \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \sup_{2N<n\in\mathbb{N}} \psi_1(n) \lesssim \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \psi_1(N),
\end{aligned}$$

and

$$(M(a^{(2)}))_{(j',k')} = \sup_{n\in\mathbb{N}_0} \frac{1}{\#Q(j',n)} \sum_{j''\in Q(j',n)} |a^{(2)}(j'',k')| \lesssim \psi_1(N) \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})}.$$

Hence,

$$\|(M(a^{(2)}))_{(\cdot,k')}\|_{\ell^p(Q(j,N))} \lesssim \psi_1(N) \#Q(j,N)^{\frac{1}{p}} \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})},$$

and

$$\|(M(a^{(2)}))_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \lesssim \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})}, \quad k' \in \mathbb{Z}.$$

By combining the estimate regarding $a^{(1)}$ and $a^{(2)}$, we have

$$\|(M(a))_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \lesssim \|a_{(\cdot,k')}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})}, \quad k' \in \mathbb{Z}. \tag{10}$$

One can follow the similar technique to obtain that

$$\|(M(a))_{(\cdot,k')}\|_{W\mathcal{M}^1_{\psi_1}(\mathbb{Z})} \lesssim \|a_{(\cdot,k')}\|_{\mathcal{M}^1_{\psi_1}(\mathbb{Z})}, \quad k' \in \mathbb{Z}.$$

The assumption $\psi_1 \lesssim \psi_2$ implies

$$\|M(a)_{(\cdot,k)}\|_{\mathcal{M}^p_{\psi_2}(\mathbb{Z})} \lesssim \|M(a)_{(\cdot,k)}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})} \lesssim \|a_{(\cdot,k)}\|_{\mathcal{M}^p_{\psi_1}(\mathbb{Z})}, \quad k \in \mathbb{Z},$$

and

$$\|M(a)_{(\cdot,k)}\|_{W\mathcal{M}^1_{\psi_2}(\mathbb{Z})} \lesssim \|M(a)_{(\cdot,k)}\|_{W\mathcal{M}^1_{\psi_1}(\mathbb{Z})} \lesssim \|a_{(\cdot,k)}\|_{\mathcal{M}^1_{\psi_1}(\mathbb{Z})}, \quad k \in \mathbb{Z}.$$

Therefore, by the assumption and definition of the norm, we have

$$\|M(a)\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})} \lesssim \|a\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})},$$

and

$$\|M(a)\|_{(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})} \lesssim \|M(a)\|_{(\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})} \lesssim \|a\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})},$$

showing that M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$ for $1<p<\infty$ and bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$.

Conversely, we assume that M is bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$ for $1<p<\infty$ and bounded from $(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})$ to $(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})$. Let $M,N\in\mathbb{N}_0$ and $j,k\in\mathbb{Z}$. Define the sequence $a:\mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ as in Lemma 2. Thus, for $1<p<\infty$, by the lemma,

$$\begin{aligned}(\psi_2(N)\psi(M))^{-1} &\sim \|a\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})}\\ &\le \|M(a)\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_2})(\mathbb{Z}\times\mathbb{Z})}\\ &\lesssim \|a\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^p_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim (\psi_1(N)\psi(M))^{-1},\end{aligned}$$

implying that $\psi_1\lesssim\psi_2$. We also apply Lemma 2 for the norm of weak type,

$$\begin{aligned}(\psi_2(N)\psi(M))^{-1} &\sim \|a\|_{(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})}\\ &\le \|M(a)\|_{(W\mathcal{M}^q_\psi)(W\mathcal{M}^1_{\psi_2})(\mathbb{Z}\times\mathbb{Z})}\\ &\lesssim \|a\|_{(\mathcal{M}^q_\psi)(\mathcal{M}^1_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim (\psi_1(N)\psi(M))^{-1},\end{aligned}$$

which also implies $\psi_1\lesssim\psi_2$. This completes the proof of the theorem. □

3. Proof of Theorem 3 and 4

We first provide some lemmas and propositions. The following two lemmas are proved in [14]. For the definition of A_∞ and related concepts, we refer the reader to [14, Section 2].

Lemma 3. *Let $1\le p<\infty$, $w\in A_\infty$, and $b\in BMO$. Then,*

$$\left(\frac{1}{w(Q)}\sum_{j\in Q}|b_j-b_Q|^p w_j\right)^{\frac{1}{p}} \lesssim \|b\|_*. \tag{11}$$

Lemma 4. *If $b\in BMO$, then $|b_Q-b_{nQ}|\lesssim\|b\|_*\ln n$ for any interval $Q=Q(j,N)$ and $n\in\mathbb{N}$ where $nQ=Q(j,nN)$.*

The next lemma gives a relation between M^m_b and M which is useful in estimating M^m_b via operator M.

Lemma 5. *Let $u,v>1$, $b\in BMO$, and $m\in\mathbb{N}$. Then,*

$$|(M^\#(M^m_b(a)))_j| \lesssim \|b\|^m_* \left((M(|M(a)|^u))^{\frac{1}{u}}_j + (M(|a|^v))^{\frac{1}{v}}_j\right), \quad j\in\mathbb{Z}. \tag{12}$$

Proof. Let $j\in\mathbb{Z}$ and the interval Q containing j where we write $Q=Q(c(Q),N_0)$ for some $c(Q)\in\mathbb{Z}$ and $N_0\in\mathbb{N}_0$. Since the map $t\mapsto t^m$ is convex, we can obtain that

$$\begin{aligned}|(M^m_b(a))_j| &= \sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_n-b_j|^m|a_n|\\ &\le 2^{m-1}\left[\sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_j-b_Q|^m|a_n| + \sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_n-b_Q|^m|a_n|\right]\end{aligned}$$

$$
\begin{aligned}
&\simeq |b_j - b_Q|^m |(M(|a|))_j| + (M(|b-b_Q|^m|a|))_j \\
&\le |b_j - b_Q|^m |(M(|a|))_j| + (M(|b-b_Q|^m|a|\chi_{2Q}))_j + (M(|b-b_Q|^m|a|\chi_{(2Q)^c}))_j \\
&:= I_1(j) + I_2(j) + I_3(j).
\end{aligned}
$$

We shall estimate the mean value over Q for each I_1, I_2, and I_3. First, by Hölder's inequality and Lemma 3, we can obtain the following estimate for I_1.

$$
\frac{1}{\#Q}\sum_{k\in Q} I_1(k) \le \left(\frac{1}{\#Q}\sum_{k\in Q}|b_k - b_Q|^{mu'}\right)^{\frac{1}{u'}}\left(\frac{1}{\#Q}\sum_{k\in Q}|(M(a))_k|^u\right)^{\frac{1}{u}} \lesssim \|b\|_*^m (M(|M(a)|^u))_j^{\frac{1}{u}}.
$$

Next, we let $v = rs > 1$ where $r, s > 1$. The boundedness of M on ℓ^rtogether with Lemma 3 ($w \equiv 1$), we obtain that

$$
\begin{aligned}
\frac{1}{\#Q}\sum_{n\in Q} I_2(n) &\le \left(\frac{1}{\#Q}\sum_{n\in Q}|(M(|b-b_Q|^m|a|\cdot\chi_{2Q}))_n|^r\right)^{\frac{1}{r}} \lesssim \frac{1}{\#Q^{\frac{1}{r}}}\left(\sum_{n\in Q}|b_n - b_Q|^{mr}|a_n|^r\right)^{\frac{1}{r}} \\
&\le \frac{1}{\#Q^{\frac{1}{r}}}\left(\sum_{n\in Q}|b_n - b_Q|^{mrs'}\right)^{\frac{1}{rs'}}\left(\sum_{n\in Q}|a_n|^{rs}\right)^{\frac{1}{rs}} \\
&\lesssim \frac{1}{\#Q^{\frac{1}{r'}-\frac{1}{rs'}}}\|b\|_*^m\left(\sum_{n\in Q}|a_n|^v\right)^{\frac{1}{v}} \lesssim \|b\|_*^m (M(|a|^v))_j^{\frac{1}{v}}.
\end{aligned}
$$

We now estimate the mean value for I_3. We first note that if $n \in \mathbb{Z}$ and $M \in \mathbb{N}_0$ such that $0 < |n-j| \le M$, then $0 < |n-j| \le 2M+1$ and

$$
\frac{1}{2M+1} \le \frac{1}{|n-j|}.
$$

Moreover, $\frac{1}{|n-j|}$ makes sense if $n \in (2Q)^c$. Indeed, for $n \in (2Q)^c$, we have $|n - c(2Q)| = |n - c(Q)| > 2N$ which implies $|n-j| \ge |n-c(Q)| - |c(Q) - j| > 2N - N = N \ge 0$ and $|n-j| > 0$. Hence,

$$
\begin{aligned}
I_3(j) &= \sup_{M\in\mathbb{N}_0}\frac{1}{(2M+1)}\sum_{n\in Q(j,M)\cap(2Q)^c}|b_n - b_Q|^m|a_n| \\
&\le \sup_{M\in\mathbb{N}_0}\sum_{n\in Q(j,M)\cap(2Q)^c}\frac{1}{|n-j|}|b_n - b_Q|^m|a_n| \\
&\le \sum_{n\in(2Q)^c}\frac{1}{|n-j|}|b_n - b_Q|^m|a_n|.
\end{aligned}
$$

For $n \in (2Q)^c$ and $j, k \in Q$, by applying the Mean Value Theorem on the map $x \mapsto \frac{1}{|n-x|}$ we have that

$$
\left|\frac{1}{|n-j|} - \frac{1}{|n-k|}\right| \le \sup_{x\in[\min\{j,k\},\max\{j,k\}]}\frac{|j-k|}{|n-x|^2}.
$$

Since $n \in (2Q)^c$, we have $|n-x| \sim |n-k|$ for $x \in [\min\{j,k\}, \max\{j,k\}]$. Hence,

$$
\left|\frac{1}{|n-j|} - \frac{1}{|n-k|}\right| \lesssim \frac{|j-k|}{|n-k|^2}. \tag{13}
$$

Therefore, Hölder's inequality implies that for $j, k \in Q$ we have

$$
|I_3(j) - I_3(k)| \le \sum_{n\in(2Q)^c}\left|\frac{1}{|n-j|} - \frac{1}{|n-k|}\right||b_n - b_Q|^m|a_n| \lesssim \sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|b_n - b_Q|^m|a_n|
$$

$$\leq \left(\sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|b_n-b_Q|^{mv'}\right)^{\frac{1}{v'}}\left(\sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|a_n|^v\right)^{\frac{1}{v}}.$$

We see that, by Lemma 4

$$\begin{aligned}\left(\sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|b_n-b_Q|^{mv'}\right)^{\frac{1}{v'}} &\lesssim \sum_{l\in\mathbb{N}}\left(\frac{N}{\#(2^{l+1}Q)^2}\sum_{n\in 2^{l+1}Q\setminus 2^lQ}|b_n-b_Q|^{mv'}\right)^{\frac{1}{v'}}\\ &\lesssim \sum_{l\in\mathbb{N}}\left(\frac{N}{\#(2^{l+1}Q)^2}\sum_{n\in 2^{l+1}Q}|b_n-b_{2^{l+1}Q}|^{mv'}\right)^{\frac{1}{v'}}\\ &\quad+\sum_{l\in\mathbb{N}}\left(\frac{N}{\#(2^{l+1}Q)^2}\sum_{n\in 2^{l+1}Q}|b_{2^{l+1}Q}-b_Q|^{mv'}\right)^{\frac{1}{v'}}\lesssim \|b\|_*^m.\end{aligned}$$

Since $j,k\in Q$, we have $|j-k|\leq 2N_0$. If $N_0=0$, then it is clear that k should be equals to j and

$$\left(\sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|a_n|^v\right)^{\frac{1}{v}}=0\leq (M(|a|^v))_j^{\frac{1}{v}}.$$

On the other hand, for $l\in\mathbb{N}$, $n\in 2^{l+1}Q\setminus 2^lQ$ implies

$$|k-n|\geq |n-c(2^lQ)|-|c(2^lQ)-k|>2^lN_0-N_0=(2^l-1)N_0\geq\frac{1}{2}(2^lN_0)\geq\frac{1}{10}\#(2^{l+1}Q),\tag{14}$$

and

$$\begin{aligned}\left(\sum_{n\in(2Q)^c}\frac{|j-k|}{|k-n|^2}|a_n|^v\right)^{\frac{1}{v}} &= \left(\sum_{l\in\mathbb{N}}\sum_{n\in 2^{l+1}Q\setminus 2^lQ}\frac{|j-k|}{|k-n|^2}|a_n|^v\right)^{\frac{1}{v}} < 100^{\frac{1}{v}}\sum_{l\in\mathbb{N}}\left(\frac{2N}{\#(2^{l+1}Q)^2}\sum_{n\in 2^{l+1}Q\setminus 2^lQ}|a_n|^v\right)^{\frac{1}{v}}\\ &\leq 200^{\frac{1}{v}}\sum_{l\in\mathbb{N}}\left(\frac{N}{\#(2^{l+1}Q)}\frac{1}{\#(2^{l+1}Q)}\sum_{n\in 2^{l+1}Q}|a_n|^v\right)^{\frac{1}{v}}\\ &\leq 200^{\frac{1}{v}}(M(|a|^v))_j^{\frac{1}{v}}\sum_{l\in\mathbb{N}}\left(\frac{N_0}{2^{l+2}N_0+1}\right)^{\frac{1}{v}}\\ &< 200^{\frac{1}{v}}(M(|a|^v))_j^{\frac{1}{v}}\sum_{l\in\mathbb{N}}\left(\frac{1}{2^{l+2}}\right)^{\frac{1}{v}}\simeq (M(|a|^v))_j^{\frac{1}{v}}.\end{aligned}$$

These imply that $|I_3(j)-I_3(k)|\lesssim\|b\|_*^m(M(|a|^v))_j^{\frac{1}{v}}$, which allows us to conclude (12). $\square$

Lemma 6. *Let $u,v>1$, $b\in BMO$, and $m\in\mathbb{N}$. Then,*

$$|(M^{\#}(M_b^m(a)))_{(j,k)}|\lesssim\|b\|_*^m\left((M(|M(a)|^u))_{(j,k)}^{\frac{1}{u}}+(M(|a|^v))_{(j,k)}^{\frac{1}{v}}\right),\quad (j,k)\in\mathbb{Z}\times\mathbb{Z}.\tag{15}$$

Proof. Let $(j,k)\in\mathbb{Z}\times\mathbb{Z}$ and the interval $Q=Q(j_0,N)$ containing j. We write

$$\begin{aligned}|(M_b^m(a))_{(j,k)}| &= \sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_n-b_j|^m|a_{(n,k)}|\\ &\leq 2^{m-1}\left[\sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_j-b_Q|^m|a_{(n,k)}|+\sup_{N\in\mathbb{N}_0}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_n-b_Q|^m|a_{(n,k)}|\right]\end{aligned}$$

$$
\begin{aligned}
&\simeq |b_j - b_Q|^m |(M(|a|))_{(j,k)}| + (M(|b-b_Q|^m|a|))_{(j,k)} \\
&\leq |b_j - b_Q|^m |(M(|a|))_{(j,k)}| + (M(|b-b_Q|^m|a|\chi_{2Q}))_{(j,k)} + (M(|b-b_Q|^m|a|\chi_{(2Q)^c}))_{(j,k)} \\
&:= J_1(j,k) + J_2(j,k) + J_3(j,k).
\end{aligned}
$$

We shall estimate the mean value over Q for each J_1, J_2, and J_3.

The Hölder's inequality, Lemma 3, and the assumption that Q containing j imply the following estimates for J_1.

$$
\begin{aligned}
\frac{1}{\#Q}\sum_{n\in Q} J_1(n,k) &\leq \left(\frac{1}{\#Q}\sum_{n\in Q}|b_n - b_Q|^{mu'}\right)^{\frac{1}{u'}}\left(\frac{1}{\#Q}\sum_{n\in Q}|(M(a))_{(n,k)}|^u\right)^{\frac{1}{u}} \\
&\lesssim \|b\|_*^m\left(\sup_{M\in\mathbb{N}_0}\frac{1}{\#Q(j,M)}\sum_{n\in Q(j,M)}|(M(a))_{(n,k)}|^u\right)^{\frac{1}{u}} \\
&\leq \|b\|_*^m (M(|M(a)|^u))_{(j,k)}^{\frac{1}{u}}.
\end{aligned}
$$

Next, we let $v = rs > 1$ where $r,s > 1$. The boundedness of M on $\ell^r(\mathbb{Z})$ with respect to the first coordinate according to Proposition 1, together with Lemma 3 ($w \equiv 1$), we obtain that

$$
\begin{aligned}
\frac{1}{\#Q}\sum_{n\in Q} J_2(n,k) &\leq \left(\frac{1}{\#Q}\sum_{n\in Q}|(M[|b-b_Q|^m|a|\cdot\chi_{2Q}])_{(n,k)}|^r\right)^{\frac{1}{r}} \lesssim \frac{1}{\#Q^{\frac{1}{r}}}\left(\sum_{n\in Q}|(b-b_Q|^m|a|\cdot\chi_{2Q})_{(n,k)}|^r\right)^{\frac{1}{r}} \\
&= \frac{1}{\#Q^{\frac{1}{r}}}\left(\sum_{n\in Q}|b_n - b_Q|^{mr}|a_{(n,k)}|^r\right)^{\frac{1}{r}} \leq \frac{1}{\#Q^{\frac{1}{r}}}\left(\sum_{n\in Q}|b_n-b_Q|^{mrs'}\right)^{\frac{1}{rs'}}\left(\sum_{n\in Q}|a_{(n,k)}|^{rs}\right)^{\frac{1}{rs}} \\
&\lesssim \frac{1}{\#Q^{\frac{1}{r}-\frac{1}{rs'}}}\|b\|_*^m\left(\sum_{n\in Q}|a_{(n,k)}|^v\right)^{\frac{1}{v}} \leq \|b\|_*^m (M(|a|^v))_{(j,k)}^{\frac{1}{v}}.
\end{aligned}
$$

It should be noted that the constants appearing in the inequalities are independent of the second coordinate, as a consequence of Proposition 1.

We now estimate the mean value for J_3. As we have seen before,

$$
\frac{1}{2M+1} \leq \frac{1}{|n-j|},
$$

for $n,j\in\mathbb{Z}$ and $M\in\mathbb{N}_0$ such that $0 < |n-j| \leq M$. Moreover, $\frac{1}{|n-j|}$ makes sense if $n \in (2Q)^c$. Then,

$$
\begin{aligned}
J_3(j,k) &= \sup_{M\in\mathbb{N}_0}\frac{1}{(2M+1)}\sum_{n\in Q(j,M)\cap(2Q)^c}|b_n - b_Q|^m|a_{(n,k)}| \\
&\leq \sup_{M\in\mathbb{N}_0}\sum_{n\in Q(j,M)\cap(2Q)^c}\frac{1}{|n-j|}|b_n-b_Q|^m|a_{(n,k)}| \\
&\leq \sum_{n\in(2Q)^c}\frac{1}{|n-j|}|b_n-b_Q|^m|a_{(n,k)}|.
\end{aligned}
$$

As we have seen before, for $n\in(2Q)^c$ and $j,j'\in Q$,

$$
\left|\frac{1}{|n-j|} - \frac{1}{|n-j'|}\right| \lesssim \frac{|j-j'|}{|n-j'|^2}. \tag{16}
$$

Therefore, (16) and Hölder's inequality imply that for $j,j'\in Q$ we have

$$
|J_3(j,k) - J_3(j',k)| \leq \sum_{n\in(2Q)^c}\left|\frac{1}{|n-j|}-\frac{1}{|n-j'|}\right||b_n-b_Q|^m|a_{(n,k)}| \lesssim \sum_{n\in(2Q)^c}\frac{|j-j'|}{|j'-n|^2}|b_n-b_Q|^m|a_{(n,k)}|
$$

$$\leq \left(\sum_{n\in(2Q)^c} \frac{|j-j'|}{|j'-n|^2} |b_n - b_Q|^{mv'} \right)^{\frac{1}{v'}} \left(\sum_{n\in(2Q)^c} \frac{|j-j'|}{|j'-n|^2} |a_{(n,k)}|^{v} \right)^{\frac{1}{v}}.$$

We have seen that

$$\left(\sum_{n\in(2Q)^c} \frac{|j-j'|}{|j'-n|^2} |b_n - b_Q|^{mv'} \right)^{\frac{1}{v'}} \lesssim \|b\|_*^m.$$

Next, since $j, j' \in Q$, we have $|j-j'| \leq 2N_0$. If $N_0 = 0$, then j' should be j and

$$\left(\sum_{n\in(2Q)^c} \frac{|j-j'|}{|j'-n|^2} |a_{(n,k)}|^{v} \right)^{\frac{1}{v}} = 0 \leq (M(|a|^v))_j^{\frac{1}{v}}.$$

Otherwise, for $l \in \mathbb{N}$, $n \in 2^{l+1}Q \setminus 2^l Q$ implies $|j'-n| > \frac{1}{10}\#(2^{l+1}Q)$ as in (14). Therefore,

$$\begin{aligned}
\left(\sum_{n\in Q^c} \frac{|j-j'|}{|j'-n|^2} |a_{(n,k)}|^{v} \right)^{\frac{1}{v}} &= \left(\sum_{l\in\mathbb{N}} \sum_{n\in 2^{l+1}Q\setminus 2^l Q} \frac{|j-j'|}{|j'-n|^2} |a_n|^{v} \right)^{\frac{1}{v}} \\
&< 100^{\frac{1}{v}} \sum_{l\in\mathbb{N}} \left(\frac{2N}{\#(2^{l+1}Q)^2} \sum_{n\in 2^{l+1}Q\setminus 2^l Q} |a_{(n,k)}|^{v} \right)^{\frac{1}{v}} \\
&\leq 200^{\frac{1}{v}} \sum_{l\in\mathbb{N}} \left(\frac{N}{\#(2^{l+1}Q)} \frac{1}{\#(2^{l+1}Q)} \sum_{n\in 2^{l+1}Q} |a_{(n,k)}|^{v} \right)^{\frac{1}{v}} \\
&\leq 200^{\frac{1}{v}} (M(|a|^v))_{(j,k)}^{\frac{1}{v}} \sum_{l\in\mathbb{N}} \left(\frac{N_0}{2^{l+2}N_0 + 1} \right)^{\frac{1}{v}} \\
&< 200^{\frac{1}{v}} (M(|a|^v))_{(j,k)}^{\frac{1}{v}} \sum_{l\in\mathbb{N}} \left(\frac{1}{2^{l+2}} \right)^{\frac{1}{v}} \simeq (M(|a|^v))_{(j,k)}^{\frac{1}{v}}.
\end{aligned}$$

These imply that $|I_3(j,k) - I_3(j',k)| \lesssim \|b\|_*^m (M(|a|^v))_{(j,k)}^{\frac{1}{v}}$, and (15) holds. The lemma is proved. □

The following proposition concerns the Layer cake formula established in [31].

Proposition 2. *If $aw^{\frac{1}{p}} \in \ell^p(\mathbb{Z})$ and $p > 0$. Then*

$$\|aw^{\frac{1}{p}}\|_{\ell^p(\mathbb{Z})} = p^{\frac{1}{p}} \left(\int_0^\infty \lambda^{p-1} w\left(\{m : |a_m| > \lambda\}\right) d\lambda \right)^{\frac{1}{p}}.$$

An application of the layer cake formula yields the following discrete analogue of the Kolmogorov–Smirnov inequality.

Proposition 3. *If $0 < \gamma < 1$, then*

$$\frac{1}{\#Q} \sum_{j\in Q} [M(a)]_j^{\gamma} \lesssim \left(\frac{\|a\|_{\ell^1(\mathbb{Z})}}{\#Q} \right)^{\gamma}, \tag{17}$$

for any discrete interval Q and $a \in \ell^1(\mathbb{Z})$.

Proof. Let Q be any discrete interval. By the layer cake formula, we have that

$$\begin{aligned}\sum_{j\in Q}[M(a)]_j^\gamma &= \gamma\int_0^\infty \lambda^{\gamma-1}\#\{j\in Q:[M(a)]_j>\lambda\}d\lambda\\ &= \gamma\int_0^K \lambda^{\gamma-1}\#\{j\in Q:[M(a)]_j>\lambda\}d\lambda+\gamma\int_K^\infty \lambda^{\gamma-1}\#\{j\in Q:[M(a)]_j>\lambda\}d\lambda\\ &:= M_1+M_2,\end{aligned}$$

where $K=\|a\|_{\ell^1(\mathbb{Z})}/\#Q$. Since $\{j\in Q:[M(a)]_j>\lambda\}\subseteq Q$, it can be seen that

$$M_1\le\gamma\int_0^K\lambda^{\gamma-1}\#Q\,d\lambda=K^\gamma\#Q=\#Q\frac{\|a\|^\gamma_{\ell^1}(\mathbb{Z})}{\#Q^\gamma}.$$

On the other hand, by the weak type (1,1) boundedness of M, we have that

$$\lambda\#\{j\in Q:[M(a)]_j>\lambda\}\lesssim\|a\|_{\ell^1(\mathbb{Z})},\quad \lambda>0,$$

which implies that

$$M_2\le\gamma\int_K^\infty\lambda^{\gamma-1}\frac{\|a\|_{\ell^1(\mathbb{Z})}}{\lambda}\,d\lambda=\|a\|_{\ell^1(\mathbb{Z})}\frac{K^{\gamma-1}}{-\gamma+1}=\frac{\#Q}{-\gamma+1}\frac{\|a\|^\gamma_{\ell^1(\mathbb{Z})}}{\#Q^\gamma}.$$

Hence,

$$\sum_{j\in Q}[M(a)]_j^\gamma\le\#Q\frac{\|a\|^\gamma_{\ell^1}(\mathbb{Z})}{\#Q^\gamma}+\frac{\#Q}{-\gamma+1}\frac{\|a\|^\gamma_{\ell^1(\mathbb{Z})}}{\#Q^\gamma},$$

which is immediately prove (17) to conclude the proposition. □

In establishing the Fefferman-Stein inequality in the classical Morrey sequence spaces in [14], we use the fact that weight $w_k=[M(\chi_Q)]_k^\gamma$ belongs to $A_1(\mathbb{Z})$ for $0<\gamma<1$ and the discrete interval Q. We can further generalize the fact by replacing the characteristic function χ_Q with a locally summable sequence a on $\mathbb{Z}$, namely one satisfying $\|a\|_{\ell^1(Q)}<\infty$ for every $Q\subseteq\mathbb{Z}$. This generalization is formulated in the following proposition.

Proposition 4. *Let $0<\gamma<1$ and $a:\mathbb{Z}\to\mathbb{C}$ be a sequence such that $\|a\|_{\ell^1(Q)}<\infty$ for every $Q\subseteq\mathbb{Z}$. Then, the weight w defined by $w_k:=[M(a)]_k^\gamma$ belongs to $A_1(\mathbb{Z})$.*

Proof. Recall that the weight w where $w_k:=[M(a)]_k^\gamma$ belongs to $A_1(\mathbb{Z})$ provided

$$(M(w))_j\lesssim w_j,\quad j\in\mathbb{Z}. \tag{18}$$

To do so, we take $j\in\mathbb{Z}$ and any discrete interval Q containing j. We decompose $a=a\cdot\chi_{5Q}+a\cdot\chi_{(5Q)^c}$. The local part $a_{\text{local}}=a\cdot\chi_{5Q}$ relies on the Kolmogorov–Smirnov inequality (17) in Proposition 3, whereas the global part $a_{\text{global}}=a\cdot\chi_{(5Q)^c}$ is handled using the geometry of $(5Q)^c$.

We first estimate the local part. From (17) and the fact that $\#Q\sim\#(5Q)$, we have that

$$\frac{1}{\#Q}\sum_{j\in Q}[M(a_{\text{local}})]_j^\gamma\lesssim\left(\frac{\|a_{\text{loc}}\|_{\ell^1(\mathbb{Z})}}{\#Q}\right)^\gamma\lesssim\left(\frac{\|a\|_{\ell^1(5Q)}}{\#(5Q)}\right)^\gamma.$$

Since $5Q$ contains j, we obtain that

$$\frac{1}{\#Q}\sum_{j\in Q}[M(a_{\text{local}})]_j^\gamma\lesssim[M(a)]_j^\gamma,$$

and

$$(M(M(a_{\text{local}})^\gamma))_j\lesssim[M(a)]_j^\gamma. \tag{19}$$

Next, we estimate the global part. If

$$[M(a_{\text{global}})]_k \lesssim (M(a))_j, \quad k \in Q, \tag{20}$$

then

$$\frac{1}{\#Q}\sum_{k\in Q}[M(a_{\text{global}})]_k^{\gamma} \lesssim [M(a)]_j^{\gamma},$$

which implies

$$(M(M(a_{\text{global}})^{\gamma}))_j \lesssim [M(a)]_j^{\gamma}. \tag{21}$$

The estimates (19) and (21) immediately imply (18). Therefore, it only remains to establish (20).

Let $k \in Q = Q(c(Q), N(Q))$ and take $Q' = Q(c(Q'), N(Q'))$ an arbitrary discrete interval containing k. We shall consider two cases: $\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| = 0$ and $\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| > 0$.

If $\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| = 0$, then it is clear that

$$\frac{1}{\#Q'}\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| \le (M(a_{\text{global}}))_j.$$

Now, suppose that $\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| > 0$. Hence, $Q' \cap (5Q)^c \neq \emptyset$ and there exists $j_0 \in Q' \cap (5Q)^c$. Moreover, $|j_0 - c(Q)| > 5N(Q)$ and

$$|j_0 - j| \ge |j_0 - c(Q)| - |c(Q) - j| > 5N(Q) - N(Q) = 4N(Q).$$

Since $j_0, k \in Q'$ and $j, k \in Q$, we have

$$2N(Q') \ge |j_0 - k| \ge |j_0 - j| - |j - k| \ge 4N(Q) - 2N(Q) = 2N(Q),$$

and $N(Q') \ge N(Q)$.

We claim that $Q \subseteq 3Q'$. Indeed, for $j' \in Q$, we have

$$|j' - c(3Q')| = |j' - c(Q')| \le |j' - k| + |k - c(Q')| \le 2N(Q) + N(Q') \le 3N(Q').$$

Since Q contains j, the fact $Q \subseteq 3Q'$ then allows us to conclude that $3Q'$ contains j. Therefore,

$$\frac{1}{\#Q'}\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| \lesssim \frac{1}{\#(3Q')}\sum_{j'\in 3Q'}|(a_{\text{global}})_{j'}| \le (M(a_{\text{global}}))_j.$$

Form the two cases,

$$\frac{1}{\#Q'}\sum_{j'\in Q'}|(a_{\text{global}})_{j'}| \lesssim (M(a_{\text{global}}))_j \le (M(a))_j,$$

for any discrete interval Q' containing k. Then, by taking the supremum over the discrete interval Q' containing k, this implies (20). Hence, the proposition is proved. □

The next lemma establishes the Fefferman-Stein inequality for generalized Morrey sequence spaces.

Lemma 7. *Let $1 \le p < \infty$ and $\psi \in G_p^{\eta}(\mathbb{Z})$ where $1 \le p < 1/\eta < \infty$. Then,*

$$\|M(a)\|_{\mathcal{M}_{\psi}^{p}(\mathbb{Z})} \lesssim \|M^{\#}(a)\|_{\mathcal{M}_{\psi}^{p}(\mathbb{Z})}. \tag{22}$$

Proof. Let $0 < \gamma < 1$ such that $1 - \eta p - \gamma < 0$. Let $Q = Q(j, N)$ be any discrete interval. As mention earlier, the weight $w_k = [M(\chi_Q)]_k^\gamma$ belongs to $A_1(\mathbb{Z})$. Then, by [14, Lemma 2.12] and the condition (3),

$$\begin{aligned}
\sum_{k\in Q}(M(a_k))^p &= \sum_{k\in\mathbb{Z}}(M(a_k))^p\,\chi_Q(k) \lesssim \sum_{k\in\mathbb{Z}}(M(a_k))^p\,(M(\chi_Q))_k^\gamma \lesssim \sum_{k\in\mathbb{Z}}(M^{\#}(a))_k^p\,(M(\chi_Q))_k^\gamma\\
&= \sum_{k\in Q}(M^{\#}(a))_k^p(M(\chi_Q))_k^\gamma + \sum_{l\in\mathbb{N}}\sum_{k\in 2^lQ\backslash 2^{l-1}Q}(M^{\#}(a))_k^p(M(\chi_Q))_k^\gamma\\
&\lesssim \sum_{k\in Q}[(M^{\#}(a))_k]^p + \sum_{l\in\mathbb{N}}\frac{1}{2^{l\gamma}}\sum_{k\in 2^lQ}(M^{\#}(a))_k^p = \sum_{l\in\mathbb{N}_0}\frac{1}{2^{l\gamma}}\sum_{k\in 2^lQ}(M^{\#}(a))_k^p\\
&\le \|M^{\#}(a)\|^p_{\ell^p_\psi(\mathbb{Z})}\sum_{l\in\mathbb{N}_0}\frac{1}{2^{l\gamma}}\#(2^lQ)\psi(2^lN)^p\\
&\lesssim \|M^{\#}(a)\|^p_{\ell^p_\psi(\mathbb{Z})}\#Q\sum_{l\in\mathbb{N}_0}\frac{1}{2^{l\gamma-l}}(2^l)^{-\eta p}\psi(N)^p\\
&\le \|M^{\#}(a)\|^p_{\ell^p_\psi(\mathbb{Z})}\psi(N)^p\#Q\sum_{l\in\mathbb{N}}\frac{1}{2^{l\gamma}}(2^l)^{1-\eta p} \simeq \|M^{\#}(a)\|^p_{\mathcal{M}^p_\psi(\mathbb{Z})}\psi(N)^p\#Q,
\end{aligned}$$

which implies (22) to conclude the lemma. □

Lemma 8. *Let $w \in A_\infty$. If $a : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ satisfies $(M^{\#}(a))_{(\cdot,k)}w_{(\cdot)}^{1/p_0} \in \ell^{p_0}(\mathbb{Z})$ for some $0 < p_0 < \infty$ and for all $k \in \mathbb{Z}$, then for every $p \in [p_0, \infty)$, we have that*

$$\sum_{j\in\mathbb{Z}}(M(a))^p_{(j,k)}w_j \lesssim \sum_{j\in\mathbb{Z}}(M^{\#}(a))^p_{(j,k)}w_j, \quad k \in \mathbb{Z}.$$

Proof. Since we concern the operator M, we may assume that $a \ge 0$ on $\mathbb{Z}$. Take $t > 0$. By Proposition the Calderón-Zygmund decomposition in discrete setting established in [31], we have a countable family of disjoint intervals $\{Q_{t,j}\}_{j\in\Lambda}$ such that $|a_{(\cdot,k)}| \le t$ on $\mathbb{Z}\setminus\cup_j Q_{t,j}$, $t \le \frac{1}{\#Q_{(t,j)}}\sum_{Q_{(t,j)}}|a_{(\cdot,k)}| < 2t$, and $\sum_j \#Q_{(t,j)} \le \frac{1}{t}\sum|a|$. Given $t > 0$, we fix $Q_0 = Q_{t/4,j_0}$ and let $A > 0$.

We shall consider two cases. For the first case, $Q_0 \subset \{n : (M^{\#}(a))_{n,k} > t/A\}$. Thus, since the intervals are disjoint, we the have that

$$\sum_{j:Q_{t,j}\subset Q_0}\#Q_{t,j} \le \#\{n : (M^{\#}(a))_{(n,k)} > t/A\}.$$

By [14, Corollary 2.2] which is applied to previous inequality,

$$\sum_{j:Q_{t,j}\subset Q_0} w(Q_{t,j}) \le w\left(\{n : (M^{\#}(a))_{(n,k)} > t/A\}\right). \tag{23}$$

Now we move to the second case, namely $Q_0 \not\subset \{n : (M^{\#}(a))_{n,k} > t/A\}$. Then, there exists $t_0 \in Q_0$ such that $t_0 \notin \{n : (M^{\#}(a))_{(n,k)} > t/A\}$. Hence,

$$\frac{1}{\#Q_0}\sum_{j'\in Q_0}|a_{(j',k)} - a_{Q_0,k}| \le (M^{\#}(a))_{(t_0,k)} \le \frac{t}{A}.$$

Since $a_{Q_0,k} \le 2(t/4) = t/2$ from the construction, we may write

$$\begin{aligned}
\sum_{j:Q_{t,j}\subset Q_0}(t - t/2)\#Q_{t,j} &\le \sum_{j:Q_{t,j}\subset Q_0}\left[\left(\frac{1}{\#Q_{t,j}}\sum_{j'\in Q_{t,j}}|a_{(j',k)}|\right) - a_{Q_0,k}\right]\#Q_{t,j}\\
&\le \sum_{j:Q_{t,j}\subset Q_0}\sum_{j'\in Q_{t,j}}|a_{(j',k)} - a_{Q_0,k}| \le \sum_{j'\in Q_0}|a_{(j',k)} - a_{Q_0,k}| \le A^{-1}t\#Q_0.
\end{aligned}$$

We apply [14, Corollary 2.2] again to the previous inequalities to obtain that

$$\sum_{j:Q_{t,j}\subset Q_0} w(Q_{t,j}) \le 2A^{-1}w(Q_0). \tag{24}$$

By adding up in all possible Q_0's we may obtain from (23) and (24) that

$$\sum_j w(Q_{t,j}) \le w\left(\{n\in\mathbb{Z} : (M^{\#}(a))_{(n,k)} > t/A\}\right) + 2A^{-1}\sum_{j''} w(Q_{t/4,j''}).$$

Next, we let $\alpha(t) = \sum_j w(Q_{t,j})$ and $\beta(t,k) = w\left(\{j' : (Ma)_{(j',k)} > t\}\right)$. By [31], we have that

$$\#\{j' : (Ma)_{(j',k)} > t\} \le \sum_j \#3Q_{t/4,j} \lesssim \sum_j \#Q_{t/C,j}.$$

Hence, by [14, Corollary 2.2] and the previous inequalities, we have $\beta(t,k) \lesssim \sum_j w(3Q_{t/4,j}) \lesssim \alpha(t/C)$. For $\alpha(t)$, we obtain

$$\alpha(t) \le w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) + 2A^{-1}\alpha(t/4).$$

By Proposition 2,

$$\sum_{j'} (Ma)^p_{(j',k)} w_{j'} = \int_0^\infty pt^{p-1}\beta(t,k)dt.$$

We first consider

$$J_N = \int_0^N pt^{p-1}\alpha(t)dt.$$

Since $(Ma)_{(\cdot,k)} w_{(\cdot)}^{\frac{1}{p_0}} \in \ell^{p_0}(\mathbb{Z})$, we may see that

$$J_N \le \int_0^N pt^{p-1}\beta(t,k)dt \lesssim \int_0^N p_0 t^{p_0-1}\beta(t,k)dt \le \sum_{j'\in\mathbb{Z}} (Ma)^{p_0}_{(j',k)} w_{j'} < \infty.$$

On the other hand,

$$\begin{aligned} J_N &\le \int_0^N pt^{p-1} w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) dt + 2A^{-1}\int_0^N pt^{p-1}\alpha(t/4)dt \\ &= \int_0^N pt^{p-1} w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) dt + CA^{-1}\int_0^{N/4} pt^{p-1}\alpha(t)dt. \end{aligned}$$

Hence,

$$J_N \le \int_0^N pt^{p-1} w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) dt + CA^{-1}J_N.$$

We take $A = 2C$ and obtain that

$$J_N \le 2\int_0^N pt^{p-1} w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) dt.$$

Therefore,

$$\int_0^\infty pt^{p-1}\alpha(t)dt = \lim I_N \le 2\int_0^\infty pt^{p-1} w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right) dt.$$

By taking this into account, Lemma 2 allows us obtaining that

$$\begin{aligned}\sum_{j'\in\mathbb{Z}}(Ma)^p_{(j',k)}w_{j'} &= \int_0^\infty pt^{p-1}\beta(t,k)dt \lesssim \int_0^\infty pt^{p-1}\alpha(t/C_2)dt \simeq \int_0^\infty pt^{p-1}\alpha(t)dt\\ &\lesssim \int_0^\infty pt^{p-1}w\left(\{j'\in\mathbb{Z} : (M^{\#}a)_{(j',k)} > t/A\}\right)dt \simeq \sum_{j'\in\mathbb{Z}}(M^{\#}(a))^p_{(j',k)}w_{j'},\end{aligned}$$

as desired. □

Under the same assumptions in Lemma 7, by using Lemma 8 we can see that for each fixed $k\in\mathbb{Z}$, the lemma yields

$$\|(M(a))_{(\cdot,k)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} \lesssim \|(M^{\#}(a))_{(\cdot,k)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})},$$

where the estimate constant is independent of k. This observation allows us to have the following lemma regarding the Fefferman-Stein inequality for generalized mixed Morrey double-sequence spaces.

Lemma 9. *Let $1\le p<\infty$ and $\psi\in G^\eta_p(\mathbb{Z})$ where $1\le p<1/\eta<\infty$. Then,*

$$\|(M(a))_{(\cdot,k)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})} \lesssim \|(M^{\#}(a))_{(\cdot,k)}\|_{\mathcal{M}^p_\psi(\mathbb{Z})}, \quad k\in\mathbb{Z}, \tag{25}$$

for any $a:\mathbb{Z}\to\mathbb{C}$ where $(M^{\#}(a))_{(\cdot,k)}\in\mathcal{M}^p_\psi(\mathbb{Z})$ for $k\in\mathbb{Z}$.

Now, we are ready to prove Theorem 3 and 4.

Proof of Theorem 3. First suppose that M^m_b is bounded on $\mathcal{M}^p_\psi(\mathbb{Z})$. Let $j\in\mathbb{Z}$ and $N\in\mathbb{N}_0$. We write

$$\mathcal{N}(j,N) = \frac{1}{\#Q(j,N)}\sum_{k\in Q(j,N)}|b_k - b_{Q(j,N)}|.$$

We shall show that $\sup_{j\in\mathbb{Z},N\in\mathbb{N}_0}\mathcal{N}(j,N)\lesssim 1$. We can see by Hölder's inequality and the definition of $b_{Q(j,N)}$ that

$$\begin{aligned}\mathcal{N}(j,N) &\le \left(\frac{1}{\#Q(j,N)}\sum_{k\in Q(j,N)}|b_k - b_{Q(j,N)}|^m\right)^{1/m}\\ &= \left(\frac{1}{\#Q(j,N)}\sum_{k\in Q(j,N)}\left|b_k - \frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}b_n\right|^m\right)^{1/m}\\ &= \left(\frac{1}{\#Q(j,N)}\sum_{k\in Q(j,N)}\left|\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}(b_k-b_n)\right|^m\right)^{1/m}.\end{aligned}$$

Since

$$\left|\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}(b_k-b_n)\right| \le \left(\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_k-b_n|^m\right)^{1/m},$$

by Hölder's inequality, we then have

$$\mathcal{N}(j,N) \le \left(\frac{1}{\#Q(j,N)}\sum_{k\in Q(j,N)}\frac{1}{\#Q(j,N)}\sum_{n\in Q(j,N)}|b_k-b_n|^m\right)^{1/m}.$$

Now, it is clear that $\#Q(j,N) \sim \#Q(k,2N)$. Moreover, for $k \in Q(j,N)$ and $n \in Q(j,N)$, we have $|n-k| \leq |n-j| + |j-k| \leq 2N$. These imply

$$\sum_{n\in Q(j,N)} |b_k - b_n|^m \leq \sum_{n\in Q(k,2N)\cap Q(j,N)} |b_k - b_n|^m = \sum_{n\in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n),$$

for $k \in Q(j,N)$. Hence, the definition of M_b^m and its boundedness on $\mathcal{M}_\psi^p(\mathbb{Z})$ as well as Lemma 1 yield that

$$\begin{aligned}
\mathcal{N}(j,N) &\lesssim \left(\frac{1}{\#Q(j,N)} \sum_{k\in Q(j,N)} \frac{1}{\#Q(k,2N)} \sum_{n\in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n)\right)^{1/m} \\
&\leq \left(\frac{1}{\#Q(j,N)} \sum_{k\in Q(j,N)} (M_b^m(\chi_{Q(j,N)}))_k\right)^{1/m} \\
&\lesssim \frac{1}{\#Q(j,N)^{1/m}} \|M_b^m(\chi_{Q(j,N)})\|_{\ell^p(Q(j,N))}^{1/m} \|1\|_{\ell^{p'}(Q(j,N))}^{1/m} \\
&\leq \frac{1}{\#Q(j,N)^{1/m}} \psi(N)^{1/m} \#Q(j,N)^{1/pm} \|M_b^m(\chi_{Q(j,N)})\|_{\mathcal{M}_\psi^p(\mathbb{Z})}^{1/m} \#Q(j,N)^{1/p'm} \\
&= \psi(N)^{1/m} \|M_b^m(\chi_{Q(j,N)})\|_{\mathcal{M}_\psi^p(\mathbb{Z})}^{1/m} \\
&\lesssim \psi(N)^{1/m} \|\chi_{Q(j,N)}\|_{\mathcal{M}_\psi^p(\mathbb{Z})}^{1/m} \\
&\sim \psi(N)^{1/m} \psi(N)^{-1/m} = 1,
\end{aligned}$$

which proves that $b \in BMO$. Conversely, we assume that $b \in BMO$. We first need to ensure that $\psi \in G_p^\eta(\mathbb{Z})$ implies $\psi^u \in G_{p/u}^{\eta u}(\mathbb{Z})$ for any $1 < u < p$. Indeed, by taking the power of u to all side of (3) and use the fact that the map $t \mapsto t^u$ is increasing on $[0,\infty)$, we have

$$\left(\frac{2N+1}{2M+1}\right)^{-\frac{u}{p}} \lesssim \frac{\psi(N)^u}{\psi(M)^u} \lesssim \left(\frac{2N+1}{2M+1}\right)^{-\eta u}, \quad N, M \in \mathbb{N}, N \geq M,$$

implying $\psi^u \in G_{p/u}^{\eta u}(\mathbb{Z})$ where $1 < p/u < 1/(\eta u) < \infty$.

Therefore, by Theorem 1, M is bounded on $\mathcal{M}_\psi^p(\mathbb{Z})$ and $\mathcal{M}_{p/u}^{\psi^u}(\mathbb{Z})$. Hence, by the fact that $|a_k| \leq (M(|a|))_k$ and Lemma 5, we have that

$$\begin{aligned}
\|M_b^m(a)\|_{\mathcal{M}_\psi^p(\mathbb{Z})} &\leq \|M(M_b^m(a))\|_{\mathcal{M}_\psi^p(\mathbb{Z})} \lesssim \|M^\#(M_b^m(a))\|_{\mathcal{M}_\psi^p(\mathbb{Z})} \\
&\lesssim \|b\|_*^m \left(\|M(M(|a|)^u)^{1/u}\|_{\mathcal{M}_\psi^p(\mathbb{Z})} + \|M(|a|^u)^{1/u}\|_{\mathcal{M}_\psi^p(\mathbb{Z})}\right) \\
&= \|b\|_*^m \left(\|M(M(|a|)^u)\|_{\mathcal{M}_{\psi^u}^{p/u}(\mathbb{Z})}^{1/u} + \|M(|a|^u)\|_{\mathcal{M}_{\psi^u}^{p/u}(\mathbb{Z})}^{1/u}\right) \\
&\lesssim \|b\|_*^m \left(\|M(|a|)^u\|_{\mathcal{M}_{\psi^u}^{p/u}(\mathbb{Z})}^{1/u} + \||a|^u\|_{\mathcal{M}_{\psi^u}^{p/u}(\mathbb{Z})}^{1/u}\right) \\
&= \|b\|_*^m \left(\|M(|a|)\|_{\mathcal{M}_\psi^p(\mathbb{Z})} + \|a\|_{\mathcal{M}_\psi^p(\mathbb{Z})}\right) \\
&\lesssim \|b\|_*^m \|a\|_{\mathcal{M}_\psi^p(\mathbb{Z})}.
\end{aligned}$$

Hence, we conclude the theorem. $\square$

Proof of Theorem 4. We first prove the statement 2. Suppose that M_b^m is bounded on $(\mathcal{M}_{\psi_2}^{p_2})(\mathcal{M}_{\psi_1}^{p_1})(\mathbb{Z}\times\mathbb{Z})$. Let $j \in \mathbb{Z}$ and $M, N \in \mathbb{N}_0$. Recall $\mathcal{N}(j,N)$ defined earlier. Our aim is to show $\sup_{j\in\mathbb{Z}, N\in\mathbb{N}_0} \mathcal{N}(j,N) \lesssim 1$.

For $j \in \mathbb{Z}$ and $M, N \in \mathbb{N}_0$, we write $A(n,l) = \chi_{Q(j,N)}(n)\,\chi_{Q(j,M)}(l)$ for $(n,l) \in \mathbb{Z} \times \mathbb{Z}$. We have seen that

$$\mathcal{N}(j,N) \leq \left(\frac{1}{\#Q(j,N)} \sum_{k \in Q(j,N)} \frac{1}{\#Q(k,2N)} \sum_{n \in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n) \right)^{1/m}. \tag{26}$$

Moreover, since

$$\begin{aligned}
\sum_{n \in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n) &= \frac{1}{\#Q(j,M)} \sum_{l \in Q(j,M)} \left[\sum_{n \in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n) \right] \chi_{Q(j,M)}(l) \\
&= \frac{1}{\#Q(j,M)} \sum_{l \in Q(j,M)} \sum_{n \in Q(k,2N)} |b_k - b_n|^m \chi_{Q(j,N)}(n) \chi_{Q(j,M)}(l) \\
&= \frac{1}{\#Q(j,M)} \sum_{l \in Q(j,M)} \sum_{n \in Q(k,2N)} |b_k - b_n|^m A(n,l),
\end{aligned}$$

then from (26) and the definition of M_b^m in the mixed setting, we have that

$$\begin{aligned}
\mathcal{N}(j,N) &\leq \left(\frac{1}{\#Q(j,N)} \sum_{k \in Q(j,N)} \frac{1}{\#Q(k,2N)} \frac{1}{\#Q(j,M)} \sum_{l \in Q(j,M)} \sum_{n \in Q(k,2N)} |b_k - b_n|^m A(n,l) \right)^{1/m} \\
&\leq \left(\frac{1}{\#Q(j,N)\#Q(j,M)} \sum_{l \in Q(j,M)} \sum_{k \in Q(j,N)} (M_b^m(A))_{(k,l)} \right)^{1/m} \\
&= \left(\frac{1}{\#Q(j,N)^m \#Q(j,M)} \|(M_b^m(A))_{(k,l)}\|_{(\ell^1,l)(\ell^1,k)(Q(j,N)\times Q(j,M))} \right)^{1/m}.
\end{aligned}$$

By applying the Hölder's inequality in mixed setting to the last estimate, the boundedness of M_b^m, and Lemma 2, we obtain

$$\begin{aligned}
\mathcal{N}(j,N) &= \left(\frac{1}{\#Q(j,N)\#Q(j,M)} \|(M_b^m(A))_{(k,l)}\|_{(\ell^1,l)(\ell^1,k)(Q(j,N)\times Q(j,M))} \right)^{1/m} \\
&\leq \frac{1}{\#Q(j,N)^{1/m}} \frac{1}{\#Q(j,M)^{1/m}} \|(M_b^m(A))_{(k,l)}\|^{1/m}_{(\ell^{p_2})(\ell^{p_1})(Q(j,N)\times Q(j,M))} \|1\|^{1/m}_{(\ell^{p_2'})(\ell^{p_1'})(Q(j,N)\times Q(j,M))} \\
&= \frac{1}{\#Q(j,N)^{\frac{1}{mp_1}}} \frac{1}{\#Q(j,M)^{\frac{1}{mp_2}}} \|(M_b^m(A))_{(k,l)}\|^{1/m}_{(\ell^{p_2})(\ell^{p_1})(Q(j,N)\times Q(j,M))} \\
&\leq \frac{1}{\#Q(j,N)^{\frac{1}{mp_1}}} \frac{1}{\#Q(j,M)^{\frac{1}{mp_2}}} \|(M_b^m(A))_{(k,l)}\|^{1/m}_{(\ell^{p_2})(\ell^{p_1})(Q(j,N)\times Q(j,M))} \\
&\leq \psi_1(N)^{1/m} \psi_2(M)^{1/m} \|(M_b^m(A))_{(k,l)}\|^{1/m}_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \\
&\lesssim \psi_1(N)^{1/m} \psi_2(M)^{1/m} \|(A)_{(k,l)}\|^{1/m}_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \\
&\leq \psi_1(N)^{1/m} \psi_2(M)^{1/m} \|\chi_{Q(j,N)}(k) \cdot \chi_{Q(j,M)}(l)\|^{1/m}_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} \sim 1,
\end{aligned}$$

which proves that $b \in BMO$.

Next, we prove the statement 1. To do so, assume that $b \in BMO$. As we have seen before, $\psi_1 \in G^{\eta}_{p_1}(\mathbb{Z})$ implies $\psi_1^u \in G^{\eta u}_{p_1/u}(\mathbb{Z})$ where $1 < p_1/u < 1/(\eta u) < \infty$. From Lemma 9, for $k \in \mathbb{Z}$,

$$\|(M_b^m(a))_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \leq \|(M(M_b^m(a)))_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \lesssim \|(M^{\#}(M_b^m(a)))_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j},$$

Now using Lemma 6 and (10), from the last inequality we obtain that

$$\begin{aligned}
&\left\| \|(M_b^m(a))_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} \\
&\lesssim \|b\|_*^m \left\| \left(\|(M(M(|a|)^u))^{1/u}_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} + \left\| \|(M(|a|^u))^{1/u}_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} \right) \\
&= \|b\|_*^m \left(\left\| \|(M(M(|a|))^u_{(j,k)})\|^{1/u}_{\mathcal{M}^{p_1/u}_{\psi^u}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} + \left\| \|M(|a|^u)\|^{1/u}_{\mathcal{M}^{p_1/u}_{\psi_1{}^u}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} \right) \\
&\lesssim \|b\|_*^m \left(\left\| \|(M(|a|))^u_{(j,k)}\|^{1/u}_{\mathcal{M}^{p_1/u}_{\psi_1^u}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} + \left\| \||a_{(j,k)}|^u\|^{1/u}_{\mathcal{M}^{p_1/u}_{\psi_1{}^u}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} \right) \\
&= \|b\|_*^m \left(\left\| \|(M(|a|))_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} + \left\| \|a_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)} \right) \\
&\lesssim \|b\|_*^m \left\| \|a_{(j,k)}\|_{\mathcal{M}^{p_1}_{\psi}(\mathbb{Z}),j} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z},k)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|M_b^m(a)\|_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})} = \left\| \|(M_b^m(a))_{(\cdot,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z})} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),k} &\lesssim \|b\|_*^m \left\| \|a_{(\cdot,k)}\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{Z})} \right\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{Z}),k} \\
&= \|b\|_*^m \|a\|_{(\mathcal{M}^{p_2}_{\psi_2})(\mathcal{M}^{p_1}_{\psi_1})(\mathbb{Z}\times\mathbb{Z})}.
\end{aligned}$$

These complete the proof of the theorem. □

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: No data is required for this research.

Funding Information: No funding is available for this research.

Acknowledgments: The author is grateful to the reviewers for their valuable comments and suggestions, which have helped improve the manuscript.

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