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Completion principles, real commutants, and positive-frequency geometry for orthogonal Hilbert representations

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Abstract: This paper develops a mathematics-first modular framework for completion quotients, Real-commutant parameter spaces, positive-frequency polar selection, temporal spectral triples, and conditional cone readouts for orthogonal Hilbert representations. The opening part states the ontological and operational principles used to organize the construction: generative histories are separated from completed records, observable dynamics is obtained by quotienting through completed-future tests, phase is selected only after a real orthogonal carrier has positive time orientation, and Lorentzian readout is treated as a downstream cone geometry obtained from length-delay inequalities. The theorem-bearing core concerns orthogonal Hilbert representations. For a real representation $U : G \to O(\mathcal{H}_{\mathbb{R}})$, with complexification V on $\mathcal{H}_{\mathbb{C}}$ and canonical conjugation $\mathcal{C}$, the Real commutant

$$\mathcal{A}_{\mathbb{R}}(U) = \{T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) : TV(g) = V(g)T \;\forall g \in G,\; T\mathcal{C} = \mathcal{C}T\},$$

is a real C^*-algebra canonically isomorphic to the real commutant of U. Symmetry-compatible orthogonal complex structures are precisely the skew-adjoint square roots of $-\operatorname{Id}$ in this algebra, naturally under real unitary intertwiners. Abelian representations yield partial spectral phase operators, orthogonal flows yield the positive-frequency polar factorization $A = J_A|A| = |A|J_A$ on the common domain $\operatorname{Dom}(A) = \operatorname{Dom}(|A|)$ after removing the kernel, and compact groups yield a Peter–Weyl/Frobenius–Schur bounded-product formula in complex, real, and quaternionic multiplicity algebras. Downstream of this phase layer, the phase-balanced real tensor product is identified with the real form of the usual complex Hilbert tensor product, and a temporal shift spectral triple is built from a concrete dense $*$-algebra with compact resolvent, bounded generator commutators, exact diagonal Connes distance $|m-n|/\kappa$ on the diagonal record algebra, and spectral dimension governed by an explicit counting-function hypothesis. The final module proves a conditional cone lemma: once a length budget, completion delay, and Euclidean readout are supplied, completed displacements lie in the finite cone $\|\Delta x\| \leq c_*\Theta$; the standard automorphism statement gives Lorentz similitudes, and a smooth Lorentzian representative requires additional smooth geometric input. The broader physical interpretation motivates the axioms but is not used as a hidden premise.

Keywords: real Hilbert space, orthogonal representation, real commutant, complex structure, positive frequency, spectral triple, Connes distance, completion quotient, Lorentzian cone

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1. Introduction

Many operator-theoretic models begin with several layers already fixed: a phase convention, a Hilbert completion, a class of observable records, and sometimes a causal cone or Lorentzian metric. The aim of this paper is to separate the dependencies between these layers in a self-contained mathematical setting. We first state a small set of ontological and operational principles, then develop the operator-theoretic core: the classification and selection of compatible complex structures inside real orthogonal Hilbert representations.

An orthogonal complex structure on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ is a bounded real-linear operator J satisfying

$$J^2 = -\operatorname{Id}, \qquad J^* = -J.$$

Such an operator turns $\mathcal{H}_{\mathbb{R}}$ into a complex Hilbert space by declaring $(a+ib)x = ax + bJx$. If $\mathcal{H}_{\mathbb{R}}$ carries an orthogonal representation $U : G \to O(\mathcal{H}_{\mathbb{R}})$, the natural mathematical questions are:

(Q1) Which complex structures are compatible with the symmetry?
(Q2) Which additional data select one of them?
(Q3) How does such a phase-selection result fit into a broader completion/projection picture in which only completed records are observable?

The answer to the first question is the Real commutant. Let $\mathcal{H}_{\mathbb{C}}$ be the complexification of $\mathcal{H}_{\mathbb{R}}$, let V be the complexified unitary representation, and let $\mathcal{C}$ denote canonical conjugation. Then

$$\mathcal{A}_{\mathbb{R}}(U) = \{T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) : TV(g) = V(g)T \ \forall g \in G, T\mathcal{C} = \mathcal{C}T\},$$

is a real C^*-algebra canonically equivalent to the real commutant of U. The standard real/complex identification of operators commuting with conjugation is used here as a functorial organizing device: the symmetry-compatible orthogonal complex structures are exactly the skew-adjoint square roots of $-\operatorname{Id}$ in $\mathcal{A}_{\mathbb{R}}(U)$. The answer to the second question is positive frequency: if $U(t) = e^{tA}$ is a strongly continuous orthogonal flow, then on $(\ker A)^{\perp}$ the polar decomposition of A gives, as an equality of closed operators on $\operatorname{Dom}(A) = \operatorname{Dom}(|A|)$,

$$A = J_A|A| = |A|J_A, \qquad |A| \geq 0,$$

with $J_A^2 = -\operatorname{Id}$. The selected J_A is the polar phase associated with the oriented flow; zero modes carry only a partial phase unless additional structure is supplied.

The third question motivates the first part of the paper. In the broader Single Monad Model and Duality of Time Theory program, one distinguishes hidden or generative order from completed observable records. That lineage runs through the author's earlier work on Ibn Arabi's cosmology, the Single Monad Model, and the Duality of Time Theory [1–4]; a more recent mathematical development formulates dual-time topological geometry and temporal asymmetry [5]. The present article is self-contained: those references provide historical and conceptual motivation only. The proofs below depend solely on the Hilbert-space, representation-theoretic, quotient-theoretic, and elementary cone-geometric hypotheses explicitly stated here.

The main mathematical contributions are:

(R1) a formal completion/projection layer based on history monoids, completed records, observable descent, and behavioral quotients;
(R2) the intrinsic Real-commutant classification of compatible orthogonal complex structures;
(R3) spectral-orientation formulas for abelian representations and positive-frequency polar selection for flows;
(R4) the compact-group classification through Peter–Weyl theory and Frobenius–Schur type;
(R5) functorial sign classes on non-self-conjugate compact pair sectors;
(R6) the standard phase-balanced tensor-product compatibility for selected real Hilbert spaces;
(R7) a temporal spectral-triple construction from a concrete dense $*$-algebra, with compact resolvent, bounded generator commutators, exact diagonal Connes distance, and computable spectral dimension under an explicit counting hypothesis;
(R8) a conditional downstream cone theorem showing that supplied length-delay data force the finite cone $\|\Delta x\| \leq c_* \Theta$, together with the standard Lorentz-similitude automorphism statement for that cone.

The individual ingredients are classical: Stone's theorem, spectral measures, Peter–Weyl theory, Frobenius–Schur type, polar decomposition, balanced tensor products, spectral triples of shift type, and elementary cone geometry. No novelty is claimed for these standard ingredients in isolation. The contribution is the modular assembly and the precise bookkeeping of dependencies: completion determines what is observable, positive-frequency dynamics selects the phase operator, and cone geometry is treated as a downstream readout layer rather than an assumed spacetime primitive. Further applications, when desired, require their own additional hypotheses and are not part of the theorem-bearing claims here.

2. Ontological and operational principles

This section states the organizing principles before the formal theorems. They are not substitutes for proofs; they identify which structures are primitive in the framework and which are reconstructed under hypotheses.

Principle 1 (Generative histories before completed records)**.** A generative history is a composable process fragment. It may be acted on, concatenated, or varied before observation. A completed record is a stabilized output after completion, projection, coarse graining, measurement, or any other observer-level closure operation. The distinction is a distinction of roles, not a claim that there are two independently measured clocks.

Principle 2 (Observable dynamics is quotient dynamics)**.** If two generative states have the same completed future behavior under a specified family of tests, then they represent the same observable state for that test family. Observable dynamics is therefore induced on a quotient of the generative state space, not necessarily on the full generative space.

Principle 3 (Positive-frequency phase after real dynamics)**.** An admissible phase operator is not introduced at the generative level. A real orthogonal carrier may acquire a compatible complex structure only when its symmetry or time-flow data select a real-linear operator J with $J^2 = -\,\mathrm{Id}$. For a nonzero-frequency orthogonal flow, positive time orientation selects such a J by $A = J|A| = |A|J$.

Principle 4 (Records and cones are downstream)**.** A causal cone is assigned to completed observable events, not to arbitrary microscopic fluctuations. If completed transitions have both a spatial readout and a completion delay, then a length-delay inequality produces a finite cone. A smooth Lorentzian metric is an additional representative of that cone under manifoldlike stabilization assumptions.

Definition 1 (Claim-status hierarchy)**.** A statement has theorem status only when it follows from explicit definitions and hypotheses. It has principle status when it fixes an explanatory direction or input class. It has reconstruction status when a familiar structure is recovered from stated data. It has translation status when a standard construction is placed into the present architecture without claiming novelty for the construction itself.

Remark 1. This discipline is important. If the completion map, test family, positive time orientation, treatment of zero modes, symplectic data, or delay law is changed, the reconstructed structure changes. The framework is therefore modular: weakening an input weakens the conclusion rather than producing an unconditional theory.

The remainder of the paper proceeds in the following order:

$$\begin{aligned}&\text{histories} \longrightarrow \text{completion quotients} \longrightarrow \text{real representations}\\ &\longrightarrow \mathcal{A}_{\mathbb{R}}(U) \longrightarrow J \longrightarrow \text{cone readout.}\end{aligned}$$

The Real-commutant results form the theorem-bearing core. The phase-balanced tensor product and temporal spectral triple show how the selected phase enters composite Hilbert structures and spectral geometry, while the quotient and cone sections explain how that core sits inside a broader completion/projection ontology. The arrows in the display are an architectural order, not a theorem that each layer canonically produces the next. In particular, the quotient formalism does not by itself construct the real orthogonal representation, and a chosen complex structure does not by itself produce a length-delay cone. Those links require the additional Hilbertization, flow, delay, and readout data isolated in §11.

3. Completion, observable descent, and behavioral quotients

Definition 2 (History monoid and action)**.** A history monoid is a monoid $(\mathsf{Hist}, \cdot, e)$. A left action of Hist on a set X is a map $(h, x) \mapsto h \cdot x$ such that $e \cdot x = x$ and $(hk) \cdot x = h \cdot (k \cdot x)$.

Definition 3 (Completion-observation data)**.** Completion-observation data consist of a generative state space X, a completed-record space Comp, a completion map $Q : X \to \mathsf{Comp}$, and possibly an observation map $\pi : \mathsf{Comp} \to \mathsf{Obs}$. The completed observable map is $\rho = \pi Q : X \to \mathsf{Obs}$.

Definition 4 (Observable descent)**.** A history $h \in \mathsf{Hist}$ descends through ρ if

$$\rho(x) = \rho(y) \quad \Longrightarrow \quad \rho(h \cdot x) = \rho(h \cdot y).$$

The action descends if every $h \in \mathsf{Hist}$ descends.

Theorem 1 (Observable semigroup descent)**.** *Assume $\rho : X \to \mathsf{Obs}$ is surjective and the history action descends through ρ. Then for each $h \in \mathsf{Hist}$ there is a unique map $\overline{h} : \mathsf{Obs} \to \mathsf{Obs}$ satisfying*

$$\overline{h}(\rho(x)) = \rho(h \cdot x).$$

The assignment $h \mapsto \overline{h}$ is a monoid action on Obs. In particular, reversible generative dynamics may descend to irreversible observable dynamics when the quotient identifies generative distinctions.

Proof. Well-definedness is exactly the descent condition. If $\rho(x) = o$, define $\overline{h}(o) = \rho(h \cdot x)$. The identity and multiplication laws follow from the monoid action:

$$\overline{h}(\overline{k}(\rho(x))) = \overline{h}(\rho(k \cdot x)) = \rho(h \cdot (k \cdot x)) = \rho((hk) \cdot x) = \overline{hk}(\rho(x)).$$

If ρ identifies different states, injectivity need not survive descent. □

Definition 5 (Completed-future tests)**.** Let $\mathcal{O}$ be a family of tests $o : \mathsf{Comp} \to Y_o$, and let $\mathsf{Hist}_c \subseteq \mathsf{Hist}$ be a class of completed future histories. Define

$$x \equiv_{\mathcal{O}} y \quad \Longleftrightarrow \quad o(Q(h \cdot x)) = o(Q(h \cdot y)),$$

for every $h \in \mathsf{Hist}_c$ and every $o \in \mathcal{O}$.

Theorem 2 (Behavioral quotient)**.** *The relation $\equiv_{\mathcal{O}}$ is an equivalence relation. Moreover, it is the coarsest quotient relation preserving all specified completed-future tests: if an equivalence relation R has the property that xRy implies equality of all tested completed futures, then $R \subseteq \equiv_{\mathcal{O}}$.*

Proof. For each pair (h, o), the map $x \mapsto o(Q(h \cdot x))$ has a kernel equivalence relation. The relation $\equiv_{\mathcal{O}}$ is the intersection of these kernels, hence is an equivalence relation. The universal property follows immediately from the definition. □

Remark 2. The quotient is selected only after Q, Hist_c, and $\mathcal{O}$ are supplied. The theorem does not remove the need to justify these inputs; it makes their consequences precise. Theorems 1 and 2 are therefore formal bookkeeping results rather than independent sources of Hilbert-space or cone geometry.

Example 1 (One-sided completed records)**.** Let $X = \{0,1\}^{\mathbb{Z}}$, let $\mathsf{Hist} = \mathbb{Z}$ act by the two-sided shift, and let $Q(x) = (x_0, x_1, x_2, \ldots)$ be the completed future tail. If $\mathcal{O}$ consists of finite cylinder tests on $\{0,1\}^{\mathbb{N}_0}$ and $\mathsf{Hist}_c = \mathbb{N}_0$, then $x \equiv_{\mathcal{O}} y$ means that all tested completed future tails agree. The invertible two-sided shift may therefore descend to the one-sided completed shift, where inverse information has been quotiented out. A real Hilbert representation is not automatic from this set-theoretic quotient; it appears only after extra model data, such as an invariant measure or a Hilbert completion of record observables, are supplied.

4. The real commutant

Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. Its complexification is $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, written $x + iy$ with $x, y \in \mathcal{H}_{\mathbb{R}}$. Canonical conjugation is the antiunitary involution

$$\mathcal{C}(x + iy) = x - iy,$$

and $\mathcal{H}_{\mathbb{R}} = \mathrm{Fix}(\mathcal{C})$. If $U : G \to O(\mathcal{H}_{\mathbb{R}})$ is an orthogonal representation, its complexification is

$$V(g)(x + iy) = U(g)x + iU(g)y.$$

Definition 6 (Real commutant)**.** The complex commutant of V is

$$\mathcal{A}(U) = \{T \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) : TV(g) = V(g)T \ \forall g \in G\}.$$

The Real commutant is the real C^*-subalgebra

$$\mathcal{A}_{\mathbb{R}}(U) = \{T \in \mathcal{A}(U) : T\mathcal{C} = \mathcal{C}T\}.$$

The real commutant of U is

$$\mathrm{Comm}_{\mathbb{R}}(U) = \{S \in \mathcal{B}(\mathcal{H}_{\mathbb{R}}) : SU(g) = U(g)S \ \forall g \in G\}.$$

Definition 7 (Compatible complex structures)**.** The space of U-compatible orthogonal complex structures is

$$\mathcal{J}(U) = \{J \in \mathrm{Comm}_{\mathbb{R}}(U) : J^* = -J, \ J^2 = -\mathrm{Id}\}.$$

For a real bounded operator S on $\mathcal{H}_{\mathbb{R}}$, the notation $S_{\mathbb{C}}$ always denotes its complex-linear extension to $\mathcal{H}_{\mathbb{C}}$:

$$S_{\mathbb{C}}(x + iy) = Sx + iSy.$$

It does not mean composition with the conjugation $\mathcal{C}$. The following identification is the standard equivalence between real operators and complex-linear operators commuting with the canonical conjugation; the point here is to use it as a natural parameterization of all compatible phase operators for a real representation [6–8].

Theorem 3 (Intrinsic Real-commutant theorem)**.** *Complexification gives an isometric real $*$-isomorphism*

$$\mathrm{Comm}_{\mathbb{R}}(U) \longrightarrow \mathcal{A}_{\mathbb{R}}(U), \qquad S \longmapsto S_{\mathbb{C}}.$$

Consequently $\mathcal{J}(U)$ is canonically the set of skew-adjoint square roots of $-\mathrm{Id}$ in $\mathcal{A}_{\mathbb{R}}(U)$.

Proof. If $S \in \mathrm{Comm}_{\mathbb{R}}(U)$, then $S_{\mathbb{C}}$ is complex-linear, commutes with $V(g)$, and commutes with canonical conjugation. Hence $S_{\mathbb{C}} \in \mathcal{A}_{\mathbb{R}}(U)$. Complexification preserves sums, products, adjoints, and norms.

Conversely, if $T \in \mathcal{A}_{\mathbb{R}}(U)$, then $T\mathcal{C} = \mathcal{C}T$, so T preserves $\mathrm{Fix}(\mathcal{C}) = \mathcal{H}_{\mathbb{R}}$. Its restriction $S = T|_{\mathcal{H}_{\mathbb{R}}}$ is bounded, real-linear, and commutes with $U(g)$. Since T is complex-linear, $T = S_{\mathbb{C}}$. The two constructions are inverse. The last statement transports the equations defining an orthogonal complex structure through this isomorphism. $\square$

Corollary 1 (Functoriality)**.** *Let $W : \mathcal{H}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$ be a real unitary intertwiner between orthogonal representations U and U'. Then*

$$\mathrm{Ad}(W_{\mathbb{C}}) : \mathcal{A}_{\mathbb{R}}(U) \to \mathcal{A}_{\mathbb{R}}(U'),$$

is an isometric real $*$-isomorphism and maps $\mathcal{J}(U)$ bijectively onto $\mathcal{J}(U')$.

Proof. The complexification $W_{\mathbb{C}}$ is a complex unitary intertwiner and commutes with canonical conjugations. Conjugation by it preserves the commutant condition, the Real condition, and the equations defining a complex structure. $\square$

Remark 3 (Parameterization property)**.** The term "universal" is used only in the following modest, functorial sense. For every real orthogonal representation U, Theorem 3 gives a natural bijection

$$\mathcal{J}(U) \longleftrightarrow \{J \in \mathcal{A}_{\mathbb{R}}(U) : J^* = -J,\ J^2 = -\mathrm{Id}\},$$

and Corollary 1 transports this bijection under real unitary intertwiners. Abelian, flow, compact, and direct-integral classifications are therefore different computations of the same Real-commutant invariant, not different scalar conventions.

Remark 4 (Existence is a separate question)**.** Theorem 3 identifies the parameter space; it does not guarantee that this space is nonempty. For instance, if the commutant contains an odd finite real type-I summand such as $M_{2r+1}(\mathbb{R})$, then that summand has no skew-adjoint square root of -1. In finite-dimensional compact decompositions this becomes the even-real-multiplicity condition in Corollary 2. More generally, the square-root existence problem belongs to the structure theory of real von Neumann algebras and is upstream of any later positive-frequency selection.

5. Abelian spectral orientations and positive-frequency flows

Let G be a second-countable locally compact abelian group with Pontryagin dual $\widehat{G}$. The spectral theorem gives a projection-valued measure E on $\widehat{G}$ such that

$$V(g) = \int_{\widehat{G}} \chi(g)\, dE(\chi).$$

Because V is the complexification of a real representation,

$$\mathcal{C}E(\Delta)\mathcal{C} = E(\Delta^{-1}).$$

Definition 8 (Spectral orientation)**.** A spectral orientation is a Borel map $\sigma : \widehat{G} \to \{-1, 0, 1\}$ satisfying $\sigma(\chi^{-1}) = -\sigma(\chi)$. Put

$$P = \sigma^{-1}(1), \quad N = \sigma^{-1}(-1), \quad F = \sigma^{-1}(0).$$

Proposition 1 (Orientation construction and partial phase)**.** *Define*

$$\mathcal{H}^0_{\mathbb{R}}(\sigma) = \mathcal{H}_{\mathbb{R}} \cap E(F)\mathcal{H}_{\mathbb{C}}, \qquad \mathcal{H}^{dyn}_{\mathbb{R}}(\sigma) = \{x + \mathcal{C}x : x \in E(P)\mathcal{H}_{\mathbb{C}}\}.$$

Then $\mathcal{H}^0_{\mathbb{R}}(\sigma)$ is orthogonal to $\mathcal{H}^{dyn}_{\mathbb{R}}(\sigma)$, and

$$P_\sigma = i(E(P) - E(N))|_{\mathcal{H}_{\mathbb{R}}},$$

is a bounded skew-adjoint U-compatible partial phase operator: its restriction $J_\sigma = P_\sigma|_{\mathcal{H}^{dyn}_{\mathbb{R}}(\sigma)}$ is an orthogonal complex structure on $\mathcal{H}^{dyn}_{\mathbb{R}}(\sigma)$, while P_σ vanishes on $\mathcal{H}^0_{\mathbb{R}}(\sigma)$. Thus P_σ is a genuine complex structure on the whole selected real space only when the zero sector is absent, or after an independent compatible complex structure has been chosen on the zero sector.

Proof. The projections $E(P), E(N), E(F)$ are orthogonal. Conjugation exchanges $E(P)\mathcal{H}_{\mathbb{C}}$ and $E(N)\mathcal{H}_{\mathbb{C}}$, while preserving $E(F)\mathcal{H}_{\mathbb{C}}$. Every real vector in the oriented part has the form $x + \mathcal{C}x$ with $x \in E(P)\mathcal{H}_{\mathbb{C}}$. For such a vector,

$$J_\sigma(x + \mathcal{C}x) = i(x - \mathcal{C}x),$$

and applying J_σ again gives $-x - \mathcal{C}x$. Skew-adjointness follows from self-adjointness of the spectral projections and the factor i. Since P_σ is a spectral multiplier, it commutes with U. On $E(F)\mathcal{H}_{\mathbb{C}}$ the multiplier is zero, so the full operator is partial rather than complex on the zero sector. □

Example 2. For $G = \mathbb{R}$, the orientation $\sigma(\lambda) = \operatorname{sgn}\lambda$ is the usual positive-frequency split. For $G = \mathbb{R}^n$, any nonzero covector ℓ gives $\sigma_\ell(\xi) = \operatorname{sgn}\ell(\xi)$ once such a direction has been supplied. This is only an abelian spectral-orientation example; restrictions imposed by adding larger symmetry groups are not pursued here.

Now let $U : \mathbb{R} \to O(\mathcal{H}_{\mathbb{R}})$ be a strongly continuous orthogonal flow with skew-adjoint generator A, so $U(t) = e^{tA}$. Let $V(t) = U(t)_{\mathbb{C}}$ denote the complexified unitary flow, not composition with $\mathcal{C}$. By Stone's theorem [7,9], $V(t) = e^{itK}$ for a self-adjoint K on $\mathcal{H}_{\mathbb{C}}$, and the generators satisfy

$$A_{\mathbb{C}} = iK \quad \text{on} \quad \operatorname{Dom}(A_{\mathbb{C}}) = \operatorname{Dom}(K).$$

Since $V(t)\mathcal{C} = \mathcal{C}V(t)$ and $\mathcal{C}$ is anti-linear, $\mathcal{C}e^{itK}\mathcal{C} = e^{-it\mathcal{C}K\mathcal{C}} = e^{itK}$, whence $\mathcal{C}K\mathcal{C} = -K$.

Let

$$\mathcal{H}_0 = \ker A, \qquad \mathcal{H}_{dyn} = \mathcal{H}_0^{\perp}.$$

Theorem 4 (Positive-frequency polar selection). *Let A_{dyn} be the restriction of A to the reducing subspace $\mathcal{H}_{dyn}$, with domain*

$$\operatorname{Dom}(A_{dyn}) = \operatorname{Dom}(A) \cap \mathcal{H}_{dyn}.$$

There is a unique bounded orthogonal complex structure J_A on $\mathcal{H}_{dyn}$ and a positive self-adjoint operator $B = |A_{dyn}|$ with $\operatorname{Dom}(B) = \operatorname{Dom}(A_{dyn})$ such that $J_A \operatorname{Dom}(B) = \operatorname{Dom}(B)$ and

$$A_{dyn} = J_A B = B J_A,$$

as equalities of closed operators on the common domain $\operatorname{Dom}(B)$. Equivalently, after complexification to the nonzero spectral subspace $E(\mathbb{R} \setminus \{0\})\mathcal{H}_{\mathbb{C}} = (\mathcal{H}_{dyn})_{\mathbb{C}}$,

$$(J_A)_{\mathbb{C}} = i \operatorname{sgn}(K)\big|_{E(\mathbb{R}\setminus\{0\})\mathcal{H}_{\mathbb{C}}} = i\big(E((0,\infty)) - E((-\infty,0))\big)\big|_{E(\mathbb{R}\setminus\{0\})\mathcal{H}_{\mathbb{C}}},$$

where E is the spectral measure of K. If $A_{dyn} = \widetilde{J}\widetilde{B} = \widetilde{B}\widetilde{J}$, where $\widetilde{J}$ is a bounded orthogonal complex structure on $\mathcal{H}_{dyn}$, $\widetilde{B}$ is positive self-adjoint with $\operatorname{Dom}(\widetilde{B}) = \operatorname{Dom}(A_{dyn})$, and $\widetilde{J}\operatorname{Dom}(\widetilde{B}) = \operatorname{Dom}(\widetilde{B})$, then $\widetilde{J} = J_A$ and $\widetilde{B} = B$.

Proof. Since A is skew-adjoint, it is closed and normal, $\ker A$ and $(\ker A)^{\perp}$ reduce A, and A_{dyn} is skew-adjoint on $\mathcal{H}_{dyn}$. Let $E_0 = E(\mathbb{R} \setminus \{0\})$. The relation $A_{\mathbb{C}} = iK$ on $\operatorname{Dom}(K)$ gives

$$(\mathcal{H}_{dyn})_{\mathbb{C}} = E_0\mathcal{H}_{\mathbb{C}}, \qquad |A_{dyn}|_{\mathbb{C}} = |K|\big|_{E_0\mathcal{H}_{\mathbb{C}}}.$$

Define on $E_0\mathcal{H}_{\mathbb{C}}$

$$J_{A,\mathbb{C}} = i \operatorname{sgn}(K) = i\big(E((0,\infty)) - E((-\infty,0))\big).$$

Because $\mathcal{C}K\mathcal{C} = -K$, one has $\mathcal{C}\operatorname{sgn}(K)\mathcal{C} = -\operatorname{sgn}(K)$; the anti-linearity of $\mathcal{C}$ then gives $\mathcal{C}J_{A,\mathbb{C}}\mathcal{C} = J_{A,\mathbb{C}}$. Hence $J_{A,\mathbb{C}}$ is the complexification of a bounded real operator J_A on $\mathcal{H}_{dyn}$.

The multiplier $i\operatorname{sgn}(\lambda)$ has modulus one and square -1 on the nonzero spectral sector. Therefore J_A is orthogonal, $J_A^2 = -\operatorname{Id}$, and $J_A^* = -J_A$. Since $J_{A,\mathbb{C}}$ and $|K|$ are Borel functions of the same self-adjoint operator, they commute strongly. Consequently $J_A \operatorname{Dom}(|A_{dyn}|) = \operatorname{Dom}(|A_{dyn}|)$, and

$$A_{dyn} = J_A|A_{dyn}| = |A_{dyn}|J_A,$$

as equalities of closed operators on $\operatorname{Dom}(|A_{dyn}|)$.

Equivalently, the polar decomposition $A_{dyn} = V_A|A_{dyn}|$ has initial space $\overline{\operatorname{Ran}|A_{dyn}|} = (\ker A_{dyn})^{\perp} = \mathcal{H}_{dyn}$ and final space $\overline{\operatorname{Ran} A_{dyn}} = (\ker A_{dyn}^*)^{\perp} = \mathcal{H}_{dyn}$. Thus the polar partial isometry is unitary on $\mathcal{H}_{dyn}$, but this uses only dense range, not surjectivity of $|A_{dyn}|$. The spectral formula above identifies this unitary with J_A. Put $B = |A_{dyn}|$.

For uniqueness, suppose $A_{dyn} = \widetilde{J}\widetilde{B} = \widetilde{B}\widetilde{J}$ with the stated domain and self-adjointness assumptions. Then

$$A_{dyn}^* A_{dyn} = (\widetilde{J}\widetilde{B})^*(\widetilde{J}\widetilde{B}) = \widetilde{B}^2,$$

because $\widetilde{J}$ is unitary and commutes with $\widetilde{B}$ on the common domain. The positive square root of $A_{dyn}^* A_{dyn}$ is unique, so

$$\widetilde{B} = |A_{dyn}|.$$

For $x \in \mathrm{Dom}(|A_{dyn}|)$,

$$(\widetilde{J} - J_A)|A_{dyn}|x = \widetilde{J}\widetilde{B}x - J_A|A_{dyn}|x = A_{dyn}x - A_{dyn}x = 0.$$

Since

$$\overline{\mathrm{Ran}\,|A_{dyn}|} = \mathcal{H}_{dyn},$$

after removal of the kernel, and since both $\widetilde{J}$ and J_A are bounded, it follows that $\widetilde{J} = J_A$ on $\mathcal{H}_{dyn}$. □

Example 3 (Finite-dimensional oscillator)**.** Let $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$ with its Euclidean inner product and let

$$A(q, p) = (\omega p, -\omega q), \qquad \omega > 0.$$

Then $A^* = -A$, $|A| = \omega I$, and Theorem 4 gives

$$J_A(q, p) = (p, -q), \qquad A = J_A|A|.$$

Thus the usual positive-frequency phase of a harmonic oscillator is recovered as the polar phase of the real rotation generator. For m identical copies the Real commutant contains $\mathcal{B}_{\mathbb{C}}(\mathbb{C}^m)$, so positive frequency selects the scalar orientation J_A while the compact multiplicity classification records the remaining symmetry-compatible operator-valued freedom.

Proposition 2 (Covariance)**.** *If W is a real unitary with $W\,\mathrm{Dom}(A) = \mathrm{Dom}(A)$ and $WAW^{-1} = aA$ for $a \neq 0$, then*

$$WJ_AW^{-1} = \mathrm{sgn}(a)J_A, \qquad W|A|W^{-1} = |a|\,|A|.$$

Thus time-orientation-preserving symmetries commute with J_A, while reversing symmetries send it to $-J_A$.

Proof. The hypothesis preserves $\ker A$ and hence $\mathcal{H}_{dyn}$. Also $|WAW^{-1}| = |a|\,|A|$. Conjugating $A = J_A|A|$ gives $aA = (WJ_AW^{-1})(|a|\,|A|)$. The polar phase is therefore $\mathrm{sgn}(a)J_A$. □

6. Compact groups and Frobenius–Schur blocks

Let G be compact and second-countable. Let $U : G \to O(\mathcal{H}_{\mathbb{R}})$ be a strongly continuous orthogonal representation on a separable real Hilbert space. Peter–Weyl theory decomposes the complexified representation into isotypic Hilbert summands

$$\mathcal{H}_{\mathbb{C}} \cong \bigoplus_{\pi \in \widehat{G}} H_\pi \otimes M_\pi, \qquad V(g) \cong \bigoplus_{\pi \in \widehat{G}} \pi(g) \otimes \mathrm{Id}_{M_\pi}.$$

Here H_π is a fixed irreducible carrier and M_π is the corresponding complex multiplicity space, equivalently the space of copies of H_π inside $\mathcal{H}_{\mathbb{C}}$. Write $\pi^\vee$ for the conjugate representation class. Canonical conjugation transports the π-isotypic block antiunitarily onto the $\pi^\vee$-isotypic block:

$$\mathcal{C}(H_\pi \otimes M_\pi) = H_{\pi^\vee} \otimes M_{\pi^\vee}.$$

Thus non-self-conjugate classes occur in exchanged pairs, and the two multiplicity spaces in a pair are identified antiunitarily by this transport.

For a self-conjugate irreducible ρ, choose an antiunitary intertwiner $j_\rho : H_\rho \to H_\rho$ with $j_\rho^2 = \varepsilon_\rho \,\mathrm{Id}$, where $\varepsilon_\rho = +1$ in real type and $\varepsilon_\rho = -1$ in quaternionic type. Relative to this carrier-side choice, the restriction of canonical conjugation to the ρ-isotypic block has the form

$$\mathcal{C}|_{H_\rho \otimes M_\rho} = j_\rho \otimes c_\rho,$$

where c_ρ is an antiunitary on M_ρ satisfying $c_\rho^2 = \varepsilon_\rho \,\mathrm{Id}$. If $\varepsilon_\rho = +1$, set

$$M_\rho^{\mathbb{R}} = \mathrm{Fix}(c_\rho),$$

so that $M_\rho \cong M_\rho^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. If $\varepsilon_\rho = -1$, write $M_\rho^{\mathbb{H}}$ for the same multiplicity Hilbert space equipped with the quaternionic structure determined by the complex scalar i and the antiunitary c_ρ. Equivalently,

$$\mathcal{B}_{\mathbb{H}}(M_\rho^{\mathbb{H}}) = \{T \in \mathcal{B}_{\mathbb{C}}(M_\rho) : Tc_\rho = c_\rho T\},$$

as a real C^*-algebra.

The auxiliary choices do not change the resulting factors except by unitary equivalence. Indeed, another carrier antiunitary has the form $j_\rho' = \alpha j_\rho$ with $|\alpha| = 1$ by Schur's lemma, and the same block conjugation is then written as $j_\rho' \otimes c_\rho'$ with $c_\rho' = \overline{\alpha} c_\rho$. Multiplication by a scalar square root identifies the real fixed spaces in real type, and gives the corresponding quaternionic unitary equivalence in quaternionic type. Thus the symbols $M_\rho^{\mathbb{R}}$, $M_\rho^{\mathbb{H}}$, and their operator algebras are canonical up to this harmless conjugacy.

Choose a representative set P for the non-self-conjugate pairs $\{\pi, \pi^\vee\}$. Let $\widehat{G}_{\mathbb{R}}$ and $\widehat{G}_{\mathbb{H}}$ denote the occurring real-type and quaternionic-type self-conjugate classes.

Theorem 5 (Compact Real-commutant formula)**.** *There is an isometric real $*$-isomorphism*

$$\mathcal{A}_{\mathbb{R}}(U) \cong \prod_{\pi \in P}^{\infty} \mathcal{B}_{\mathbb{C}}(M_\pi) \times \prod_{\rho \in \widehat{G}_{\mathbb{R}}}^{\infty} \mathcal{B}_{\mathbb{R}}(M_\rho^{\mathbb{R}}) \times \prod_{\tau \in \widehat{G}_{\mathbb{H}}}^{\infty} \mathcal{B}_{\mathbb{H}}(M_\tau^{\mathbb{H}}).$$

The products are bounded ℓ^∞-products: a family belongs to the product precisely when the supremum of the operator norms of its components is finite, and the product norm is this supremum. Consequently, compatible orthogonal complex structures are exactly the bounded products of skew-adjoint square roots of $-\,\mathrm{Id}$ *in these complex, real, and quaternionic multiplicity algebras.*

Proof. By Schur's lemma on each irreducible carrier, any bounded operator commuting with $V(G)$ preserves the isotypic decomposition and has the block form

$$T = \bigoplus_{\pi \in \widehat{G}} \mathrm{Id}_{H_\pi} \otimes T_\pi, \qquad T_\pi \in \mathcal{B}_{\mathbb{C}}(M_\pi).$$

On an infinite Hilbert direct sum this formula defines a bounded operator exactly when $\sup_\pi \|T_\pi\| < \infty$, and then $\|T\| = \sup_\pi \|T_\pi\|$. This proves the bounded ℓ^∞-product description of the complex commutant.

Impose the Real condition $T\mathcal{C} = \mathcal{C}T$. For a non-self-conjugate pair $\{\pi, \pi^\vee\}$, let $C_\pi : H_\pi \otimes M_\pi \to H_{\pi^\vee} \otimes M_{\pi^\vee}$ denote the antiunitary transport induced by $\mathcal{C}$. The Real condition on the two-block sum is equivalent to

$$(\mathrm{Id} \otimes T_{\pi^\vee}) C_\pi = C_\pi (\mathrm{Id} \otimes T_\pi),$$

so $T_{\pi^\vee}$ is determined by T_π. Hence one free complex operator $T_\pi \in \mathcal{B}_{\mathbb{C}}(M_\pi)$ remains for each unordered non-self-conjugate pair.

For a self-conjugate block, write $\mathcal{C} = j_\rho \otimes c_\rho$. Then

$$(\mathrm{Id} \otimes T_\rho)(j_\rho \otimes c_\rho) = (j_\rho \otimes c_\rho)(\mathrm{Id} \otimes T_\rho),$$

is equivalent to $T_\rho c_\rho = c_\rho T_\rho$ on the multiplicity space. If $c_\rho^2 = +\mathrm{Id}$, this says exactly that T_ρ is the complexification of a bounded real-linear operator on $M_\rho^{\mathbb{R}} = \mathrm{Fix}(c_\rho)$, giving $\mathcal{B}_{\mathbb{R}}(M_\rho^{\mathbb{R}})$. If $c_\tau^2 = -\mathrm{Id}$, the antiunitary c_τ supplies the second quaternionic scalar and the same commutation relation says exactly that T_τ is quaternionic-linear on $M_\tau^{\mathbb{H}}$, giving $\mathcal{B}_{\mathbb{H}}(M_\tau^{\mathbb{H}})$. Combining these pair, real-type, and quaternionic-type blocks with the bounded product norm proves the stated real $*$-isomorphism. The final statement follows by applying $J^* = -J$ and $J^2 = -\mathrm{Id}$ componentwise in this product algebra. □

Corollary 2 (Finite-dimensional existence criterion)**.** *In the finite-dimensional compact case, a global compatible orthogonal complex structure exists exactly when every real-type multiplicity block has even real dimension. Non-self-conjugate pair sectors and nonzero quaternionic-type sectors carry compatible complex structures without this even-real-dimension obstruction.*

Proof. A real vector space admits an operator $J^2 = -\mathrm{Id}$ only in even real dimension. This gives the obstruction in real-type factors. Non-self-conjugate sectors occur in conjugate pairs, and quaternionic spaces contain quaternionic-linear square roots of -1. The criterion follows from Theorem 5. □

7. Sign classes and functorial complex carriers

Let $[\pi] = \{\pi, \pi^\vee\}$ be a non-self-conjugate pair. The corresponding real pair sector is

$$\mathcal{H}_{\mathbb{R}}[\pi] = \mathcal{H}_{\mathbb{R}} \cap \left(H_\pi \otimes M_\pi \oplus H_{\pi^\vee} \otimes M_{\pi^\vee}\right).$$

Proposition 3 (Canonical sign class)**.** *The sector $\mathcal{H}_{\mathbb{R}}[\pi]$ carries a canonical unordered pair $\{\pm J_{[\pi]}\}$ of G-compatible orthogonal complex structures. A signed operator $J_{[\pi]}$ is obtained only after choosing an orientation of the pair, i.e. after choosing which of π and $\pi^\vee$ is the positive member. The associated two-dimensional real $*$-subalgebra*

$$\mathbb{C}_{[\pi]} = \mathrm{span}_{\mathbb{R}}\{\mathrm{Id}, J_{[\pi]}\} \subseteq \mathcal{B}(\mathcal{H}_{\mathbb{R}}[\pi]),$$

is independent of this sign choice.

Proof. Choose an orientation by declaring π to be positive. On the complexified pair block define $J_{[\pi]}$ as multiplication by $+i$ on $H_\pi \otimes M_\pi$ and by $-i$ on $H_{\pi^\vee} \otimes M_{\pi^\vee}$. This operator commutes with $V(G)$, commutes with canonical conjugation because conjugation exchanges the two blocks, and restricts to a real orthogonal complex structure on $\mathcal{H}_{\mathbb{R}}[\pi]$. Reversing the orientation of the pair interchanges the two blocks and replaces $J_{[\pi]}$ by $-J_{[\pi]}$. Hence the unordered sign class and the real algebra generated by either sign are canonical, while a signed complex structure is not. □

Remark 5. The sign class is the preferred scalar carrier orientation on a non-self-conjugate pair. It is generally only a subfamily of the full Real-commutant classification: when $\dim M_\pi > 1$, Theorem 5 also permits arbitrary skew-adjoint unitary square roots of $-I$ inside the multiplicity algebra $\mathcal{B}_{\mathbb{C}}(M_\pi)$.

Corollary 3. *An orientation of the non-self-conjugate part of $\widehat{G}$, namely a choice of one representative from each pair, turns the sign classes into a global complex structure on the non-self-conjugate sector.*

Proposition 4 (Operator-linearity)**.** *On a non-self-conjugate pair sector $\mathcal{H}_{\mathbb{R}}[\pi]$, every bounded real-linear operator commuting with G is linear over the carrier algebra $\mathbb{C}_{[\pi]}$.*

Proof. Let R be a bounded real-linear operator on $\mathcal{H}_{\mathbb{R}}[\pi]$ commuting with G, and let $R_{\mathbb{C}}$ be its complexification. Since π and $\pi^\vee$ are inequivalent complex irreducibles, Schur's lemma forbids off-diagonal maps between the two isotypic components. Thus

$$R_{\mathbb{C}} = \mathrm{Id}_{H_\pi} \otimes T_\pi \oplus \mathrm{Id}_{H_{\pi^\vee}} \otimes T_{\pi^\vee},$$

where $T_{\pi^\vee}$ is the conjugation-transport of T_π because $R_{\mathbb{C}}\mathcal{C} = \mathcal{C}R_{\mathbb{C}}$. The chosen $J_{[\pi]}$ is scalar multiplication by $+i$ on the first summand and by $-i$ on the second. Both multiplicity operators are complex-linear on their respective summands, so $R_{\mathbb{C}}$ commutes with $J_{[\pi],\mathbb{C}}$. Restricting back to $\mathcal{H}_{\mathbb{R}}[\pi]$ gives $RJ_{[\pi]} = J_{[\pi]}R$, which is exactly $\mathbb{C}_{[\pi]}$-linearity. $\square$

Example 4 (Circle group with multiplicity)**.** For $G = S^1$, $\widehat{G} \cong \mathbb{Z}$ and conjugation sends n to $-n$. Consider m copies of the real two-dimensional rotation of frequency $n > 0$. After complexification, the nonzero-frequency sector is

$$H_n \otimes \mathbb{C}^m \;\oplus\; H_{-n} \otimes \mathbb{C}^m,$$

where $H_{\pm n}$ are the one-dimensional character spaces $z \mapsto z^{\pm n}$. Theorem 5 gives the Real commutant of this pair sector as $\mathcal{B}_{\mathbb{C}}(\mathbb{C}^m)$. The canonical oriented phase $J_{[n]}$ corresponds to $+iI_m$ in this factor; reversing the orientation gives $-iI_m$. When $m = 1$, these are the only compatible orthogonal complex structures. For $m > 1$, the full classification also allows any skew-adjoint unitary square root of $-I$ in $\mathcal{B}_{\mathbb{C}}(\mathbb{C}^m)$, for example $Q\,\mathrm{diag}(iI_r, -iI_{m-r})Q^*$ with $Q \in U(m)$. Thus the example displays both the canonical sign class of Proposition 3 and the larger multiplicity freedom described by Theorem 5.

Example 5 (Dihedral obstruction)**.** Let C_3 act on $\mathbb{R}^2$ by rotation through $2\pi/3$. The representation is a conjugate character pair and admits $\{\pm J\}$. Enlarging to the dihedral group adds a reflection that swaps the pair and reverses orientation. Therefore no orthogonal complex structure commutes with the full dihedral action.

Example 6 ($SU(2)$)**.** Every irreducible representation of $SU(2)$ is self-conjugate. The highest-weight representation of weight n is real type for even n and quaternionic type for odd n. Thus compatible complex structures for $SU(2)$-representations come from real multiplicity choices on even blocks and quaternionic-linear choices on odd blocks.

8. Phase-balanced composition

The Real-commutant construction selects complex structures system by system. If two selected real systems are composed, the local phase operators must represent one shared scalar. The following phase-balanced tensor product is the usual complex Hilbert tensor product written in real form; it is included to show compatibility with the selected phase convention, not as a new tensor-product construction.

Definition 9 (Phase-balanced tensor product)**.** Let $(\mathcal{H}_{\mathbb{R}}, J_H)$ and $(\mathcal{K}_{\mathbb{R}}, J_K)$ be real Hilbert spaces with orthogonal complex structures. Let $\mathcal{H}_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathcal{K}_{\mathbb{R}}$ be the real Hilbert tensor product. Define

$$N_J = \overline{\mathrm{span}_{\mathbb{R}}\{(J_H x) \otimes y - x \otimes (J_K y) : x \in \mathcal{H}_{\mathbb{R}},\ y \in \mathcal{K}_{\mathbb{R}}\}}.$$

The phase-balanced tensor product is

$$\mathcal{H}_{\mathbb{R}}\widehat{\otimes}_J\mathcal{K}_{\mathbb{R}} = (\mathcal{H}_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathcal{K}_{\mathbb{R}})/N_J.$$

Theorem 6 (Standard compatibility with the complex tensor product)**.** *Let $\mathcal{H}_{J_H}$ and $\mathcal{K}_{J_K}$ denote the complex Hilbert spaces obtained from $(\mathcal{H}_{\mathbb{R}}, J_H)$ and $(\mathcal{K}_{\mathbb{R}}, J_K)$. There is a canonical real unitary isomorphism*

$$(\mathcal{H}_{J_H}\widehat{\otimes}_{\mathbb{C}}\mathcal{K}_{J_K})_{\mathbb{R}} \cong \mathcal{H}_{\mathbb{R}}\widehat{\otimes}_J\mathcal{K}_{\mathbb{R}}.$$

Under this isomorphism multiplication by i is induced equally by $J_H \otimes I$ and by $I \otimes J_K$.

Proof. The algebraic complex tensor product is the algebraic real tensor product modulo the complex balancing relation

$$(ix) \otimes y = x \otimes (iy).$$

Substituting $ix = J_H x$ and $iy = J_K y$ gives exactly the generators whose closed span is N_J. Passing to the Hilbert quotient and completing therefore gives the stated Hilbert tensor product. The common complex structure is well defined because $J_H \otimes I - I \otimes J_K$ vanishes in the quotient. $\square$

Corollary 4 (Universal property). *Let $(L_{\mathbb{R}}, J_L)$ be another selected real Hilbert space. A bounded real bilinear map $b : \mathcal{H}_{\mathbb{R}} \times \mathcal{K}_{\mathbb{R}} \to L_{\mathbb{R}}$ factors uniquely through a bounded real linear map $\mathcal{H}_{\mathbb{R}} \widehat{\otimes}_J \mathcal{K}_{\mathbb{R}} \to L_{\mathbb{R}}$ if and only if*

$$b(J_H x, y) = b(x, J_K y),$$

for all x, y. If in addition $J_L b(x, y) = b(J_H x, y)$, then the induced map is complex-linear.

Proof. Every bounded real bilinear map induces a bounded real linear map on $\mathcal{H}_{\mathbb{R}} \widehat{\otimes}_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$. It descends to the quotient exactly when it vanishes on N_J, which is equivalent to the displayed balancing identity. The final condition says precisely that the descended map intertwines the quotient complex structure with J_L. □

9. A positive-frequency temporal spectral triple

This section is an operator-theoretic model of a completed record direction in the sense of spectral geometry [10,11]. It is not needed for the Real-commutant classification, but it illustrates how positive-frequency phase selection can be recorded in metric spectral data.

Let $\ell^2(\mathbb{N}_0)$ have orthonormal basis $(e_n)_{n \geq 0}$. Let $Se_n = e_{n+1}$ be the unilateral shift and put $\partial = S - I$. Let $(\mu_n)_{n \geq 0}$ be a real sequence satisfying

$$|\mu_n| \to \infty, \qquad L_\mu := \sup_n |\mu_{n+1} - \mu_n| < \infty.$$

Let $Me_n = \mu_n e_n$, with maximal domain. On

$$\mathcal{H}_\tau = \ell^2(\mathbb{N}_0) \otimes \mathbb{C}^2$$

define

$$D_\tau = M \otimes \sigma_3 + \kappa \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}, \qquad \kappa > 0,$$

with domain $\mathrm{Dom}(D_\tau) = \mathrm{Dom}(M) \otimes \mathbb{C}^2$, since the off-diagonal term is bounded. The algebra acts on $\mathcal{H}_\tau$ through the representation $a \mapsto a \otimes I_2$ for diagonal operators and $S \mapsto S \otimes I_2$. Let

$$\mathcal{A}_{lip} = \left\{ a = \mathrm{diag}(a_n) : (a_n) \in \ell^\infty, \sup_n |a_{n+1} - a_n| < \infty \right\}.$$

Define $\mathcal{A}_\tau^{alg}$ to be the unital $*$-algebra generated algebraically by $\mathcal{A}_{lip}$, S, and S^*. No bounded-commutator condition is included in this definition; boundedness is proved below for the generators and hence for the algebra by the Leibniz rule. Its norm closure in $\mathcal{B}(\ell^2(\mathbb{N}_0))$ is, by definition, the C^*-algebra $C^*(\mathcal{A}_{lip}, S)$, so $\mathcal{A}_\tau^{alg}$ is dense there.

Theorem 7 (Temporal spectral triple). *The triple $(\mathcal{A}_\tau^{alg}, \mathcal{H}_\tau, D_\tau)$ is a spectral triple: D_τ is self-adjoint, has compact resolvent, and $[D_\tau, a]$ is bounded for every $a \in \mathcal{A}_\tau^{alg}$. In particular, every $a = \mathrm{diag}(a_n) \in \mathcal{A}_{lip}$ satisfies*

$$\|[D_\tau, a]\| = \kappa \sup_n |a_{n+1} - a_n|.$$

Proof. The operator $M \otimes \sigma_3$ is self-adjoint on $\mathrm{Dom}(M) \otimes \mathbb{C}^2$. Since $\partial = S - I$ is bounded, the off-diagonal difference term is bounded and self-adjoint; hence D_τ is self-adjoint by bounded perturbation. Because $|\mu_n| \to \infty$, $M \otimes \sigma_3$ has compact resolvent, and compactness of the resolvent is preserved by bounded perturbation.

For a diagonal $a \in \mathcal{A}_{lip}$, the diagonal multiplier commutes with $M \otimes \sigma_3$, and

$$[\partial, a] = [S, a], \qquad [S, a]e_n = (a_n - a_{n+1})e_{n+1}.$$

Similarly,

$$[\partial^*, a] = [S^*, a], \qquad [S^*, a]e_n = \begin{cases} (a_n - a_{n-1})e_{n-1}, & n \geq 1, \\ 0, & n = 0. \end{cases}$$

Hence

$$\|[\partial, a]\| = \|[\partial^*, a]\| = \sup_n |a_{n+1} - a_n|.$$

Since

$$[D_\tau, a] = \kappa \begin{pmatrix} 0 & [\partial^*, a] \\ [\partial, a] & 0 \end{pmatrix},$$

the norm of this off-diagonal block operator is $\kappa \max\{\|[\partial, a]\|, \|[\partial^*, a]\|\}$, giving the displayed equality for $\|[D_\tau, a]\|$. For the shift generators,

$$[M, S]e_n = (\mu_{n+1} - \mu_n)e_{n+1}, \qquad [M, S^*]e_n = (\mu_{n-1} - \mu_n)e_{n-1},$$

with the evident convention at $n = 0$, so $[M, S]$ and $[M, S^*]$ are bounded by L_μ. Also $[\partial, S] = 0$ and $[\partial, S^*]$ is finite rank, hence bounded; the same holds for adjoints. Therefore $[D_\tau, S]$ and $[D_\tau, S^*]$ are bounded. The Leibniz rule $[D, ab] = [D, a]b + a[D, b]$ then gives bounded commutators for all algebraic words in the generators. □

Theorem 8 (Exact diagonal Connes distance). *Let δ_n be the pure state on the diagonal record algebra $\mathcal{A}_{lip}$ given by evaluation at n. The Connes distance computed on this diagonal algebra, using the Lipschitz seminorm $L(a) = \|[D_\tau, a]\|$, is*

$$d_{diag}(\delta_m, \delta_n) = \frac{|m - n|}{\kappa}.$$

This theorem is not a distance statement for the larger noncommutative algebra containing shifts.

Proof. For a real diagonal element $a = \mathrm{diag}(a_n) \in \mathcal{A}_{lip}$, Theorem 7 gives

$$L(a) = \|[D_\tau, a]\| = \kappa \sup_n |a_{n+1} - a_n|.$$

Thus $L(a) \le 1$ implies $|a_m - a_n| \le |m - n|/\kappa$ by telescoping. Conversely, assume $m < n$ and set

$$a_j = \frac{1}{\kappa} \min\{\max\{j, m\}, n\}.$$

This bounded Lipschitz diagonal sequence has $L(a) \le 1$ and $a_n - a_m = (n - m)/\kappa$. Hence the supremum is exactly $|m - n|/\kappa$. □

Proposition 5 (Spectral dimension from a counting law). *Let*

$$N_\mu(\lambda) = \#\{n \ge 0 : |\mu_n| \le \lambda\}.$$

If $N_\mu(\lambda) \sim C^{-d}\lambda^d$ as $\lambda \to \infty$, with $C > 0$ and $d > 0$, then the eigenvalue counting function of $|D_\tau|$ satisfies

$$N_{D_\tau}(\lambda) \sim 2C^{-d}\lambda^d.$$

In particular, this conclusion holds when $|\mu_n|$ is eventually nondecreasing and $|\mu_n| \sim Cn^{1/d}$. The effective spectral dimension is then d.

Proof. Let R be the norm of the bounded off-diagonal difference part of D_τ. By the min-max principle for self-adjoint operators with compact resolvent,

$$N_{M\otimes\sigma_3}(\lambda - R) \le N_{D_\tau}(\lambda) \le N_{M\otimes\sigma_3}(\lambda + R),$$

for large λ. Since $N_{M\otimes\sigma_3}(\lambda) = 2N_\mu(\lambda)$ and the assumed counting law is stable under the fixed shifts $\lambda \mapsto \lambda \pm R$, the stated asymptotic follows. □

Remark 6. The metric theorem is deliberately restricted to the diagonal record algebra. Extending the same distance interpretation to a larger algebra containing shifts requires additional Lipschitz-contracting expectations or comparable hypotheses. The dimension theorem is an operator-growth statement; it is not, by itself, a derivation of physical spatial dimension.

10. Conditional cone geometry from length-delay data

This section gives a conditional mathematical residue of the completion-delay idea. It is independent of the Real-commutant classification, and it begins only after a spatial readout and a delay functional have been supplied. Finite-propagation estimates, including Lieb–Robinson-type bounds, are useful motivation for inequalities of this form [12,13]; however, no such estimate is proved or used here. The theorem below uses only the abstract length-budget and completion-delay inequalities in Definitions 10 and 11.

Definition 10 (Weighted transformation monoid)**.** Let M be a monoid acting on a pseudometric space (X, δ) by maps $F_u : X \to X$. A weight is a function $w : M \to \mathbb{R}_{\geq 0}$ such that $w(e) = 0$ and

$$w(uv) \leq w(u) + w(v).$$

A length budget is a constant $\ell_* > 0$ such that

$$\delta(F_u x, x) \leq w(u)\ell_*$$

for all admissible $u \in M$ and $x \in X$.

Definition 11 (Completion delay)**.** A completion-delay functional is a map $\Theta : M \to \mathbb{R}_{\geq 0}$. It is compatible with the length budget at speed $c_* > 0$ if

$$w(u)\ell_* \leq c_*\Theta(u),$$

for all completed transformations u.

Theorem 9 (Conditional finite completion cone)**.** *Assume a weighted transformation monoid acts on (X, δ) with length budget ℓ_*, and assume a compatible completion delay Θ with speed c_*. Then*

$$\delta(F_u x, x) \leq c_*\Theta(u),$$

for every completed transformation u and every admissible x. If there is a spatial readout map $\chi : X \to \mathbb{R}^d$ satisfying

$$\|\chi(F_u x) - \chi(x)\| \leq \delta(F_u x, x),$$

then each completed displacement

$$E_x(u) = \big(\Theta(u), \chi(F_u x) - \chi(x)\big),$$

lies in the pointed cone

$$K^+_{c_*} = \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d : \|y\| \leq c_* t\}.$$

Equivalently,

$$Q_{c_*}(t, y) = c_*^2 t^2 - \|y\|^2 \geq 0.$$

Proof. The first inequality follows by combining $\delta(F_u x, x) \leq w(u)\ell_*$ with $w(u)\ell_* \leq c_*\Theta(u)$. The readout inequality then gives $\|\chi(F_u x) - \chi(x)\| \leq c_*\Theta(u)$, which is exactly membership in $K^+_{c_*}$. □

Proposition 6 (Standard linear symmetries of the cone)**.** *For $d \geq 2$, let L be an invertible linear map that preserves the pointed future cone setwise, meaning $L(K^+_{c_*}) = K^+_{c_*}$. Then L is a Lorentz similitude: after the change of coordinate $z_0 = c_* t$, it has the form $\lambda\Lambda$, where $\lambda > 0$ and $\Lambda \in O^+(1, d)$ preserves the future cone. If a common rod-and-clock scale is fixed, the scale factor λ is removed.*

Proof. In the coordinate $(z_0, y) = (c_* t, y)$, the cone is the standard future cone $K^+ = \{(z_0, y) : z_0 \geq 0,\ q(z_0, y) \geq 0\}$ of the quadratic form $q(z_0, y) = z_0^2 - \|y\|^2$. Since L is a homeomorphism and $L(K^+) = K^+$, it maps the topological boundary $\partial K^+ = \{q = 0,\ z_0 \geq 0\}$ onto itself. Thus the homogeneous quadratic form $q \circ L$ vanishes on the same boundary null cone as q. For $d \geq 2$, the real null cone determines the Lorentz quadratic form up to a nonzero scalar; because L maps the interior of the future cone to itself, this scalar is positive. Hence $L^* q = \lambda^2 q$ for some $\lambda > 0$, and $\lambda^{-1} L \in O^+(1, d)$.

For $d = 1$ there is no exceptional obstruction, but the preceding null-cone-determination sentence is replaced by an elementary two-ray argument: in light-cone coordinates the cone is $\{u \geq 0,\ v \geq 0\}$, and its linear automorphisms rescale the two boundary rays, possibly interchanging them. These maps are again positive scalars times elements of $O^+(1, 1)$. The proposition is stated for $d \geq 2$ to use the standard higher-dimensional second-order cone formulation. □

Proposition 7 (Smooth representative from supplied smooth data)**.** *Let M be a smooth manifold with a one-form ϑ, a horizontal distribution $H = \ker \vartheta$, a positive definite metric h on H, and a constant $\kappa > 0$. Define*

$$g = -\kappa^2 \vartheta \otimes \vartheta + h,$$

by extending h to vanish on the vertical line complementary to H. Then the inequality

$$h(u_H, u_H) \leq \kappa^2 \vartheta(u)^2,$$

is equivalent to $g(u, u) \leq 0$. Thus g is a smooth Lorentzian representative of the local cone defined by this length-delay inequality whenever ϑ is nonzero and h has rank d.

Proof. Every tangent vector decomposes as $u = u_H + u_V$ with $u_H \in H$. By definition,

$$g(u, u) = -\kappa^2 \vartheta(u)^2 + h(u_H, u_H).$$

Therefore $g(u, u) \leq 0$ is exactly the displayed inequality. Since h is positive on H and the ϑ-direction has negative coefficient, the signature is $(-, +, \ldots, +)$. □

Remark 7. The cone theorem does not claim that every completion/projection system produces a manifoldlike sector, nor that Lorentz symmetry has been newly derived. It proves only that once a length budget, delay law, and Euclidean readout are supplied, a finite second-order cone follows. Proposition 6 records the standard linear automorphism group of that cone, and Proposition 7 requires the additional smooth one-form, horizontal distribution, and horizontal metric stated in its hypotheses.

11. Modular synthesis and open problems

The preceding modules can be summarized as a conditional synthesis. A natural class of systems to which all inputs can be applied is a completed-record orthogonal flow: a tuple consisting of a history action (Hist, X), completion data $Q : X \to \mathsf{Comp}$ and tests $\mathcal{O}$, a real orthogonal Hilbert representation U with an oriented time-flow generator A, and a completed-event readout (Θ, χ) satisfying the length-delay inequalities. The modules do not assert that every physical or dynamical system supplies all of these data, nor that the resulting chain is unique; they say what follows when the data are present.

Proposition 8 (Modular synthesis)**.** *Assume:*

- (I1) *a history action and a completion-observation map satisfying observable descent;*
- (I2) *a completed-future test family defining a behavioral quotient;*
- (I3) *a real orthogonal representation U and, where required, a positive time-oriented orthogonal flow;*
- (I4) *compact or abelian representation hypotheses when the corresponding classification formula is invoked;*
- (I5) *a length budget, completion delay, and spatial readout when cone geometry is asserted.*

Then the corresponding outputs are:

- (O1) *a descended observable monoid action and behavioral quotient;*

(O2) *the Real-commutant parameterization of compatible complex structures;*
(O3) *spectral-orientation and positive-frequency selection formulas, with zero modes treated as partial-phase sectors unless separately completed;*
(O4) *compact-group existence and sign-obstruction criteria;*
(O5) *a finite completion cone and, under supplied smooth stabilization data, a Lorentzian metric representative.*

Proof. Output (O1) follows from Theorems 1 and 2; (O2) from Theorem 3; (O3) from Proposition 1 and Theorem 4; (O4) from Theorem 5, Corollary 2, and Proposition 3; and (O5) from Theorem 9 and Proposition 7. This is an assembly of conditional implications, not a separate uniqueness theorem. □

Open mathematical problems directly suggested by the preceding technical points include:

(P1) extending the compact and abelian computations to a fully choice-free measurable type-I theory: fixed central/disjoint disintegrations should carry canonical multiplicity-side antiunitary transport gauge classes, while the main remaining issue is descent across different measurable realizations;
(P2) describing the topology and connected components of the space of skew-adjoint square roots of $-\,\mathrm{Id}$ in infinite-dimensional real C^*-algebras and in the bounded products appearing in Theorem 5;
(P3) extending positive-frequency selection to semidirect products $K \rtimes \mathbb{R}$, especially when time-reversing symmetries act on the sign of J_A;
(P4) finding concrete completed-record systems for which the quotient data, real representation, positive-frequency flow, and length-delay readout are all naturally present;
(P5) proving model-specific propositions that turn finite-propagation estimates, such as Lieb–Robinson bounds plus record-distinguishability assumptions, into the abstract length-budget and delay hypotheses used in Theorem 9;
(P6) characterizing when completion-delay cones admit canonical smooth Lorentzian representatives rather than merely supplied ones.

12. Conclusion

The paper has placed three constructions in one mathematical chain. First, completed records and observable quotients explain how observable dynamics may descend from a more detailed generative action. Second, the Real commutant $\mathcal{A}_{\mathbb{R}}(U)$ parameterizes all symmetry-compatible orthogonal complex structures of a real Hilbert representation when such structures exist, while positive-frequency polar decomposition selects a canonical phase on the nonzero-frequency part of an orthogonal time flow. Third, once length-delay data on completed events are supplied, completed displacements lie in a finite cone; the associated Lorentz-similitude statement and any smooth Lorentzian metric representative require the additional hypotheses explicitly stated in the cone section.

The physical language of DTT and SMM motivates the ordering, but the mathematical results do not depend on accepting that ontology. What is proved is conditional: given the input layers, the corresponding quotient, phase, tensor, spectral, and cone structures follow. Changing the completion map or test family changes the behavioral quotient; reversing the time orientation changes J_A to $-J_A$; retaining zero modes without extra structure gives only a partial phase. Further applications require additional data, such as compatible symplectic forms, and are downstream of the present proofs; changing those additional inputs changes the downstream application. Changing the delay law or readout changes the finite cone and its possible smooth representative.

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