Open Journal of
Mathematical Sciences

PSR Press

Article

Cubic curves and conics associated to the sphere S^4

Mircea Crasmareanu

Faculty of Mathematics, University "Al. I. Cuza", 700506, Iaşi, Romania; mcrasm@uaic.ro

Academic Editor: Waqas Nazeer
Received: 08 December 2025; Accepted: 02 January 2026; Published: 15 January 2026

Abstract: This note introduces a 1-parameter of cubic curves naturally associated to the sphere S^4 considered in the unique 5-dimensional irreducible representation space of $SO(3)$. Eight examples are discussed with the last two being elliptic curves. Also, two conics are defined naturally in our setting by a special basis of the Lie algebra $sl(3, \mathbb{R})$.

Keywords: the Euclidean spheres S^4 and S^2, cubic curve, cubic discriminant, conics

MSC: 14H52, 53A04.

1. Introduction

The paper [1] of Nigel Hitchin starts with a very interesting approach of the unit sphere S^4. More precisely, this sphere is not considered directly in the Euclidean space $\mathbb{E}^5 := (\mathbb{R}^5, g_{can})$ but is the unit sphere in a five-dimensional linear space of real 3×3 matrices. Concretely, the space is the intersection $Sym_0(3) := Sym(3) \cap sl(3, \mathbb{R})$ where $Sym(3)$ is the six-dimensional space of symmetric matrices while $sl(3, \mathbb{R})$ is the usual Lie algebra of the special Lie group $SL(3, \mathbb{R})$ of dimension 8. The space $Sym_0(3)$ arises naturally in the Cartan decomposition:

$$sl(3, \mathbb{R}) = Sym_0(3) \oplus so(3); \quad dim : 8 = 5 + 3.$$

Considering a diagonal matrix as being defined by $diag(\lambda_1, \lambda_2, \lambda_3)$ we use the traceless property of elements of $sl(3, \mathbb{R})$ in order to define the cubic curve $\mathcal{C} : y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$. Hence, the present work focuses on the study of the 1-parameter family of cubics $\mathcal{C} = \mathcal{C}(u)$, $u \in [0, 2\pi)$ provided by S^4 as unit sphere in $Sym(3) \cap sl(3, \mathbb{R})$ endowed with the inner product $Trace(B_1 B_2)$. So, we compute the Weierstrass coefficients p, q and the discriminant Δ of $\mathcal{C}(u)$. Also, a main subject is that of examples and hence an elliptic curve is discussed after two singular cubics. We finish the first part with a non-existence result of Euclidean cubic polynomials for our setting.

In the second part of this paper we consider conics naturally provided by the symmetric matrices from a specific basis of $sl(3, \mathbb{R})$. We point out that this research continues that of [2] where cubics and conics are geodesically associated to the points of a geometric surface.

2. Cubic curves from the matrix approach of S^4

Nigel Hitchin introduces in [1] a 1-parameter family of Einstein metrics on the standard sphere S^4. The parameter is an integer $k \geq 3$ and each metric is self-dual, $SO(3)$-invariant, of positive scalar curvature, and noncompact but admits a compactification as a metric with conical singularities. In fact, for $k = 3$, there is no singularity at all, and the metric is the standard one on the 4-sphere.

The underlying sphere S^4 is considered as the unit sphere in the 5-dimensional linear space $Sym_0(3) := Sym(3) \cap sl(3, \mathbb{R})$ of symmetric and traceless 3×3 real matrices B, with $SO(3)$ acting by conjugation. The invariant inner product of $Sym_0(3)$ is defined as $Trace(B_1 B_2)$ and hence S^4 corresponds to the matrices $B \in Sym_0(3)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ satisfying:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1. \end{cases} \tag{1}$$

The intersection of the plane $\Pi : \lambda_1 + \lambda_2 + \lambda_3 = 0$ with the sphere S^2 provided by the second equation (1) yields the ellipse:

$$E : (\lambda_2 - \lambda_1)^2 + 3(\lambda_1 + \lambda_2)^2 = 2, \tag{2}$$

having the eccentricity:

$$\mathfrak{e} = \sqrt{\frac{2}{3}} \simeq 0.81649. \tag{3}$$

The inverse $\frac{1}{\mathfrak{e}} = \sqrt{\frac{3}{2}} \simeq 1.2247$ is the eccentricity of the self-complementary hyperbolas, conform [3], while the intersection of the plane Π with the general rotational ellipsoid $\mathcal{E} : \frac{\lambda_1^2}{a^2} + \frac{\lambda_2^2}{b^2} + \frac{\lambda_3^2}{c^2} = 1$, for $a, b, c > 0$, is the ellipse:

$$E(a,b,c) : \left(\frac{1}{a^2} + \frac{1}{c^2}\right)\lambda_1^2 + \frac{2}{c^2}\lambda_1\lambda_2 + \left(\frac{1}{b^2} + \frac{1}{c^2}\right)\lambda_2^2 - 1 = 0,$$

with the eccentricity:

$$\mathfrak{e}^2 = \mathfrak{e}^2(a,b,c) = \frac{2\sqrt{(a^2-b^2)^2c^4 + 4a^4b^4}}{2a^2b^2 + c^2(a^2+b^2) + \sqrt{(a^2-b^2)^2c^4 + 4a^4b^4}}.$$

It results that the eigenvalues λ defines a curve on the sphere S^2:

$$C : \bar{r}(\varphi) = (\lambda_1, \lambda_2, \lambda_3)(\varphi) := \left(\frac{1}{2}(\mathfrak{e}\sin\varphi - \sqrt{2}\cos\varphi), \frac{1}{2}(\mathfrak{e}\sin\varphi + \sqrt{2}\cos\varphi), -\mathfrak{e}\sin\varphi\right), \quad \varphi \in [0, 2\pi). \tag{4}$$

We point out that in [1, p. 192] the eigenvalues are expressed with rational functions of

$$t = \left|\tan\left(\frac{3\pi}{4} + \frac{\varphi}{2}\right)\right| = \left|\frac{\tan\frac{\varphi}{2} - 1}{\tan\frac{\varphi}{2} + 1}\right|,$$

but we prefer a trigonometrical approach.

Due to the first identity (1) we introduce now a parametrized cubic curve naturally associated to this setting:

$$\mathcal{C}(\varphi) : y^2 = P_\varphi(x) := (x - \lambda_1(\varphi))(x - \lambda_2(\varphi))(x - \lambda_3(\varphi)), \quad \varphi \in [0, 2\pi), \tag{5}$$

and then we have immediately the main theoretical result of this note:

Proposition 1. *The Weierstrass coefficients p, q of the cubic curve $\mathcal{C}(\varphi) : y^2 = x^3 + px + q$ are:*

$$p = constant = -\frac{1}{2}, \quad q = q(\varphi) = -\frac{\sqrt{6}}{18}\sin(3\varphi) \in \left[q_{min} = -\frac{1}{3\sqrt{6}}, q_{max} = \frac{1}{3\sqrt{6}}\right]. \tag{6}$$

The discriminant $\Delta := 4p^3 + 27q^2$ of $\mathcal{C}(\varphi)$ is:

$$\Delta(\varphi) = -\frac{1}{2}\cos^2(3\varphi) \in \left[-\frac{1}{2}, 0\right]. \tag{7}$$

The discriminant map Δ has an unique fixed point ($\Delta(\varphi) = \varphi$), namely $\varphi_0 \simeq -0.257079$.

Proof. We have directly:

$$\Delta = 27 \cdot \frac{6}{18^2}\sin^2(3\varphi) - \frac{1}{2} = \frac{1}{2}(\sin^2(3\varphi) - 1). \tag{8}$$

The value of the fixed point is obtained by using WolframAlpha as solution $\cos^2(3\varphi) = -2\varphi$. It is easy to prove the uniqueness of this fixed point: let us consider the function $f : \mathbb{R} \to R$, $f(x) := 2x - \cos^2(3x)$. This function is smooth and it has the derivative $f'(x) = 2 - 3\sin(6x)$. For $x \in \left[-\frac{1}{2}, 0\right]$ it results $6x \in [-3, 0] \subset (-\pi, 0]$ and hence $\sin(6x) < 0$ which means that the restriction of f to the interval $\left[-\frac{1}{2}, 0\right]$ is strictly increasing. □

Remark 1. i) For the sake of completeness we provide also the expressions in t:

$$q(t) = \frac{(1-t^2)(1-14t^2+t^4)}{3\sqrt{6}(1+t^2)^3}, \quad \Delta(t) = -\frac{2t^2(t^2-3)^2(3t^2-1)^2}{(1+t^2)^6}, \quad t \in [0, +\infty]. \tag{9}$$

ii) We have the periodicity: $\Delta(\varphi + \pi) = \Delta(\varphi)$ which means that the function $\Delta(\cdot)$ corresponds to a function on the projective space $\mathbb{R}P^1$. Indeed, with $\left(\cos\varphi = \frac{u}{\sqrt{u^2+v^2}}, \sin\varphi = \frac{v}{\sqrt{u^2+v^2}}\right)$ we have the function:

$$\Delta : \mathbb{R}P^1 \to \mathbb{R}, \quad [u,v] \to \Delta([u,v]) = -\frac{u^2(u^2-3v^2)^2}{2(u^2+v^2)^3}. \tag{10}$$

iii) Due to the expression of the coefficient q we note the trigonometrical identity:

$$\sin(3\varphi) = 4\sin\varphi\sin\left(\varphi + \frac{\pi}{3}\right)\sin\left(\varphi + \frac{2\pi}{3}\right), \tag{11}$$

which was used in [4] in order to prove the famous Morley's theorem (1899).

iv) A very useful remark of the anonymous referee is that the 1-parameter cubic curve $\mathcal{C}(\varphi)$ is based only on the spectral data $(\lambda_1, \lambda_2, \lambda_3)$. Hence, it remains a (very interesting) open problem if the dependence of 3φ captures some geometrical or representation-theoretic information(s) about the Hitchin's matrix-model viewpoint of the initial round sphere S^4.

Our main interest is in studying remarkable examples. Firstly, we remark that are six singular cases provided by the vanishing of Δ from (7); in fact, for the singular angles $3\varphi \in \left\{\frac{(2k+1)\pi}{2}; k = 0, ..., 5\right\}$ we derive only *two* singular curves while a periodicity of 3 in the values of the parameter k yields antipodal points on S^2.

Example 1. For $k = 1$ we have $\varphi = \frac{\pi}{2}$ (or $t = 0$), the cubic curve $\mathcal{C}\left(\frac{\pi}{2}\right) = \mathcal{C}_1 : y^2 = x^3 - \frac{x}{2} + \frac{1}{3\sqrt{6}}$ is singular and corresponds to the point $\bar{r}\left(\frac{\pi}{2}\right) = \mathfrak{e}\left(\frac{1}{2}, \frac{1}{2}, -1\right) \in S^2$.

Example 2. Similarly, for $k = 4$ we have $\varphi = \frac{3\pi}{2}$ (or $t = +\infty$), the cubic curve $\mathcal{C}\left(\frac{3\pi}{2}\right) = \mathcal{C}_2 : y^2 = x^3 - \frac{x}{2} - \frac{1}{3\sqrt{6}}$ is singular and corresponds to the point $\bar{r}\left(\frac{3\pi}{2}\right) = -\bar{r}\left(\frac{\pi}{2}\right)$.

Example 3. For $k = 0$ we have $\varphi = \frac{\pi}{6}$ (or $t = \frac{1}{\sqrt{3}}$) and the cubic curve $\mathcal{C}\left(\frac{\pi}{6}\right) = \mathcal{C}_2$ corresponds to the point $\bar{r}\left(\frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{\mathfrak{e}-\sqrt{6}}{2}, \frac{\mathfrak{e}+\sqrt{6}}{2}, -\mathfrak{e}\right) \in S^2$.

Example 4. Similarly, for $k = 3$ we have $\varphi = \frac{7\pi}{6}$ (or $t = \sqrt{3}$) and the cubic curve $\mathcal{C}\left(\frac{7\pi}{6}\right) = \mathcal{C}_1$ corresponds to the point $\bar{r}\left(\frac{7\pi}{6}\right) = -\bar{r}\left(\frac{\pi}{6}\right)$.

Example 5. For $k = 2$ we have $\varphi = \frac{5\pi}{6}$ and the cubic curve $\mathcal{C}_1$ corresponds to the point $\bar{r}\left(\frac{5\pi}{6}\right) = \mathfrak{e}\left(1, -\frac{1}{2}, -\frac{1}{2}\right) \in S^2$.

Example 6. For $k = 5$ we have $\varphi = \frac{11\pi}{6}$ and the cubic curve $\mathcal{C}_2$ corresponds to the point $\bar{r}\left(\frac{11\pi}{6}\right) = -\bar{r}\left(\frac{5\pi}{6}\right)$.

Example 7. For $\varphi = 0$ (or $t = 1$) we have the elliptic curve $\mathcal{C}(0) : y^2 = x^3 - \frac{x}{2}$ with $\Delta = -\frac{1}{2}$ and correspond to the equatorial point $\bar{r}(0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \in S^2$. This elliptic curve appears in the LMFDB Database as https://www.lmfdb.org/EllipticCurve/Q/256/c/1 with the equation:

$$\mathcal{C}(0) : Y^2 = X^3 - 8X, \quad \Delta = -2^{11} = -2048, \quad X = 2^2 x, \quad Y = 2^3 y, \tag{12}$$

and is a CM-elliptic curve. We remark that Susumo Okubo showed in [5] that the space of 3×3 traceless complex matrices can be endowed with a multiplication, derived from the usual matrix multiplication, in such a way that it becomes a non-unital composition algebra.

In [6] we introduce the notion of *Euclidean polynomial* as being a monic polynomial for which the sum of squares of its roots is equal with the sum of squares of its proper coefficients; for the monic polynomial P_φ of (5) this means that $p^2 + q^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ which means that $q^2 = \frac{3}{4}$. We obtain a non-existence result as follows:

Proposition 2. *There are no Euclidean polynomials P_φ of (5)-type.*

Proof. From $q^2 = \frac{3}{4}$ it results $q_\pm = \pm\frac{\sqrt{3}}{2}$ but $q_+ > q_{max}$. The value q_- is greater than q_{min} but the cubic equation provided by the expression of q from (6):

$$X\left(\frac{4}{3}X^2 - 1\right) = \frac{2}{e}q_- = -\frac{3}{\sqrt{2}}, \tag{13}$$

has only one real solution $X \simeq -1.3796 < -1$ and hence it can not be a $\sin\varphi$. □

3. The geometry of the spherical curve C and two conics from $Sym_0(3)$

In this section we firstly study the Frenet geometry of curve C considered as curve in the Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, \langle\cdot,\cdot\rangle_{canonic})$:

$$C : \bar{r}(\varphi) = \mathfrak{e}\left(-\cos\left(\varphi + \frac{\pi}{6}\right), \cos\left(\varphi - \frac{\pi}{6}\right), -\sin\varphi\right) \in S^2. \tag{14}$$

Its tangent vector field is:

$$T(u) = \bar{r}'(\varphi) = \mathfrak{e}\left(\sin\left(\varphi + \frac{\pi}{6}\right), -\sin\left(\varphi - \frac{\pi}{6}\right), -\cos\varphi\right) \in S^2. \tag{15}$$

Its binormal vector field is constant since $\bar{r}$ belongs to the plane $\Pi : \lambda_1 + \lambda_2 + \lambda_3 = 0$:

$$B(u) = \bar{r}'(\varphi) \times \bar{r}''(\varphi) = -\frac{\mathfrak{e}}{\sqrt{2}}(1,1,1) \in S^2. \tag{16}$$

Then its torsion is zero while the curvature function is constant:

$$k(\varphi) = \frac{\|\bar{r}'(\varphi) \times \bar{r}''(\varphi)\|}{\|\bar{r}'(\varphi)\|^3} = 1, \tag{17}$$

which confirm that in the given plane Π this curve is the unit circle centered in $O(0,0,0)$. Its normal vector field is:

$$N(\varphi) = B(\varphi) \times T(\varphi) = \mathfrak{e}\left(\cos\left(\varphi + \frac{\pi}{6}\right), -\cos\left(\varphi - \frac{\pi}{6}\right), \sin\varphi\right) = -\bar{r}(\varphi) \in S^2. \tag{18}$$

We have studied recently other classes of spherical curves in both S^2 and S^3 in [7], [8] and [9].

Example 8. We point out that an interesting problem is when the matrix $Frenet(\varphi) = (T(\varphi), N(\varphi), B(\varphi)) \in SO(3)$ is symmetric. A straightforward computation yields the characterization $Frenet(\varphi) \in Sym(3)$ if and only if $\varphi = -\frac{\pi}{4}$ when:

$$Frenet\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1-\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} & -1 \\ \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} & -1 \\ -1 & -1 & -1 \end{pmatrix}, \quad TraceFrenet\left(-\frac{\pi}{4}\right) = -1. \tag{19}$$

The corresponding elliptic curve is:

$$\mathcal{C}\left(-\frac{\pi}{4}\right) : y^2 = x^3 - \frac{x}{2} + \frac{1}{6\sqrt{3}}, \quad \Delta\left(-\frac{\pi}{4}\right) = -\frac{1}{4}. \tag{20}$$

Secondly, we associate two conics to our linear space $Sym_0(3)$. To do this, we recall after [10, p. 13] that the semisimple Lie algebra $sl(3,\mathbb{R})$ is spanned by the matrices:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = -E_1^t, \quad D = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{21}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_1 = -P_2^t, \quad R_2 = P_1^t, \tag{22}$$

where the subscript t means the transposition map. The motivation for this choice of basis consists in the presence of E_2 and D as elements in $Sym_0(3)$.

With the approach of [3] a matrix $\Gamma \in Sym(3)$ defines naturally a conic in the Euclidean plane $\mathbb{E}^2$ of coordinates (x,y) through the equation:

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \cdot \Gamma \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{23}$$

In conclusion, there are two associated conics:

i) E_2 is the degenerate hyperbola consisting in the pair of bisectrices $B_1 : y = x$, $B_2 : y = -x$,

ii) D is the circle centered in the origin $O(0,0) \in \mathbb{R}$ and having the radius $R = \sqrt{2}$.

The symmetric matrix (19) yields a hyperbola with the eccentricity:

$$\tilde{e} = \sqrt{1 + \frac{1}{\sqrt{3}}} \simeq 1.2559. \tag{24}$$

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: No data is required for this research.

Funding Information: No funding is available for this research.

Acknowledgments: The author is extremely indebted to an anonymous referee which has improved significantly a first version of this paper.

References

[1] Hitchin, N. J. (1996). *A new Family of Einstein Metrics. de Bartolomeis, Paolo (Ed.) et al., Manifolds and Geometry.* Cambridge: Cambridge University Press. Symp. Math. 36, 190–222. Zbl 0858.53038

[2] Crasmareanu, M. (2025). Cubics and conics geodesically associated to the points of a geometric surface. In *Annales Mathematicae et Informaticae* (pp. 46-54).

[3] Crasmareanu, M. (2021). Magic conics, their integer points and complementary ellipses. *Analele Ştiinţifice ale Universităţii'Al. I. Cuza'din Iaşi. Matematică, 67*(1), 129-148.

[4] Hirschhorn, M. D. (1998). A trigonometric identity and Morley's theorem. *Australian Mathematical Society Gazette, 25*, 198-199.

[5] Okubo, S. (1978). Pseudo-quartenion and pseudo-octonion algebras. *Hadronic J.;(United States), 1*(4), 1250–1278.

[6] Crasmareanu, M. (2022). The diagonalization map as submersion, the cubic equation as immersion and Euclidean polynomials. *Mediterranean Journal of Mathematics, 19*(2), 65.

[7] Crasmareanu, M. (2024). The flow-geodesic curvature and the flow-evolute of spherical curves. *Turkish Journal of Mathematics, 48*(4), 749-756.

[8] Crasmareanu, M. (2024). The Darboux mate and the higher order curvatures of spherical Legendre curves. In *Geometry of Submanifolds and Applications* (pp. 197-207). Singapore: Springer Nature Singapore.

[9] Crasmareanu, M. (2025). Some remarks on the moment curves in $\mathbb{R}^4$ and $\mathbb{R}^3$. *Istanbul Journal of Mathematics, 3*(1), 34-38.

[10] Chapovskyi, Y. Y., Koval, S. D., & Zhur, O. (2024). Subalgebras of Lie algebras. Example of $\mathfrak{sl}_3(\mathbb{R})$ revisited. arXiv preprint arXiv:2403.02554.