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Different hypergeometric representations for classical orthogonal polynomial systems

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Abstract: Classical orthogonal polynomials of the Askey-Wilson scheme have many different properties, e.g. they satisfy differential and recurrence equations and they have hypergeometric representations, Rodrigues formulas, generating functions, moment representations, etc. In this paper we concentrate on finding multiple hypergeometric representations for the polynomial sequences belonging to the classical continuous and classical discrete classes that are defined on a linear lattice. Currently such a database is not available. Using computer algebra, especially Zeilberger's algorithm, it is possible to prove such identities and therefore the paper is accompanied by a Maple worksheet containing derivations or proofs of all given identities, most of which are new.

Keywords: classical continuous orthogonal polynomials, classical discrete orthogonal polynomials, hypergeometric representations, computer algebra

MSC: 33C20, 33F10, 68W30.

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1. Introduction

Given: a *scalar product*

$$\langle f,g\rangle := \int_\alpha^\beta f(x)g(x)\,d\mu(x),$$

with non-negative Borel measure $\mu(x)$ supported in the interval $[\alpha,\beta]$. The following special cases are most important:

- *absolutely continuous* measure $d\mu(x) = \rho(x)\,dx$ with weight function $\rho(x)$,
- *discrete* measure $\mu(x) = \rho(x)$ supported in $\mathbb{Z}$.

A system of polynomials $(P_n(x))_{n\geq 0}$

$$P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots, \quad k_n \neq 0, \tag{1}$$

is called an *orthogonal polynomial sequence* (OPS) with respect to the *positive-definite* measure $d\mu(x)$, if

$$\langle P_m, P_n\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ h_n > 0 & \text{if } m = n. \end{cases}$$

In this paper we will refer to the OPS in [1], which encyclopedically presents the OPS of the so-called Askey-Wilson scheme. The Askey-Wilson scheme contains OPS that are defined on linear and quadratic lattices and have hypergeometric or q-hypergeometric representations, whereas in this paper only OPS on linear lattices are covered.

Euler was the first to show some interest in hypergeometric series, and Pfaff, a teacher of Gauss, discovered an important transformation [2, (2.2.6)], connecting several ${}_2F_1$ (or Gauss) hypergeometric series, given by:

$${}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the Pochhammer symbol. Gauss recognized the relevance of these hypergeometric series and showed that many special functions have ${}_2F_1$ hypergeometric representations, e.g. the logarithmic function, arcsin, arctan, and also some classical orthogonal polynomials like Jacobi polynomials. Different representations of especially the Gauss hypergeometric function, are used in the study of Coulomb wave functions [3]. We also refer the reader to [4] where the computation of hypergeometric representations is discussed.

Our aim is to provide a database with different hypergeometric representations of the OPS defined on linear lattices. Such a database does not exist; one can find *some* representations in any of the underlying literature, but there is no place where one can find all of them at once.

In order to obtain new hypergeometric representations for the ${}_2F_1$ hypergeometric polynomials, the following transformations are helpful:

(1) the Pfaff transformation [2, (2.2.6)], which states that

$${}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) = \frac{1}{(1-z)^a}\, {}_2F_1\left(\left.\begin{matrix}a, c-b\\ c\end{matrix}\right| \frac{z}{z-1}\right) = \frac{1}{(1-z)^b}\, {}_2F_1\left(\left.\begin{matrix}c-a, b\\ c\end{matrix}\right| \frac{z}{z-1}\right); \tag{2}$$

(2) the Euler transformation [2, (2.2.7)]:

$${}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(\left.\begin{matrix}c-a, c-b\\ c\end{matrix}\right| z\right); \tag{3}$$

(3) the transformation provided in [2, (2.3.14)], also attributed to Pfaff:

$${}_2F_1\left(\left.\begin{matrix}-n, b\\ c\end{matrix}\right| x\right) = \frac{(c-b)_n}{(c)_n}\, {}_2F_1\left(\left.\begin{matrix}-n, b\\ b+1-n-c\end{matrix}\right| 1-x\right). \tag{4}$$

In §2 and §3 of our paper we provide more information on continuous and discrete OPS in general. In §4, we discuss the generalized hypergeometric series and provide the algorithm used to find some (related) representations. In §5 and §6, we provide the database with *different* hypergeometric representations for the classical continuous and discrete OPS, respectively. The proofs for all our given representations can be found on https://www.mathematik.uni-kassel.de/~koepf/Publikationen.

2. Classical continuous orthogonal polynomials

The *classical* continuous OPS $(P_n(x))_{n\geqq 0}$ can be defined as the polynomial solutions of the *differential equation*:

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) - \lambda_n P_n(x) = 0\,. \tag{5}$$

Substituting (1) into (5), we conclude:

- $n = 1$ yields $\tau(x) = dx + e,\ d \neq 0$,
- $n = 2$ yields $\sigma(x) = ax^2 + bx + c$,
- The coefficient of x^n yields $\lambda_n = n(a(n-1) + d)$.

These classical continuous families can be classified (modulo linear transformations) according to the following scheme [5,6]

- $\sigma(x) = 0$ powers x^n,
- $\sigma(x) = 1$ *Hermite* polynomials,
- $\sigma(x) = x$ *Laguerre* polynomials,
- $\sigma(x) = 1 - x^2$ *Jacobi* polynomials,

- $\sigma(x) = x^2$ *Bessel* polynomials,
- $\sigma(x) = 1 + x^2$ *Masjed-Jamei* polynomials.

The corresponding weight function $\rho(x)$ satisfies the *Pearson Differential Equation*

$$\frac{d}{dx}\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x) \; .$$

Hence the weight function is given by

$$\rho(x) = \frac{C}{\sigma(x)} e^{\int \frac{\tau(x)}{\sigma(x)} dx} \; .$$

3. Classical discrete orthogonal polynomials

Analogously to the continuous case, the *classical discrete* OPS can be defined as solutions of the *difference equation* [7]:

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) - \lambda_n P_n(x) = 0,$$

where $\Delta f(x) = f(x+1) - f(x)$ and $\nabla f(x) = f(x) - f(x-1)$ denote the forward and backward difference operators. As in the continuous case, we get

- $n = 1$ yields $\tau(x) = dx + e, d \neq 0$,
- $n = 2$ yields $\sigma(x) = ax^2 + bx + c$,
- The coefficient of x^n yields $\lambda_n = n(a(n-1) + d)$.

The classical discrete families can be classified (modulo linear transformations) according to the following scheme [7], see also [8]:

- $\sigma(x) = 0$ falling factorials $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$,
- $\sigma(x) = 1$ shifted *Charlier* polynomials,
- $\sigma(x) = x$ *Charlier*, *Meixner*, *Krawtchouk* polynomials,
- $\deg(\sigma(x), x) = 2$ *Hahn* polynomials.

The corresponding discrete weight function $\rho(x)$ satisfies the Pearson difference equation

$$\Delta\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x) \; .$$

Hence it is given by the term ratio

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x+1)} \; .$$

4. Hypergeometric series and representations

The power series

$${}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty}, A_k \, z^k \; ,$$

whose summands $\alpha_k = A_k z^k$ have a rational term ratio

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{A_{k+1} \, z^{k+1}}{A_k \, z^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \frac{z}{(k+1)} \; ,$$

is called the *generalized hypergeometric series*. The summand $\alpha_k = A_k z^k$ of a hypergeometric series is called a *hypergeometric term*. It is well-known and can be proved in many different ways that all the above OPS can be represented by generalized hypergeometric series.

For the coefficients of the generalized hypergeometric series one gets the following formula

$${}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

using the *Pochhammer symbol* $(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$.

If a hypergeometric representation

$$S(n) := \sum_{k=0}^{\infty} F(n,k),$$

is given, where $F(n,k)$ is a hypergeometric term w.r.t. both n and k, i.e. $\frac{F(n+1,k)}{F(n,k)} \in \mathbb{K}(n,k)$ and $\frac{F(n,k+1)}{F(n,k)} \in \mathbb{K}(n,k)$, where $\mathbb{K}(n,k)$ denotes the field of rational functions in the variables n and k over $\mathbb{K}$, mostly a transcendental extension of $\mathbb{Q}$ (adjoining some variables), Zeilberger's algorithm [9,10] computes the desired normal form, namely a holonomic recurrence equation (linear, homogeneous, with polynomial coefficients) for $S(n)$ and enough initial values. If two different hypergeometric representations are given, satisfying the same recurrence equation and having enough identical initial values, then this proves that the two representations represent the same family. This paper is accompanied by a Maple worksheet proving all the given identities obtained from this method, using in particular the *Maple* command `sumrecursion` from the `hsum` package given in [9].

Note that Zeilberger-type algorithms and the `hsum` package are also used in the on-line resource [11], where the OPS of the Askey-Wilson scheme are defined and their normal forms can be computed instantaneously.

We want to show that some of the representations that we develop are related to each other and we would like to generate new ones. For this purpose, we use the following idea. If one hypergeometric representation of an OPS is given, the following algorithm generates another one.

Algorithm 1. (Reversion) Let the hypergeometric representation

$$P_n = \sum_{k=0}^{n} F(n,k),$$

with hypergeometric term $F(n,k)$ w.r.t. both n and k be given. Then by reversing the order of summation, we get another hypergeometric representation

$$P_n = \sum_{j=0}^{n} F(n,n-j), \ .$$

Proof. Since by assumption $F(n,k)$ is a hypergeometric term, i.e. $\frac{F(n,k+1)}{F(n,k)} \in \mathbb{K}(n,k)$ and $\frac{F(n+1,k)}{F(n,k)} \in \mathbb{K}(n,k)$, after reversion the summand is $G(n,j) = F(n,n-j)$, and clearly $\frac{G(n,j+1)}{G(n,j)} \in \mathbb{K}(n,j)$ and $\frac{G(n+1,j)}{G(n,j)} \in \mathbb{K}(n,j)$. ☐

5. Hypergeometric representations for the classical continuous OPS

5.1. Jacobi polynomials

The Jacobi polynomials are given by [1, (9.8.1)] as

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) =& \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\, \frac{1-x}{2}\right) \\ =& \binom{n+\alpha}{n} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\, \frac{1-x}{2}\right). \end{aligned} \tag{6}$$

Note that (6) forms a power series with point of development $x_0 = 1$. This is not surprising since the orthogonality interval is $[-1,1]$. By using symmetry relation [12, Table 18.6.1]

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n \, P_n^{(\beta,\alpha)}(x), \tag{7}$$

we obtain another hypergeometric representation, given by

$$P_n^{(\alpha,\beta)}(x)=(-1)^n\binom{n+\beta}{n}{}_2F_1\left(\left.\begin{matrix}-n,n+\alpha+\beta+1\\ \beta+1\end{matrix}\right|\frac{1+x}{2}\right), \tag{8}$$

which constitutes a power series representation with point of development $x_0 = -1$. Using the Petkovšek-van-Hoeij algorithm [9, Chapter 9] one can prove that the power series representation of $P_n^{(\alpha,\beta)}(x)$ with $x_0 = 0$ is not hypergeometric. However, there are many other representations available that are not necessarily power series, obtained from [13, (22.5.43), (22.5.44), (22.5.45)], respectively:

$$\begin{aligned}P_n^{(\alpha,\beta)}(x)=&\binom{2n+\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\alpha\\ -2n-\alpha-\beta\end{matrix}\right|\frac{2}{1-x}\right) &(9)\\ =&\binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\beta\\ \alpha+1\end{matrix}\right|\frac{x-1}{x+1}\right) &(10)\\ =&\binom{n+\beta}{n}\left(\frac{x-1}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\alpha\\ \beta+1\end{matrix}\right|\frac{x+1}{x-1}\right). &(11)\end{aligned}$$

We use Algorithm 1 to show that some of the above representations are related to each other, as well as to generate new representations. It is easy to observe that (9) is the reversion of (6), (11) is the reversion of (10) and the reversion of (8) is the new representation

$$P_n^{(\alpha,\beta)}(x)=\binom{2n+\alpha+\beta}{n}\left(\frac{1+x}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\beta\\ -2n-\alpha-\beta\end{matrix}\right|\frac{2}{1+x}\right). \tag{12}$$

By using the symmetry relation (7), we don't get any representations other than (6) and (8)–(12); therefore, to the best of our knowledge, these are the only finite hypergeometric representations for the Jacobi polynomials.

However, besides the finite representations which have an upper parameter $-n$ and therefore not more than $(n+1)$ summands, there are also some infinite hypergeometric series representations that we would like to consider in the remaining part of this section. Please note that we can also compute recurrence equations for the following infinite series representations, but the initial values for these examples are infinite series.

By substituting $a=-n,\ b=n+\alpha+\beta+1,\ c=\alpha+1,\ z=1/2-x/2$ or $a=-n,\ b=n+\alpha+\beta+1,\ c=\beta+1,\ z=1/2+x/2$ into [12, (15.8.6) and (15.8.7)], we again obtain the representations in (6) and (8)–(12). The same substitution into (2) and (3) yields the following additional series representations for the Jacobi polynomials:

$$\begin{aligned}P_n^{(\alpha,\beta)}(x)&=\binom{n+\alpha}{n}\left(\frac{x+1}{2}\right)^{-\alpha-\beta-n-1}{}_2F_1\left(\left.\begin{matrix}\alpha+1+n,n+\alpha+\beta+1\\ \alpha+1\end{matrix}\right|\frac{x-1}{x+1}\right)\\ &=\binom{n+\alpha}{n}\left(\frac{x+1}{2}\right)^{-\beta}{}_2F_1\left(\left.\begin{matrix}\alpha+1+n,-n-\beta\\ \alpha+1\end{matrix}\right|\frac{1-x}{2}\right)\\ &=(-1)^n\binom{n+\beta}{n}\left(\frac{1-x}{2}\right)^{-\alpha-\beta-n-1}{}_2F_1\left(\left.\begin{matrix}n+\beta+1,n+\alpha+\beta+1\\ \beta+1\end{matrix}\right|\frac{1+x}{x-1}\right)\\ &=(-1)^n\binom{n+\beta}{n}\left(\frac{1-x}{2}\right)^{-\alpha}{}_2F_1\left(\left.\begin{matrix}n+\beta+1,-\alpha-n\\ \beta+1\end{matrix}\right|\frac{x+1}{2}\right).\end{aligned}$$

These 4 representations are new, as well as the representation (12). Please note that the normal forms of the representations (6) and (8)–(12) are computed in the *Maple* worksheet `Different_hypergeometric_representations.mw` which can be accessed at https://www.mathematik.uni-kassel.de/~koepf/Publikationen. Also note that Algorithms 2.2 and 2.8 in [9], that are implemented in the `hsum` package, can be used for the identification of the hypergeometric series. For the above infinite series representations, we also compute recurrence equations in the mentioned Maple worksheet, and the initial values are computed by Maple's `simplify` command.

5.2. Gegenbauer polynomials

When the parameters of the Jacobi polynomials are identical, i.e., $\alpha=\beta=\lambda-\frac{1}{2}$, we get the Gegenbauer polynomials [1, (9.8.19)]

$$C_n^{(\lambda)}(x)=\frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x), \tag{13}$$

with hypergeometric representation [1, (9.8.19)]

$$\begin{aligned} C_n^{(\lambda)}(x)=&\frac{(2\lambda)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}\right|\frac{1-x}{2}\right)\\ =&\binom{n+2\lambda-1}{n}\,{}_2F_1\left(\left.\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}\right|\frac{1-x}{2}\right), \end{aligned} \tag{14}$$

corresponding to (6). From the remaining Jacobi representations (8)–(12), we now have

$$C_n^{(\lambda)}(x)=\binom{n+2\lambda-1}{n}(-1)^n{}_2F_1\left(\left.\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}\right|\frac{1+x}{2}\right) \tag{15}$$

$$=\binom{n+\lambda-1}{n}2^n\,(x-1)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\lambda+\frac{1}{2}\\ -2n-2\lambda+1\end{matrix}\right|\frac{2}{1-x}\right) \tag{16}$$

$$=\binom{n+2\lambda-1}{n}\left(\frac{1+x}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\lambda+\frac{1}{2}\\ \lambda+1/2\end{matrix}\right|\frac{x-1}{x+1}\right) \tag{17}$$

$$=\binom{n+2\lambda-1}{n}\left(\frac{x-1}{2}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-n-\lambda+\frac{1}{2}\\ \lambda+\frac{1}{2}\end{matrix}\right|\frac{x+1}{x-1}\right) \tag{18}$$

$$=\binom{n+\lambda-1}{n}2^n\,(x+1)^n\,{}_2F_1\left(\left.\begin{matrix}-n,-n-\lambda+\frac{1}{2}\\ -2n-2\lambda+1\end{matrix}\right|\frac{2}{x+1}\right). \tag{19}$$

Whereas the power series expansion at $x_0=0$ of the Jacobi family is not hypergeometric, this is no longer true if $\alpha=\beta=\lambda-\frac{1}{2}$, and the Gegenbauer polynomials have the additional hypergeometric power series representation [12, (15.9.3)]

$$\begin{aligned} C_n^{(\lambda)}(x)=&(2x)^n\frac{(\lambda)_n}{n!}{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}n,\frac{1}{2}(1-n)\\ 1-\lambda-n\end{matrix}\right|\frac{1}{x^2}\right)\\ =&(2x)^n\binom{n+\lambda-1}{n}\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}n,\frac{1}{2}(1-n)\\ 1-\lambda-n\end{matrix}\right|\frac{1}{x^2}\right). \end{aligned} \tag{20}$$

By treating polynomials with degrees $2n$ and $2n+1$ separately, this representation leads to the two representations (cf. [1, p. 221])

$$C_{2n}^{(\lambda)}(x)=(2x)^{2n}\binom{2n+\lambda}{2n}\,{}_2F_1\left(\left.\begin{matrix}-n,\frac{1}{2}(1-2n)\\ 1-\lambda-2n\end{matrix}\right|\frac{1}{x^2}\right), \tag{21}$$

and

$$C_{2n+1}^{(\lambda)}(x)=(2x)^{2n+1}\binom{2n+1+\lambda}{2n+1}\,{}_2F_1\left(\left.\begin{matrix}-n,-\frac{1}{2}(2n+1)\\ \lambda-2n\end{matrix}\right|\frac{1}{x^2}\right). \tag{22}$$

Note that the latter representations can again be reversed by Algorithm 1 to obtain

$$C_{2n}^{(\lambda)}(x) = (-1)^n \binom{n+\lambda-1}{n} \, {}_2F_1\left(\left.\begin{matrix} -n, n+\lambda \\ \frac{1}{2} \end{matrix}\right| x^2\right), \tag{23}$$

and

$$C_{2n+1}^{(\lambda)}(x) = 2\lambda \, x \, (-1)^n \binom{n+\lambda}{n} \, {}_2F_1\left(\left.\begin{matrix} -n, n+\lambda+1 \\ \frac{3}{2} \end{matrix}\right| x^2\right). \tag{24}$$

We have collected seven different hypergeometric representations for the Gegenbauer polynomials. If we take into account that we also have representations for polynomials of even and odd degrees, eight different hypergeometric representations for the Gegenbauer polynomials are available. Representation (19), that stems from the new Jacobi representation (12), as well as (23) and (24), considered as one representation, are new.

5.3. Legendre polynomials

The Legendre polynomials are a particular case of the Jacobi and Gegenbauer polynomials and are defined by [1, (9.8.62)]

$$P_n(x) = P_n^{(0,0)}(x) = C_n^{(\frac{1}{2})}(x) = {}_2F_1\left(\left.\begin{matrix} -n, n+1 \\ 1 \end{matrix}\right| \frac{1-x}{2}\right). \tag{25}$$

From the hypergeometric representations for the Gegenbauer polynomials (14)–(24), we therefore obtain the following representations for the Legendre polynomials:

$$\begin{aligned}
P_n(x) =& (-1)^n \, {}_2F_1\left(\left.\begin{matrix} -n, n+1 \\ 1 \end{matrix}\right| \frac{1+x}{2}\right) && (26)\\
=& \binom{2n}{n} \left(\frac{x-1}{2}\right)^n {}_2F_1\left(\left.\begin{matrix} -n, -n \\ -2n \end{matrix}\right| \frac{2}{1-x}\right) && (27)\\
=& \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\left.\begin{matrix} -n, -n \\ 1 \end{matrix}\right| \frac{x-1}{x+1}\right) && (28)\\
=& \left(\frac{x-1}{2}\right)^n {}_2F_1\left(\left.\begin{matrix} -n, -n \\ 1 \end{matrix}\right| \frac{x+1}{x-1}\right) && (29)\\
=& \binom{2n}{n} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\left.\begin{matrix} -n, -n \\ -2n \end{matrix}\right| \frac{2}{1+x}\right) && (30)\\
=& \binom{2n}{n} \frac{x^n}{2^n} \, {}_2F_1\left(\left.\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ \frac{1}{2}-n \end{matrix}\right| \frac{1}{x^2}\right), && (31)
\end{aligned}$$

where (30) is obtained from the new Jacobi representation (12).

For polynomials of even degree, we also have

$$\begin{aligned}
P_{2n}(x) =& \binom{4n}{2n} \frac{x^{2n}}{4^n} {}_2F_1\left(\left.\begin{matrix} -n, -n+\frac{1}{2} \\ \frac{1}{2}-2n \end{matrix}\right| \frac{1}{x^2}\right) && (32)\\
=& \binom{2n}{n} \frac{(-1)^n}{4^n} {}_2F_1\left(\left.\begin{matrix} -n, n+\frac{1}{2} \\ \frac{1}{2} \end{matrix}\right| x^2\right), && (33)
\end{aligned}$$

and when the degree is odd, we obtain the following representations:

$$\begin{aligned}
P_{2n+1}(x) =& \binom{4n+2}{2n+1} \frac{x^{2n+1}}{2 \cdot 4^n} {}_2F_1\left(\left.\begin{matrix} -n, -\frac{1}{2}-n \\ -\frac{1}{2}-2n \end{matrix}\right| \frac{1}{x^2}\right) && (34)\\
=& x \left(\frac{3}{2}\right)_n \frac{(-1)^n}{n!} {}_2F_1\left(\left.\begin{matrix} -n, \frac{3}{2}+n \\ \frac{3}{2} \end{matrix}\right| x^2\right). && (35)
\end{aligned}$$

In the particular case of the Legendre polynomials, further hypergeometric representations are known (cf. [9, Exercise 13.11])

$$P_n(x) = x^n {}_2F_1\left(\begin{matrix}-\frac{n}{2}, -\frac{n-1}{2} \\ 1\end{matrix}\middle| 1-\frac{1}{x^2}\right), \tag{36}$$

and therefore, a polynomial with even degree can be represented as:

$$P_{2n}(x) = x^{2n} {}_2F_1\left(\begin{matrix}-n, -n+\frac{1}{2} \\ 1\end{matrix}\middle| 1-\frac{1}{x^2}\right), \tag{37}$$

and when the degree is odd, we have the representation:

$$P_{2n+1}(x) = x^{2n+1} {}_2F_1\left(\begin{matrix}-n, -n-\frac{1}{2} \\ 1\end{matrix}\middle| 1-\frac{1}{x^2}\right). \tag{38}$$

Again we can use Algorithm 1 and obtain one more independent hypergeometric representation for each of P_{2n} and P_{2n+1}:

$$P_{2n}(x) = \binom{-\frac{1}{2}}{n} (-1)^n (x^2-1)^n {}_2F_1\left(\begin{matrix}-n, -n \\ \frac{1}{2}\end{matrix}\middle| \frac{x^2}{x^2-1}\right), \tag{39}$$

$$P_{2n+1}(x) = \binom{-\frac{1}{2}}{n} (-1)^n (x^2-1)^n \, x \, {}_2F_1\left(\begin{matrix}-n, -n \\ \frac{3}{2}\end{matrix}\middle| \frac{x^2}{x^2-1}\right). \tag{40}$$

New representations for the Legendre polynomials are therefore (30), (39) and (40).

5.4. Chebyshev polynomials

When we let $\alpha = \beta = -\frac{1}{2}$ in the formula of the Jacobi polynomials, we have the Chebyshev polynomials of the first kind $T_n(x)$ [1, (9.8.35)], defined as

$$T_n(x) = \frac{4^n}{\binom{2n}{n}} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = {}_2F_1\left(\begin{matrix}-n, n \\ \frac{1}{2}\end{matrix}\middle| \frac{1-x}{2}\right). \tag{41}$$

The hypergeometric representations (8)–(12) of the Jacobi polynomials therefore yield

$$\begin{aligned}
T_n(x) =& (-1)^n {}_2F_1\left(\begin{matrix}n, -n \\ \frac{1}{2}\end{matrix}\middle| \frac{1+x}{2}\right) && (42)\\
=& 2^{n-1} (x-1)^n {}_2F_1\left(\begin{matrix}-n+\frac{1}{2}, -n \\ 1-2n\end{matrix}\middle| \frac{2}{x-1}\right) && (43)\\
=& \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix}-n+\frac{1}{2}, -n \\ \frac{1}{2}\end{matrix}\middle| \frac{x-1}{x+1}\right) && (44)\\
=& \left(\frac{x-1}{2}\right)^n {}_2F_1\left(\begin{matrix}-n+\frac{1}{2}, -n \\ \frac{1}{2}\end{matrix}\middle| \frac{x+1}{x-1}\right) && (45)\\
=& 2^{n-1} (x+1)^n {}_2F_1\left(\begin{matrix}-n+\frac{1}{2}, -n \\ 1-2n\end{matrix}\middle| \frac{2}{1+x}\right). && (46)
\end{aligned}$$

The Chebyshev polynomials of the second kind $U_n(x)$ [1, (9.8.36)] are special cases of the Jacobi polynomials, with $\alpha = \beta = \frac{1}{2}$, and

$$U_n(x) = \frac{n+1}{\binom{n+\frac{1}{2}}{n}} P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x) = (n+1) \cdot {}_2F_1\left(\begin{matrix}-n, n+2 \\ \frac{3}{2}\end{matrix}\middle| \frac{1-x}{2}\right). \tag{47}$$

The representations (8)–(12) for the Jacobi polynomials therefore yield

$$U_n(x) = (-1)^n\,(n+1)\cdot {}_2F_1\left(\left.\begin{matrix}-n, n+2\\ \frac{3}{2}\end{matrix}\right|\frac{x+1}{2}\right) \tag{48}$$

$$= 2^n\,(x-1)^n\,{}_2F_1\left(\left.\begin{matrix}-n-\frac{1}{2}, -n\\ -2\,n-1\end{matrix}\right|\frac{2}{1-x}\right) \tag{49}$$

$$= \left(\frac{1+x}{2}\right)^n\cdot(n+1)\cdot{}_2F_1\left(\left.\begin{matrix}-n-\frac{1}{2}, -n\\ \frac{3}{2}\end{matrix}\right|\frac{x-1}{x+1}\right) \tag{50}$$

$$= \left(\frac{x-1}{2}\right)^n\cdot(n+1)\cdot{}_2F_1\left(\left.\begin{matrix}-n-\frac{1}{2}, -n\\ \frac{3}{2}\end{matrix}\right|\frac{x+1}{x-1}\right) \tag{51}$$

$$= 2^n\,(x+1)^n\,{}_2F_1\left(\left.\begin{matrix}-n-\frac{1}{2}, -n\\ -2\,n-1\end{matrix}\right|\frac{2}{1+x}\right). \tag{52}$$

The Chebyshev polynomials of the second kind are also of Gegenbauer type for $\lambda = 1$, (cf. (14) and (47))

$$U_n(x) = \frac{\left(\frac{3}{2}\right)_n}{\binom{n+\frac{1}{2}}{n}\,n!}\,C_n^{(1)}(x) = C_n^{(1)}(x)\,.$$

Therefore, the given representation (20) is

$$U_n(x) = 2^n\,x^n{}_2F_1\left(\left.\begin{matrix}-\frac{n}{2}+\frac{1}{2}, -\frac{n}{2}\\ -n\end{matrix}\right|\frac{1}{x^2}\right), \tag{53}$$

and for even and odd indices we get, according to (21)–(24),

$$U_{2n}(x) = 4^n\,x^{2n}{}_2F_1\left(\left.\begin{matrix}-n+\frac{1}{2}, -n\\ -2n\end{matrix}\right|\frac{1}{x^2}\right) \tag{54}$$

$$= (-1)^n\,{}_2F_1\left(\left.\begin{matrix}n+1, -n\\ \frac{1}{2}\end{matrix}\right|x^2\right), \tag{55}$$

and

$$U_{2n+1}(x) = 2^{2n+1}\,x^{2n+1}{}_2F_1\left(\left.\begin{matrix}-n-\frac{1}{2}, -n\\ -2n-1\end{matrix}\right|\frac{1}{x^2}\right) \tag{56}$$

$$= 2\,x\,(-1)^n\,(n+1)\,{}_2F_1\left(\left.\begin{matrix}-n, n+2\\ \frac{3}{2}\end{matrix}\right|x^2\right). \tag{57}$$

The new representation of the Chebychev polynomials of the first kind, obtained from (12), is (46) and the representations of the Chebychev polynomials of the second kind that are new, are (52) (obtained from (12)), as well as (55) and (57), that can be considered as one representation.

5.5. Laguerre polynomials

The generalized Laguerre polynomials have the hypergeometric representation [1, (9.12.1)]

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \alpha+1\end{matrix}\right|x\right) \tag{58}$$

$$= \binom{n+\alpha}{n}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \alpha+1\end{matrix}\right|x\right),$$

which forms a power series with point of development $x_0 = 0$. This is compatible with the fact that the Laguerre polynomials are orthogonal in the interval $[0, \infty)$.

We use Algorithm 1 to reverse the above series and get

$$L_n^{(\alpha)}(x) = \frac{(-1)^n\, x^n}{n!}\, {}_2F_0\left(\left.\begin{matrix} -n, -n-\alpha \\ - \end{matrix}\right| -\frac{1}{x}\right). \tag{59}$$

Hence the generalized Laguerre polynomials have two different hypergeometric representations, one of the form ${}_1F_1$ and another one of the form ${}_2F_0$. The representation (59) is new.

5.6. Bessel polynomials

The Bessel polynomials have the hypergeometric representations, both given in [1, (9.13.1)]

$$\begin{aligned} y_n(x; a) =& {}_2F_0\left(\left.\begin{matrix} -n, n+a+1 \\ - \end{matrix}\right| -\frac{x}{2}\right) && (60) \\ =&(n+a+1)_n \left(\frac{x}{2}\right)^n {}_1F_1\left(\left.\begin{matrix} -n \\ -2n-a \end{matrix}\right| \frac{2}{x}\right). && (61) \end{aligned}$$

Of course it turns out that (61) is the reversion of (60). Again, the Bessel polynomials have two different hypergeometric representations, one of the form ${}_1F_1$ and another one of the form ${}_2F_0$.

5.7. Masjed-Jamei polynomials

In [6] Masjed-Jamei defined the following finite OPS

$$\begin{aligned} &J_n^{(p,q)}(x; a, b, c, d) = (-1)^n (ab+cd+i(ad-bc))^n (n+1-2p)_n \\ &\cdot \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a^2+c^2}{ab+cd+i(ad-bc)}\right)^k {}_2F_1\left(\left.\begin{matrix} k-n, p-n-i\frac{q}{2} \\ 2p-2n \end{matrix}\right| \frac{2(ad-bc)}{ad-bc-i(ab+cd)}\right) x^k, \end{aligned}$$

orthogonal on the whole line $\mathbb{R}$. In [14, Proposition 2.2], the following two hypergeometric representations were given

$$\begin{aligned} J_n^{(p,q)}(x; a, b, c, d) =& \frac{2^n (ad-bc)^n \left(-p+1-i\frac{q}{2}\right)_n}{i^n} \\ &\cdot {}_2F_1\left(\left.\begin{matrix} -n, n+1-2p \\ -p+1-i\frac{q}{2} \end{matrix}\right| \frac{i(a^2+c^2)}{2(ad-bc)}\left(x - \frac{i(ad-bc)-(ab+cd)}{a^2+c^2}\right)\right) \\ =& \frac{2^n (ad-bc)^n \left(-p+1+i\frac{q}{2}\right)_n}{(-i)^n} \\ &\cdot {}_2F_1\left(\left.\begin{matrix} -n, n+1-2p \\ -p+1+i\frac{q}{2} \end{matrix}\right| -\frac{i(a^2+c^2)}{2(ad-bc)}\left(x + \frac{i(ad-bc)+(ab+cd)}{a^2+c^2}\right)\right). && (62) \end{aligned}$$

The identity

$$J_n^{(p,q)}(x; a, b, c, d) = \frac{2^n (ad-bc)^n n!}{(-i)^n}\, P_n^{\left(-p+i\frac{q}{2}, -p-i\frac{q}{2}\right)}\left(\frac{i\left((a^2+c^2)x+ab+cd\right)}{ad-bc}\right),$$

connecting the polynomial family $J_n^{(p,q)}(x;a,b,c,d)$ directly with the Jacobi polynomials with complex parameters has been established in [14]. The hypergeometric representations (9)–(12) of the Jacobi polynomials therefore yield the following new representations for the Masjed-Jamei polynomials:

$$\begin{aligned}
J_n^{(p,q)}(x;a,b,c,d) &= (-1)^n\,(n-2p+1)_n\left(a^2+c^2\right)^n\left(x+\frac{ab+cd+i\,(ad-bc)}{a^2+c^2}\right)^n\\
&\quad\cdot\ {}_2F_1\left(\left.\begin{matrix}-n,p-i\frac{q}{2}-n\\ -2\,n+2\,p\end{matrix}\right|\frac{2\,i\,(ad-bc)}{a^2+c^2}\left(x+\frac{ab+cd+i\,(ad-bc)}{a^2+c^2}\right)^{-1}\right)\\
&= (-1)^n\,(n-2p+1)_n\left(a^2+c^2\right)^n\left(x-\frac{i\,(ad-bc)-ab-cd}{a^2+c^2}\right)^n\\
&\quad\cdot\ {}_2F_1\left(\left.\begin{matrix}-n,-n+p+i\frac{q}{2}\\ -2\,n+2\,p\end{matrix}\right|\frac{-2\,i\,(ad-bc)}{a^2+c^2}\left(x-\frac{i\,(ad-bc)-ab-cd}{a^2+c^2}\right)^{-1}\right)\\
&= (-1)^n\,(-p+i\frac{q}{2}+1)_n\left(a^2+c^2\right)^n\left(x-\frac{i\,(ad-bc)-ab-cd}{a^2+c^2}\right)^n\\
&\quad\cdot\ {}_2F_1\left(\left.\begin{matrix}-n,-n+p+i\frac{q}{2}\\ -p+i\frac{q}{2}+1\end{matrix}\right|\frac{x+\frac{ab+cd+i(ad-bc)}{a^2+c^2}}{x-\frac{i(ad-bc)-ab-cd}{a^2+c^2}}\right)\\
&= (-1)^n\left(-p-i\frac{q}{2}+1\right)_n\left(a^2+c^2\right)^n\left(x+\frac{ab+cd+i\,(ad-bc)}{a^2+c^2}\right)^n\\
&\quad\cdot\ {}_2F_1\left(\left.\begin{matrix}-n,p-i\frac{q}{2}-n\\ -p-i\frac{q}{2}+1\end{matrix}\right|\frac{x-\frac{i(ad-bc)-ab-cd}{a^2+c^2}}{x+\frac{ab+cd+i(ad-bc)}{a^2+c^2}}\right).
\end{aligned}$$

5.8. Hermite Polynomials

The Hermite polynomials have the hypergeometric representation [1, (9.15.1)]

$$H_n(x) = (2x)^n{}_2F_0\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n-1}{2}\\ -\end{matrix}\right|-\frac{1}{x^2}\right), \tag{63}$$

and by re-writing $H_n(x)$ as a polynomial of even degree and a polynomial of odd degree, we obtain, respectively,

$$H_{2n}(x) = (2x)^{2n}{}_2F_0\left(\left.\begin{matrix}-n,-n+\frac{1}{2}\\ -\end{matrix}\right|-\frac{1}{x^2}\right) \tag{64}$$

and

$$H_{2n+1}(x) = (2x)^{2n+1}{}_2F_0\left(\left.\begin{matrix}-n,-n-\frac{1}{2}\\ -\end{matrix}\right|-\frac{1}{x^2}\right). \tag{65}$$

By reversion of (64) and (65), we obtain

$$H_{2n}(x) = n!\,(-1)^n\binom{2n}{n}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \frac{1}{2}\end{matrix}\right|x^2\right) \tag{66}$$

and

$$H_{2n+1}(x) = 2\,x\left(\frac{3}{2}\right)_n(-4)^n\,{}_1F_1\left(\left.\begin{matrix}-n\\ \frac{3}{2}\end{matrix}\right|x^2\right). \tag{67}$$

We note that (63) can also be obtained from [13, (22.5.54)], (66) from [13, (22.5.56)] and (67) from [13, (22.5.57)].

6. Hypergeometric representations for the classical discrete OPS

We provide different hypergeometric representations for the Charlier, Meixner, Krawtchouk and Hahn polynomials. These polynomials are expansions in $x(x-1)\cdots(x-n+1)$, since $-x$ appears as upper parameter in the hypergeometric representations of each of these sequences.

6.1. Charlier Polynomials

The Charlier polynomials lie on the ${}_2F_0$ / ${}_1F_1$ plane of the Askey scheme of hypergeometric polynomials, together with the Laguerre polynomials, that have a ${}_1F_1$ hypergeometric representation. Charlier polynomials are defined [1, Section 9.14] in terms of the ${}_2F_0$ hypergeometric function as

$$C_n(x;a) = {}_2F_0\left(\left.\begin{matrix}-n,-x\\ -\end{matrix}\right| -\frac{1}{a}\right), \tag{68}$$

It turns out that the reverse of representation (68), constructed by Algorithm 1, is the following ${}_1F_1$ hypergeometric representation.

$$C_n(x;a) = \frac{(-1)^n(x-n+1)_n}{a^n}\,{}_1F_1\left(\left.\begin{matrix}-n\\ x-n+1\end{matrix}\right| a\right), \tag{69}$$

which was already given in [9, p. 141].

From the transformation given in [15, p. 409]: ${}_1F_1(a;b;x) = e^x\,{}_1F_1(b-a;b;-x)$, we get the following (infinite) representation, which is a new representation of these polynomials:

$$C_n(x;a) = \frac{(-1)^n(-n+x+1)_n}{a^n}\,e^a\,{}_1F_1\left(\left.\begin{matrix}x+1\\ x-n+1\end{matrix}\right| -a\right).$$

6.2. Meixner polynomials

Meixner polynomials [1, (9.10.1)] are defined in terms of the ${}_2F_1$ hypergeometric function

$$M_n(x;\beta,c) = {}_2F_1\left(\left.\begin{matrix}-n,-x\\ \beta\end{matrix}\right| 1-\frac{1}{c}\right), \tag{70}$$

$c \neq 0$ and $\beta \neq -1,-2,\ldots,-n+1$. By reversing representation (70) using Algorithm 1, we obtain

$$M_n(x;\beta,c) = \frac{(-x)_n}{(\beta)_n}\left(\frac{1-c}{c}\right)^n {}_2F_1\left(\left.\begin{matrix}1-n-\beta,-n\\ x+1-n\end{matrix}\right| \frac{c}{c-1}\right), \tag{71}$$

which is also the representation that we obtain using the first transformation given in [12, (15.8.6)].

Using transformations (2), (3) and (4), we obtain the following hypergeometric representations, respectively:

$$M_n(x;\beta,c) = \left(\frac{1}{c}\right)^n {}_2F_1\left(\left.\begin{matrix}-n,x+\beta\\ \beta\end{matrix}\right| 1-c\right); \tag{72}$$

$$= \left(\frac{1}{c}\right)^{\beta+n+x} {}_2F_1\left(\left.\begin{matrix}\beta+n,\beta+x\\ \beta\end{matrix}\right| 1-\frac{1}{c}\right); \tag{73}$$

$$= \frac{(\beta+x)_n}{(\beta)_n}\,{}_2F_1\left(\left.\begin{matrix}-n,-x\\ 1-n-x-\beta\end{matrix}\right| \frac{1}{c}\right) \tag{74}$$

By interchanging the upper parameters in the representation in (70), we obtain another representation for the Meixner polynomials:

$$M_n(x;\beta,c) = \left(\frac{1}{c}\right)^x {}_2F_1\left(\left.\begin{matrix}-x,n+\beta\\ \beta\end{matrix}\right| 1-c\right).$$

This representation, as well as the representation in (73), define an infinite series and reversion is not possible.

The reverse of representations (72) and (74), constructed by Algorithm 1, are, respectively:

$$M_n(x;\beta,c) = \frac{(x+\beta)_n}{(\beta)_n}\left(\frac{c-1}{c}\right)^n {}_2F_1\left(\left.\begin{matrix}-n, 1-n-\beta\\ 1-n-x-\beta\end{matrix}\right| \frac{1}{1-c}\right) \tag{75}$$

$$= \frac{(-x)_n}{(\beta)_n}\left(\frac{1}{c}\right)^n {}_2F_1\left(\left.\begin{matrix}-n, x+\beta\\ -n+1+x\end{matrix}\right| c\right). \tag{76}$$

The reversion in (75) can also be obtained from the (second) transformation given in [12, (15.8.7)] and (76) can also be obtained from the (second) transformation given in [12, (15.8.6)].

All the hypergeometric representations given here, except (70), are new, i.e., we have 7 new ${}_2F_1$ hypergeometric representations for the Meixner polynomials.

6.3. Krawtchouk polynomials

The Krawtchouk polynomials [1, (9.11.1)] are defined on a finite set of integers and are related to the Meixner polynomials. For $N \in \mathbb{N}$, they are defined as

$$K_n(x;p,N) = {}_2F_1\left(\left.\begin{matrix}-n, -x\\ -N\end{matrix}\right| \frac{1}{p}\right),\ n = 0, 1, \ldots, N. \tag{77}$$

The reversion of the above representation, obtained from Algorithm 1, is:

$$K_n(x;p,N) = \frac{(-1)^n(-x)_n}{p^n(-N)_n}{}_2F_1\left(\left.\begin{matrix}-n, -n+N+1\\ x-n+1\end{matrix}\right| p\right). \tag{78}$$

By using transformations (2), (3) and (4), we obtain the following hypergeometric representations for the Krawtchouk polynomials, respectively:

$$K_n(x;p,N) = \left(1-\frac{1}{p}\right)^n {}_2F_1\left(\left.\begin{matrix}-n, -N+x\\ -N\end{matrix}\right| \frac{1}{1-p}\right); \tag{79}$$

$$= \left(1-\frac{1}{p}\right)^{-N+n+x} {}_2F_1\left(\left.\begin{matrix}-N+n, -N+x\\ -N\end{matrix}\right| \frac{1}{p}\right) \tag{80}$$

$$= \frac{(x-N)_n}{(-N)_n}{}_2F_1\left(\left.\begin{matrix}-n, -x\\ -n+N-x+1\end{matrix}\right| 1-\frac{1}{p}\right). \tag{81}$$

Again, as was the case for the Meixner polynomials, by interchanging the upper parameters in (77), we obtain another (infinite) representation for the Krawtchouk polynomials:

$$K_n(x;p,N) = \left(1-\frac{1}{p}\right)^x {}_2F_1\left(\left.\begin{matrix}-x, -N+n\\ -N\end{matrix}\right| \frac{1}{1-p}\right).$$

We can find reversions for (79) and (81) by using Algorithm 1:

$$K_n(x;p,N) = \frac{(x-N)_n}{(-N)_n p^n}{}_2F_1\left(\left.\begin{matrix}-n, -n+N+1\\ -n-x+N+1\end{matrix}\right| 1-p\right) \tag{82}$$

$$= \frac{(-x)_n}{(-N)_n}\left(\frac{p-1}{p}\right)^n {}_2F_1\left(\left.\begin{matrix}-n, x-N\\ -n+x+1\end{matrix}\right| \frac{p}{p-1}\right). \tag{83}$$

We therefore have eight ${}_2F_1$ hypergeometric representations for the Krawtchouk polynomials as well, of which the last seven are new.

Krawtchouk polynomials have many applications in mathematics and Ismail [16, p.184] discusses the role of these polynomials in coding theory, especially when the value of $\frac{1}{1-p}$ is an integer, e.g., when

$p = 2, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. When $p = \frac{1}{2}$, the polynomials $K_n\left(x; \frac{1}{2}, N\right)$ are called the symmetric Krawtchouk polynomials and of interest is the number of integer zeros these polynomials have, see e.g. [17,18]. We list the hypergeometric representations of these polynomials here:

$$\begin{aligned}
K_n\left(x; \frac{1}{2}, N\right) &= {}_2F_1\left(\left.\begin{matrix}-n, -x\\ -N\end{matrix}\right| 2\right)\\
&= \frac{(-2)^n(-x)_n}{(-N)_n}\, {}_2F_1\left(\left.\begin{matrix}-n, -n+N+1\\ x-n+1\end{matrix}\right| \frac{1}{2}\right)\\
&= (-1)^n\, {}_2F_1\left(\left.\begin{matrix}-n, -N+x\\ -N\end{matrix}\right| 2\right)\\
&= (-1)^x\, {}_2F_1\left(\left.\begin{matrix}-x, -N+n\\ -N\end{matrix}\right| 2\right)\\
&= \frac{(x-N)_n}{(-N)_n}\, {}_2F_1\left(\left.\begin{matrix}-n, -x\\ -n+N-x+1\end{matrix}\right| -1\right)\\
&= \frac{2^n(x-N)_n}{(-N)_n}\, {}_2F_1\left(\left.\begin{matrix}-n, -n+N+1\\ -n-x+N+1\end{matrix}\right| \frac{1}{2}\right)\\
&= (-1)^{-N+n+x}\, {}_2F_1\left(\left.\begin{matrix}-N+n, -N+x\\ -N\end{matrix}\right| 2\right)\\
&= (-1)^n\, \frac{(-x)_n}{(-N)_n}\, {}_2F_1\left(\left.\begin{matrix}-n, x-N\\ -n+x+1\end{matrix}\right| -1\right).
\end{aligned}$$

6.4. Hahn polynomials

The Hahn polynomials [1, (9.5.1)] lie on the ${}_3F_2$ plane of the Askey scheme of hypergeometric orthogonal polynomials and are defined as

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -x\\ -N, \alpha+1\end{matrix}\right| 1\right). \tag{84}$$

More hypergeometric representations for the Hahn polynomials can be obtained from:
(i) [9, p. 141] and the representation is:

$$Q_n(x; \alpha, \beta, N) = \frac{(\alpha+x+1)_n(x-N)_n}{(\alpha+1)_n(-N)_n}\, {}_3F_2\left(\left.\begin{matrix}-n, -x, \beta+N+1-x\\ N+1-x-n, -\alpha-x-n\end{matrix}\right| 1\right); \tag{85}$$

(ii) the symmetry relation (cf. [19, 1.15]):

$$(\alpha+1)_n\, Q_n(x, \alpha, \beta, N) = (-1)^n(\beta+1)_n\, Q_n(N-x, \beta, \alpha, N),$$

namely:

$$Q_n(x; \alpha, \beta, N) = (-1)^n \frac{(\beta+1)_n}{(\alpha+1)_n}\, {}_3F_2\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -N+x\\ \beta+1, -N\end{matrix}\right| 1\right); \tag{86}$$

(iii) the transformation [12, (16.4.11)], given also as [2, Corollary 3.3.5], attributed to Kummer:

$${}_3F_2\left(\left.\begin{matrix}a, b, c\\ d, e\end{matrix}\right| 1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}\, {}_3F_2\left(\left.\begin{matrix}a, d-b, d-c\\ d, d+e-b-c\end{matrix}\right| 1\right),$$

which holds when $\Re(e-a)>0$, and the representation is:

$$Q_n(x;\alpha,\beta,N)=\frac{(-1)^n(\beta+N+1-x)_n}{(\alpha+1)_n} \cdot {}_3F_2\left(\begin{matrix}-n,-n-N-\alpha-\beta-1,x-N\\ -N,-n-N-\beta+x\end{matrix}\middle|\,1\right) \tag{87}$$

(iv) and lastly, the transformation given by Sheppard [2, Corollary 3.3.4]:

$$ {}_3F_2\left(\begin{matrix}-n,a,b\\ d,e\end{matrix}\middle|\,1\right)=\frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\;{}_3F_2\left(\begin{matrix}-n,a,a+b-n-d-e+1\\ a-n-d+1,a-n-e+1\end{matrix}\middle|\,1\right),$$

and the representation is:

$$Q_n(x;\alpha,\beta,N)=\frac{(-n-\beta)_n(-n-N-\alpha-\beta-1)_n}{(-N)_n(\alpha+1)_n} \cdot {}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,N-x+\beta+1\\ N+\alpha+\beta+2,\beta+1\end{matrix}\middle|\,1\right) \tag{88}$$

Using Algorithm 1, we obtain the following reverse presentations for (84), (85), (86), (87) and (88), respectively:

$$Q_n(x;\alpha,\beta,N)=\frac{(-x)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right)_n(-4)^n}{(\alpha+1)_n(\alpha+\beta+1)_n(-N)_n} \cdot {}_3F_2\left(\begin{matrix}-n,-n+N+1,-n-\alpha\\ x-n+1,-2n-\alpha-\beta\end{matrix}\middle|\,1\right) \tag{89}$$

$$Q_n(x;\alpha,\beta,N)=\frac{(-1)^n\,(-x)_n(\beta+N+1-x)_n}{(\alpha+1)_n(-N)_n} \cdot {}_3F_2\left(\begin{matrix}-n,\alpha+x+1,x-N\\ -n-\beta-N+x,x-n+1\end{matrix}\middle|\,1\right) \tag{90}$$

$$Q_n(x;\alpha,\beta,N)=\frac{(x-N)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right)_n4^n}{(\alpha+1)_n(\alpha+\beta+1)_n(-N)_n} \cdot {}_3F_2\left(\begin{matrix}-n,-n+N+1,-n-\beta\\ N-x-n+1,-2n-\alpha-\beta\end{matrix}\middle|\,1\right) \tag{91}$$

$$Q_n(x;\alpha,\beta,N)=\frac{(N+\alpha+\beta+2)_n(-N+x)_n}{(-N)_n(\alpha+1)_n} \cdot {}_3F_2\left(\begin{matrix}-n,1-n+N,\beta+1+N-x\\ N+\alpha+\beta+2,N+1-x-n\end{matrix}\middle|\,1\right) \tag{92}$$

$$Q_n(x;\alpha,\beta,N)=\frac{(\beta+N+1-x)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right)_n\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right)_n(-4)^n}{(\alpha+1)_n(\alpha+\beta+1)_n(-N)_n} \cdot {}_3F_2\left(\begin{matrix}-n,-\alpha-\beta-n-N-1,-n-\beta\\ -n-N+x-\beta,-2n-\alpha-\beta\end{matrix}\middle|\,1\right) \tag{93}$$

In the Hahn case, we have 8 new hypergeometric representations, namely (86) - (93).

Data Availability: All data related to this work are accessible on https://www.mathematik.uni-kassel.de/~koepf/Publikationen.

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The Maple worksheet `Multiple_hypergeometric_representations.mw` proving the results of this paper can be downloaded from https://www.mathematik.uni-kassel.de/~koepf/Publikationen.

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