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Exact Norms of the Hilbert Integral Operator on Mellin-Adapted Lorentz–Gamma Spaces

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Abstract: We determine the exact norm of the Hilbert integral operator on a Mellin-adapted, two-parameter Lorentz–Gamma scale. The adapted norm is $\|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = \|U_p f\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R},\mathrm{d}t)}$, $U_p f(t) = e^{t/p} f(e^t)$, where the Lorentz–Gamma norm is the displayed F^{**}-based norm. For $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$, $\|\mathbf{H}\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q} \to \mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = \frac{\pi}{\sin(\pi/p)}$. Under the logarithmic isometry U_p, the operator $\mathbf{H}$ becomes convolution with $k_p(t) = e^{t/p}/(1+e^t)$. On every fully symmetric Banach function space with the Fatou property, convolution by a nonnegative L^1-kernel has an operator norm on the same space equal to the total mass of the kernel. The long-interval argument establishing the reverse estimate requires no additional asymptotic assumption on the fundamental function. Applying this principle gives the stated Hilbert norm, including the fine-index endpoints $q = 1$ and $q = \infty$. Explicit multiplicative plateaux form extremizing sequences. Moreover, the norm is not attained when $1 \le q < \infty$, whereas a nonnegative, nonzero extremizer exists for every choice of logarithmic parameters when $q = \infty$. We also record short classical L^p and L^2 consistency checks. The numerical constants in the principal adapted-space norm and in its attainment assertions refer to the displayed F^{**}-based $\Gamma^{\boldsymbol{\alpha}}_{p,q}$-norm; the classical sections use the usual Lebesgue norms.

Keywords: Hilbert integral operator, Stieltjes transform, Lorentz–Gamma space, Mellin convolution, exact operator norm, extremizer, endpoint attainment, non-attainment, spectral deficit

MSC: 26D15; 42A85; 44A15; 46E30; 47G10.

1. Introduction

For $1 < p < \infty$, the Hilbert integral operator

$$(\mathbf{H}f)(x) = \int_0^\infty \frac{f(y)}{x+y}\,\mathrm{d}y, \tag{1}$$

satisfies

$$\|\mathbf{H}f\|_{L^p(\mathbb{R}_+)} \le C_p \|f\|_{L^p(\mathbb{R}_+)}, \qquad C_p = \frac{\pi}{\sin(\pi/p)}. \tag{2}$$

The constant is sharp. Equivalently, for nonnegative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$, with $p' = p/(p-1)$, see [1,2]:

$$\int_0^\infty \int_0^\infty \frac{f(y)g(x)}{x+y}\,\mathrm{d}y\,\mathrm{d}x \le C_p \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{3}$$

Exact operator norms for other Hilbert-type kernels continue to be studied; see [3]. That work concerns different kernels and weighted Lebesgue spaces.

This paper asks for an exact norm on one fixed f^{**}-based Lorentz–Gamma scale with a broken logarithmic parameter. Standard background on Lorentz–Zygmund and rearrangement-invariant spaces can be found in [4,5]; the one-weight $\Gamma_{r,\phi}$ convention is developed in [6]. Broken slowly varying parameters and recent work on Lorentz–Gamma spaces appear, respectively, in [7] and [8]. The operator in (1) is the Stieltjes operator. Direct estimates for the Stieltjes operator are given in [9,10].

The relevant convolution literature has several different levels of generality. O'Neil's classical theorem treats Young-type inequalities on Lorentz spaces [11]. Křepela characterizes convolution inequalities

between weighted spaces of type $\Gamma^r(v)$ and $\Gamma^s(w)$, including infinite indices, and identifies the optimal rearrangement-invariant multiplier space [12]. Kerman and Spektor study inequalities $\rho(T_k f) \le C\sigma(f)$ between two general rearrangement-invariant norms in [13]; that framework is broader than the same-space positive-kernel situation used here. Their preprint [14] treats boundedness of positive integral operators between one-weight Lorentz–Gamma spaces with finite integral indices and reduces the problem to weighted Stieltjes inequalities.

The cited operator-theoretic works provide general boundedness results and results concerning optimal domains and multiplier spaces. To the best of our knowledge, they do not contain the exact same-space Hilbert norm for the Mellin pullback considered here together with its attainment dichotomy. More precisely, in the notation of [12], our finite fine-index spaces are weighted spaces of type $\Gamma^q(w)$, and the endpoint is a weighted $\Gamma^\infty(v)$ space with $v(s) = s^{1/p}\ell_\alpha(s)$. The endpoint comparison closest to the present paper is [12, Lemma 3.5], where $\Gamma^\infty(v)$ is treated through an auxiliary weight envelope, together with [12, Theorem 3.6], where that framework is used in the convolution inequality. Those results belong to a more general weighted convolution theory. The point here is narrower: for the fixed Mellin pullback of the Hilbert kernel we keep the displayed same-space norm, compute the exact numerical constant, write explicit normalized multiplicative plateaux, and separate finite-index non-attainment from genuine endpoint attainment.

Thus our contribution is the following combination: the exact norm for the displayed broken-logarithmic f^{**}-norm, the scale-specific normalization of the endpoint profile $F^{**}(s) = a(s)^{-1}$ for large s, actual attainment of the norm at $q = \infty$, and non-attainment at every finite fine index. No novelty is asserted for general rearrangement-invariant convolution inequalities, the positive-kernel mass principle, or the regular-variation tools used in the endpoint asymptotic discussion. Finally, the first index in [14] is the power of f^{**}; this corresponds to q in our notation, while p occurs in the factor $s^{1/p}$.

Set

$$(U_p f)(t) = e^{t/p} f(e^t). \tag{4}$$

Then

$$U_p \mathbf{H} U_p^{-1} F = k_p * F, \qquad k_p(t) = \frac{e^{t/p}}{1+e^t}. \tag{5}$$

Also,

$$\|k_p\|_1 = \int_0^\infty \frac{s^{1/p-1}}{1+s}\, \mathrm{d}s = \frac{\pi}{\sin(\pi/p)}. \tag{6}$$

Full symmetry yields the upper estimate, while long intervals recover $\|k_p\|_1$ and therefore establish sharpness.

The main theorem is the exact norm formula for the displayed norm. An equivalent norm defines the same set of functions but need not preserve an isometric numerical statement, so no operator-norm assertion is made for an unspecified equivalent f^*-based convention. The endpoints in this paper are always the fine-index endpoints $q = 1$ and $q = \infty$, not the power endpoints $p = 1$ and $p = \infty$.

§2–§6 contain the space construction, the convolution lemma, the norm computation, the plateaux, the fine-index attainment dichotomy, and the classical applications.

2. Zygmund-Weighted Lorentz–Gamma Spaces

2.1. Decreasing Rearrangements

Throughout the paper, rearrangements are taken on $(\mathbb{R}, \mathrm{d}t)$ or, equivalently under the logarithmic map, on $(\mathbb{R}_+, \mathrm{d}x/x)$. For a measurable function F on $\mathbb{R}$, define its distribution function and decreasing rearrangement by

$$d_F(\lambda) = \big|\{t \in \mathbb{R} : |F(t)| > \lambda\}\big|, \qquad F^*(s) = \inf\{\lambda > 0 : d_F(\lambda) \le s\},$$

and set

$$F^{**}(s) = \frac{1}{s}\int_0^s F^*(u)\, \mathrm{d}u, \qquad s > 0. \tag{7}$$

We use the standard facts that $F^* \le F^{**}$, that $(F+G)^{**} \le F^{**} + G^{**}$, and that equimeasurable functions have equal rearrangements; see [5].

A rearrangement-invariant Banach function space $X(\mathbb{R})$ is a Banach function lattice whose norm depends only on the distribution of the function. Its fundamental function is

$$\phi_X(s) = \|\mathbf{1}_E\|_X, \qquad |E| = s. \tag{8}$$

The definition is independent of the measurable set E. We call X *fully symmetric* if $G \prec\prec F$ (Hardy–Littlewood submajorization) implies $\|G\|_X \le \|F\|_X$. Equivalently, $G^{**}(s) \le F^{**}(s)$ for all $s > 0$ is sufficient for the norm inequality.

2.2. A Two-Sided Logarithmic Weight

Let $\boldsymbol{\alpha} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. Broken logarithmic weights of this form are standard slowly varying parameters; see [7]. Define

$$\ell_{\boldsymbol{\alpha}}(s) = \begin{cases} (1 + |\log s|)^{\alpha_0}, & 0 < s \le 1, \\ (1 + \log s)^{\alpha_\infty}, & s > 1. \end{cases} \tag{9}$$

Here p fixes the principal power scale $s^{1/p}$, q is the fine index that aggregates this scale, and α_0, α_∞ are the logarithmic corrections at small and large distribution sizes, respectively. For $1 < p < \infty$ and $1 \le q < \infty$, define

$$\|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} = \left(\int_0^\infty \left[s^{1/p} \ell_{\boldsymbol{\alpha}}(s) F^{**}(s) \right]^q \frac{\mathrm{d}s}{s} \right)^{1/q}. \tag{10}$$

For $q = \infty$, put

$$\|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,\infty}} = \sup_{s>0} s^{1/p} \ell_{\boldsymbol{\alpha}}(s) F^{**}(s). \tag{11}$$

The supremum in (11) is the ordinary supremum. For a function of finite norm, F^{**} is finite and continuous on every compact subinterval of $(0,\infty)$, because $sF^{**}(s) = \int_0^s F^*(u)\,\mathrm{d}u$. Hence the ordinary and essential suprema agree.

For $q < \infty$, our norm is the one-weight Lorentz–Gamma norm $\Gamma_{q,w}$ in the convention of [6], with

$$w(s) = s^{q/p-1} \ell_{\boldsymbol{\alpha}}(s)^q.$$

In the notation of [12], this is a weighted space of type $\Gamma^q(w)$; at the supremum fine-index endpoint it is $\Gamma^\infty(v)$ with $v(s) = s^{1/p}\ell_{\boldsymbol{\alpha}}(s)$. Thus the underlying weighted Γ-spaces are established objects. What is fixed here, and matters for the numerical result, is the precise same-space norm and its Mellin pullback. For every $1 \le q \le \infty$, the corresponding customary Lorentz–Zygmund functional replaces F^{**} by F^*. We record the weighted Hardy comparison needed for this background identification. Put $a(s) = s^{1/p}\ell_{\boldsymbol{\alpha}}(s)$. For every $1 \le q \le \infty$ there is a finite constant $C = C(p, q, \boldsymbol{\alpha})$, independent of F, such that

$$\|aF^{**}\|_{L^q((0,\infty), \mathrm{d}s/s)} \le C \, \|aF^*\|_{L^q((0,\infty), \mathrm{d}s/s)}. \tag{12}$$

Indeed, write $s = e^t$, $u = e^r$, $b(t) = \ell_{\boldsymbol{\alpha}}(e^t)$, and $A(t) = e^{t/p} b(t) F^*(e^t)$. Since $\mathrm{d}s/s = \mathrm{d}t$,

$$\begin{aligned} e^{t/p} b(t) F^{**}(e^t) &= e^{t/p} b(t) e^{-t} \int_{-\infty}^t F^*(e^r) e^r \, \mathrm{d}r \\ &= \int_{-\infty}^t e^{-(t-r)/p'} \frac{b(t)}{b(r)} A(r) \, \mathrm{d}r. \end{aligned}$$

For any $0 < \eta < 1/p'$, the broken logarithmic form of b gives

$$\frac{b(t)}{b(r)} \le C_\eta e^{\eta(t-r)}, \qquad r \le t. \tag{13}$$

To see this, split the half-plane $r \le t$ into the three regions $r \le t \le 0$, $r \le 0 \le t$, and $0 \le r \le t$. In each region the quotient $b(t)/b(r)$ is bounded by a fixed power of $1 + t - r$, and every such power is bounded by $C_\eta e^{\eta(t-r)}$. Hence $a(e^t)F^{**}(e^t) \le h_\eta * A(t)$, where $h_\eta(\tau) = C_\eta e^{-(1/p'-\eta)\tau}\mathbf{1}_{[0,\infty)}(\tau) \in L^1(\mathbb{R})$. Young's inequality on $\mathbb{R}$ gives (12) for every $1 \le q \le \infty$, including the supremum case. The reverse functional comparison follows from $F^* \le F^{**}$. Thus the two functionals define the same set of functions, but they are not the same norm. The exact norm argument below is carried out directly with the F^{**}-based norm. The distinction is essential when an exact numerical operator norm is requested. In the basic case $\boldsymbol{\alpha} = 0$ and $q = p$, Hardy's inequality gives the explicit comparison

$$\|F\|_{L^p(\mathbb{R})} \le \|F\|_{\Gamma^{\mathbf{0}}_{p,p}} \le p' \|F\|_{L^p(\mathbb{R})}, \qquad p' = \frac{p}{p-1}. \tag{14}$$

Thus $\Gamma^{\mathbf{0}}_{p,p} = L^p(\mathbb{R})$ as a set, while $\Gamma^{\mathbf{0}}_{p,p}$ carries the particular F^{**}-based norm in (10).

Proposition 1 (Banach and Symmetry Structure)**.** *For $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$, formulas (10)–(11) define a rearrangement-invariant Banach function norm with the Fatou property. The resulting space $\Gamma^{\boldsymbol{\alpha}}_{p,q}$ is fully symmetric. Moreover, every $F \in \Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R})$ is locally integrable.*

Proof. We use the Banach-function-norm axioms of [5, Chapter 1, §1]: (P1) the norm axioms, (P2) the lattice property, (P3) the Fatou property, (P4) finiteness on indicators of finite-measure sets, and (P5) local integral control on such sets. We verify all five properties below and also establish rearrangement invariance and full symmetry as part of the argument. Compare also the Lorentz–Gamma framework in [6].

Homogeneity and definiteness are immediate. The lattice property follows because $|F| \le |G|$ almost everywhere implies $F^* \le G^*$ and hence $F^{**} \le G^{**}$. For the triangle inequality, use

$$(F+G)^{**}(s) \le F^{**}(s) + G^{**}(s), \qquad s > 0.$$

Minkowski's inequality gives the assertion when $q < \infty$, and the pointwise supremum estimate gives it when $q = \infty$.

We verify the Fatou property next. Suppose that $0 \le F_n \uparrow F$ almost everywhere. For each $s > 0$, the Hardy–Littlewood extremal identity gives, with both sides allowed to equal ∞,

$$sF_n^{**}(s) = \sup_{|E|=s} \int_E F_n.$$

The Hardy–Littlewood inequality gives one direction, while level sets, completed by a measurable subset of one level set when necessary, give the reverse direction; see [5, Chapter 2, §2]. Thus $F_n^{**}(s) \le F^{**}(s)$. If $F^{**}(s) < \infty$, then for every $\varepsilon > 0$ there is a measurable set E_ε with $|E_\varepsilon| = s$ such that

$$\int_{E_\varepsilon} F > sF^{**}(s) - \varepsilon.$$

Monotone convergence on this fixed set gives

$$\liminf_{n\to\infty} sF_n^{**}(s) \ge \lim_{n\to\infty} \int_{E_\varepsilon} F_n = \int_{E_\varepsilon} F > sF^{**}(s) - \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives the required lower bound in the finite case. If $F^{**}(s) = \infty$, then for every $M > 0$ the same extremal identity provides a measurable set E_M with $|E_M| = s$ and $\int_{E_M} F > sM$. Applying monotone convergence on E_M yields

$$\liminf_{n\to\infty} F_n^{**}(s) \ge M.$$

Since M is arbitrary, the limit is infinite. We have therefore proved $F_n^{**}(s) \uparrow F^{**}(s)$ for every $s > 0$. Monotone convergence then gives

$$\|F_n\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \uparrow \|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}},$$

when $q < \infty$. When $q = \infty$, put $a_n(s) = s^{1/p}\ell_{\boldsymbol{\alpha}}(s)F_n^{**}(s)$. Since $a_n(s) \uparrow a(s)$, the extended-real supremum identity gives

$$\lim_{n\to\infty}\sup_{s>0} a_n(s) = \sup_n \sup_{s>0} a_n(s) = \sup_{s>0}\sup_n a_n(s) = \sup_{s>0} a(s),$$

where all suprema are taken in $[0,\infty]$.

If $E \subset \mathbb{R}$ is measurable and $|E| = L < \infty$, then

$$(\mathbf{1}_E)^{**}(s) = \min\{1, L/s\}.$$

Consequently, for $q < \infty$,

$$\|\mathbf{1}_E\|^q_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} = \int_0^L s^{q/p-1}\ell_{\boldsymbol{\alpha}}(s)^q \, \mathrm{d}s + L^q \int_L^\infty s^{-q/p'-1}\ell_{\boldsymbol{\alpha}}(s)^q \, \mathrm{d}s.$$

The first integral is finite because $q/p > 0$, and the second because $q/p' > 0$. For $q = \infty$,

$$\|\mathbf{1}_E\|_{\Gamma^{\boldsymbol{\alpha}}_{p,\infty}} = \max\left\{\sup_{0<s\le L} s^{1/p}\ell_{\boldsymbol{\alpha}}(s), L\sup_{s>L} s^{-1/p'}\ell_{\boldsymbol{\alpha}}(s)\right\} < \infty.$$

Thus the norm is finite on characteristic functions of all finite-measure sets.

If $G \prec\prec F$, then $G^{**}(s) \le F^{**}(s)$ for every $s > 0$, so the space is fully symmetric. Finally, let $E \subset \mathbb{R}$ be measurable with $0 < |E| = L < \infty$. The Hardy–Littlewood inequality gives

$$\int_E |F| \le \int_0^L F^*(s) \, \mathrm{d}s = LF^{**}(L).$$

If $q = \infty$, then

$$LF^{**}(L) \le L^{1-1/p}\ell_{\boldsymbol{\alpha}}(L)^{-1}\,\|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,\infty}}.$$

If $q < \infty$, then for $L \le s \le 2L$, $F^{**}(s) \ge F^{**}(2L) \ge \frac{1}{2}F^{**}(L)$, and therefore

$$\|F\|^q_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \ge 2^{-q}F^{**}(L)^q \int_L^{2L} s^{q/p-1}\ell_{\boldsymbol{\alpha}}(s)^q \, \mathrm{d}s.$$

The last integral is positive and finite, so $F^{**}(L)$ is controlled by the norm and $\int_E |F| < \infty$. Hence $F \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Thus (P1)–(P5) hold. The Riesz–Fischer argument for Banach function norms in [5, Chapter 1, §1] gives completeness: if $\sum_n \|F_n\| < \infty$, then $\sum_n |F_n|$ belongs to the space by the Fatou property, and the partial sums converge in norm. For $1 < q < \infty$, the norm is a special weighted Lorentz–Gamma norm in the sense of [6]; the cases $q = 1$ and $q = \infty$ have been verified directly above. □

These are f^{**}-based Lorentz–Gamma spaces with Zygmund-type corrections. The notation $\Gamma^{\boldsymbol{\alpha}}_{p,q}$ is used deliberately: no isometric identification with every convention for $L^{p,q}(\log L)^{\boldsymbol{\alpha}}$ is intended.

For a measurable set E with $|E| = L$, one has

$$(\mathbf{1}_E)^{**}(s) = \begin{cases} 1, & 0 < s \le L, \\ L/s, & s > L. \end{cases} \tag{15}$$

Consequently, when $q < \infty$,

$$\begin{aligned}\phi_{p,q,\boldsymbol{\alpha}}(L)^q &:= \|\mathbf{1}_E\|^q_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \\ &= \int_0^L s^{q/p-1}\ell_{\boldsymbol{\alpha}}(s)^q \, \mathrm{d}s + L^q \int_L^\infty s^{-q/p'-1}\ell_{\boldsymbol{\alpha}}(s)^q \, \mathrm{d}s. \end{aligned} \tag{16}$$

Lemma 1 (Fundamental-Function Asymptotics). *Let* $1 < p < \infty$, $1 \le q \le \infty$, *and* $\boldsymbol{\alpha} \in \mathbb{R}^2$. *As* $L \to \infty$,

$$\phi_{p,q,\boldsymbol{\alpha}}(L) \sim c_{p,q}\, L^{1/p}(1+\log L)^{\alpha_\infty}, \tag{17}$$

where

$$c_{p,q} = \left(\frac{p+p'}{q}\right)^{1/q} \quad (1 \le q < \infty), \qquad c_{p,\infty} = 1.$$

Proof. Assume first that $q < \infty$. The function $L_0(s) = (1+\log s)^{q\alpha_\infty}$ is slowly varying at infinity. The integral and tail forms of Karamata's theorem state, for every $\rho > 0$ [15, Theorem 1.5.11],

$$\int_1^L s^{\rho-1} L_0(s)\, \mathrm{d}s \sim \frac{L^\rho L_0(L)}{\rho}, \qquad \int_L^\infty s^{-\rho-1} L_0(s)\, \mathrm{d}s \sim \frac{L^{-\rho} L_0(L)}{\rho}.$$

Taking $\rho = q/p$ and $\rho = q/p'$, respectively, gives

$$\int_0^L s^{q/p-1} \ell_{\boldsymbol{\alpha}}(s)^q\, \mathrm{d}s \sim \frac{p}{q} L^{q/p}(1+\log L)^{q\alpha_\infty},$$
$$L^q \int_L^\infty s^{-q/p'-1} \ell_{\boldsymbol{\alpha}}(s)^q\, \mathrm{d}s \sim \frac{p'}{q} L^{q/p}(1+\log L)^{q\alpha_\infty}.$$

The portion of the first integral over $(0,1)$ is a fixed finite constant and is therefore negligible. Adding the two asymptotics gives (17).

For $q = \infty$, (15) yields

$$\phi_{p,\infty,\boldsymbol{\alpha}}(L) = \max\left\{ \sup_{0<s\le L} s^{1/p}\ell_{\boldsymbol{\alpha}}(s), L \sup_{s>L} s^{-1/p'}\ell_{\boldsymbol{\alpha}}(s) \right\}.$$

We verify the fine-index endpoint asymptotics directly, rather than appealing to an unqualified monotonicity statement for arbitrary regularly varying functions. For $s > 1$, set

$$A(s) = s^{1/p}(1+\log s)^{\alpha_\infty}, \qquad B(s) = s^{-1/p'}(1+\log s)^{\alpha_\infty}.$$

Their logarithmic derivatives are

$$\frac{A'(s)}{A(s)} = \frac{1}{s}\left(\frac{1}{p} + \frac{\alpha_\infty}{1+\log s}\right), \qquad \frac{B'(s)}{B(s)} = \frac{1}{s}\left(-\frac{1}{p'} + \frac{\alpha_\infty}{1+\log s}\right).$$

Hence A is increasing and B is decreasing for all sufficiently large s. On $(0,1]$, the function $s^{1/p}(1+|\log s|)^{\alpha_0}$ is bounded and tends to zero as $s \downarrow 0$, whereas $A(L) \to \infty$. It follows that

$$\sup_{0<s\le L} s^{1/p}\ell_{\boldsymbol{\alpha}}(s) \sim A(L).$$

Similarly, for all sufficiently large L, continuity and eventual monotonicity give

$$\sup_{s>L} s^{-1/p'}\ell_{\boldsymbol{\alpha}}(s) = \lim_{s\downarrow L} B(s) = B(L).$$

The value is the limit as s approaches the left endpoint L from the right and need not be attained on the open interval. Therefore both terms in the maximum are asymptotic to $L^{1/p}(1+\log L)^{\alpha_\infty}$, which proves (17) for $q = \infty$. □

Remark 1 (Small- and Large-Measure Parameters). The parameter α_0 controls the small-distribution-measure geometry, while α_∞ controls the large-distribution-measure asymptotics and the explicit normalization of long plateaux. These are distribution sizes on $(\mathbb{R}, \mathrm{d}t)$, equivalently on $(\mathbb{R}_+, \mathrm{d}x/x)$; they do not refer directly to the

physical limits $x \downarrow 0$ and $x \to \infty$. The fixed-shift fundamental-function ratio used below is in fact automatic for every fully symmetric Banach function space, independently of these asymptotics.

2.3. The Multiplicative Group and the Adapted Scale

Equip $\mathbb{R}_+$ with its Haar measure

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{x}. \tag{18}$$

The logarithmic map $x \mapsto t = \log x$, with inverse $t \mapsto x = e^t$, is measure preserving from $(\mathbb{R}_+, \mathrm{d}\mu)$ to $(\mathbb{R}, \mathrm{d}t)$. Thus the rearrangement of a function F on $(\mathbb{R}_+, \mathrm{d}\mu)$ agrees with that of $t \mapsto F(e^t)$ on $\mathbb{R}$. Although $\mathrm{d}x$ and $\mathrm{d}x/x$ give different distribution functions, they have the same null sets on $\mathbb{R}_+$. Consequently, an almost-everywhere statement in logarithmic coordinates transfers to an almost-everywhere statement for Lebesgue measure $\mathrm{d}x$.

For $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$, define the Mellin-adapted Lorentz–Gamma space with Zygmund-type weights

$$\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}_+) = \left\{ f : \; x^{1/p} f(x) \in \Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}_+, \mathrm{d}\mu) \right\}, \tag{19}$$

with norm

$$\|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = \left\| x^{1/p} f(x) \right\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}_+, \mathrm{d}\mu)}. \tag{20}$$

Equivalently, with U_p as in (4),

$$\|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = \left\| U_p f \right\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R})}. \tag{21}$$

The exponent $1/p$ is the natural choice that makes U_p an isometry on the classical $L^p(\mathbb{R}_+, \mathrm{d}x)$ scale and simultaneously converts the homogeneous kernel into an integrable translation kernel on $\mathbb{R}$. In particular, (14) implies that $\mathfrak{X}^{\mathbf{0}}_{p,p} = L^p(\mathbb{R}_+, \mathrm{d}x)$ as a set, with

$$\|f\|_{L^p(\mathbb{R}_+, \mathrm{d}x)} \le \|f\|_{\mathfrak{X}^{\mathbf{0}}_{p,p}} \le p' \, \|f\|_{L^p(\mathbb{R}_+, \mathrm{d}x)}.$$

The middle expression is the adapted f^{**}-based norm, not the usual Lebesgue norm.

3. An Exact-Norm Principle for Positive Convolution

The truncation below defines convolution on the whole Banach function space, including the two fine-index endpoints, before the norm is computed. The same-space mass identity is a standard positive-convolution phenomenon; we include a proof to make the whole-space definition and the fine-index endpoints self-contained. Classical Lorentz convolution inequalities go back to [11], weighted Γ-space convolution is treated in [12], and the general rearrangement-invariant framework of [13] is substantially broader. The elementary observation below removes what might otherwise appear to be an additional asymptotic hypothesis in the same-space lower-bound argument.

Lemma 2 (Automatic Fixed-Shift Ratio)**.** *If $X(\mathbb{R})$ is a fully symmetric Banach function space with fundamental function ϕ_X, then, for every fixed $a > 0$,*

$$\lim_{L \to \infty} \frac{\phi_X(L - a)}{\phi_X(L)} = 1, \qquad L > a. \tag{22}$$

Proof. For $0 < s < t$,

$$\frac{s}{t} \mathbf{1}_{[0,t]} \prec\prec \mathbf{1}_{[0,s]}.$$

Full symmetry and the lattice property therefore give

$$\frac{s}{t} \phi_X(t) \le \phi_X(s) \le \phi_X(t).$$

With $s = L - a$ and $t = L$,

$$1 - \frac{a}{L} \le \frac{\phi_X(L - a)}{\phi_X(L)} \le 1,$$

and the conclusion follows. □

Lemma 3 (Truncation and Fatou Construction)**.** *Let $X(\mathbb{R})$ be a fully symmetric Banach function space with the Fatou property, let $K \in L^1(\mathbb{R})$ be nonnegative, and let $F \in X$ be nonnegative. Set*

$$K_n = K\mathbf{1}_{[-n,n]}, \qquad F_n = \min\{F, n\}\mathbf{1}_{[-n,n]}, \qquad T_nF = K_n * F_n.$$

Then $T_nF(t)$ is nondecreasing in n for every t. If a function $G \in X$ satisfies

$$(T_nF)^{**}(s) \le G^{**}(s), \qquad s > 0,\ n \ge 1,$$

then the pointwise limit $T_KF = \lim_n T_nF$ belongs to X and $\|T_KF\|_X \le \|G\|_X$.

For an arbitrary real or complex $H \in X$, suppose that the preceding construction applied to $|H|$ gives $T_K|H| < \infty$ almost everywhere. Then

$$T_KH(t) = \int_{\mathbb{R}} K(u)H(t-u)\ \mathrm{d}u,$$

for almost every t, the integral is absolutely convergent there, and the resulting map $H \mapsto T_KH$ is linear on equivalence classes.

Proof. Monotonicity follows pointwise from $K_n \uparrow K$ and $F_n \uparrow F$. Full symmetry gives $\|T_nF\|_X \le \|G\|_X$, and the Fatou property yields $T_KF \in X$ with the same bound. For an arbitrary $H \in X$, monotone convergence identifies $T_K|H|$ with $\int K(u)|H(t-u)|\ \mathrm{d}u$. If this quantity is finite almost everywhere, the signed or complex convolution is absolutely convergent there, and ordinary properties of the Lebesgue integral give linearity in H. □

Theorem 1 (Exact Mass Principle)**.** *Let $X(\mathbb{R})$ be a fully symmetric Banach function space with the Fatou property, and let $K \in L^1(\mathbb{R})$ be nonnegative. Then, for every $F \in X$, the integral*

$$(T_KF)(t) = \int_{\mathbb{R}} K(u)F(t-u)\ \mathrm{d}u,$$

is absolutely convergent for almost every t, defines an element of X, and

$$\|T_K\|_{X\to X} = \|K\|_{L^1(\mathbb{R})}. \tag{23}$$

Proof. For $F \ge 0$, consider the truncations in Lemma 3. For every measurable set $E \subset \mathbb{R}$ of finite measure $|E| = s$, Tonelli's theorem and the Hardy–Littlewood inequality give

$$\begin{aligned}\int_E T_nF(t)\ \mathrm{d}t &\le \int_{\mathbb{R}} K_n(u) \int_{E-u} F_n(v)\ \mathrm{d}v\ \mathrm{d}u \\ &\le \|K\|_1 \int_0^s F^*(r)\ \mathrm{d}r.\end{aligned}$$

Consequently,

$$(T_nF)^{**}(s) \le \|K\|_1\, F^{**}(s), \qquad s > 0.$$

By full symmetry, $\|T_nF\|_X \le \|K\|_1 \|F\|_X$. Applying Lemma 3 with $G = \|K\|_1 F$ gives $T_KF \in X$ and

$$\|T_KF\|_X \le \|K\|_1 \|F\|_X.$$

For a real or complex $F \in X$, apply the same construction to $|F|$. Then the convolution is absolutely convergent almost everywhere,

$$|T_KF| \le T_K|F|, \qquad \|T_KF\|_X \le \|K\|_1 \|F\|_X,$$

and Lemma 3 gives linearity. Thus $\|T_K\|_{X\to X} \le \|K\|_1$.

For the reverse inequality, let $F_R = \mathbf{1}_{[-R,R]}$. Fix $A > 0$ and set

$$m_A = \int_{-A}^{A} K(u)\, \mathrm{d}u.$$

If $|t| \le R - A$, then $t - u \in [-R, R]$ for every $|u| \le A$, and hence $(K * F_R)(t) \ge m_A$. The lattice property and rearrangement invariance give

$$\frac{\|K * F_R\|_X}{\|F_R\|_X} \ge m_A \frac{\phi_X(2R - 2A)}{\phi_X(2R)}.$$

Letting $R \to \infty$ and using Lemma 2 gives $\|T_K\|_{X\to X} \ge m_A$. Finally, $m_A \uparrow \|K\|_1$ as $A \to \infty$. □

Remark 2. The positivity assumption can be replaced by the requirement that K have a constant complex phase almost everywhere. For a general sign-changing kernel, the upper bound $\|T_K\| \le \|K\|_1$ remains true but need not be sharp.

Corollary 1. *Let $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$. If $K \ge 0$ belongs to $L^1(\mathbb{R})$, then*

$$\|K * \cdot\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}\to\Gamma^{\boldsymbol{\alpha}}_{p,q}} = \|K\|_1. \tag{24}$$

Proof. Apply Proposition 1 and Theorem 1. Lemma 1 is not needed for sharpness; it supplies the explicit plateau normalization. □

Remark 3 (Why Long Intervals are Canonical)**.** The lower-bound construction is a Følner argument for the additive group $\mathbb{R}$. A long interval is almost invariant under every fixed translation, so averaging it against a probability density produces almost no norm loss. After returning to $\mathbb{R}_+$, these intervals become multiplicative annuli (e^{-R}, e^{R}), which explains the form of the extremizing sequences below.

4. Sharp Hilbert Integral Norms

4.1. Logarithmic Conjugation

For a bounded compactly supported measurable f on $\mathbb{R}_+$, set $F = U_p f$. All integrals below are then absolutely convergent, and a direct calculation gives

$$\begin{aligned}(U_p \mathbf{H} f)(t) &= e^{t/p} \int_0^\infty \frac{f(y)}{e^t + y}\, \mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{e^{(t-s)/p}}{1 + e^{t-s}} F(s)\, \mathrm{d}s = (k_p * F)(t),\end{aligned} \tag{25}$$

where

$$k_p(u) = \frac{e^{u/p}}{1 + e^u}. \tag{26}$$

The kernel is strictly positive on $\mathbb{R}$. With $z = e^u$,

$$\|k_p\|_1 = \int_0^\infty \frac{z^{1/p-1}}{1+z}\, \mathrm{d}z = B\left(\frac{1}{p}, 1 - \frac{1}{p}\right) = \Gamma\left(\frac{1}{p}\right)\Gamma\left(1 - \frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)}. \tag{27}$$

Theorem 2 (Abstract Mellin-Pullback Theorem)**.** *Let $Y(\mathbb{R})$ be a fully symmetric Banach function space with the Fatou property and let $1 < p < \infty$. Define*

$$\mathcal{X}_{p,Y} = \{f : U_p f \in Y\}, \qquad \|f\|_{\mathcal{X}_{p,Y}} = \|U_p f\|_Y.$$

Then the integral defining $\mathbf{H}f$ *converges absolutely for almost every* $x > 0$*, defines a bounded operator on* $\mathcal{X}_{p,Y}$*, and*

$$\|\mathbf{H}\|_{\mathcal{X}_{p,Y}\to\mathcal{X}_{p,Y}} = \frac{\pi}{\sin(\pi/p)}.$$

Proof. Apply Theorem 1 to $K = k_p$ and use (25)–(27). The truncation construction shows that $k_p * |U_p f|$ is finite almost everywhere. Under $y = e^s$, this quantity equals

$$e^{t/p}\int_0^\infty \frac{|f(y)|}{e^t + y}\,\mathrm{d}y,$$

so the original integral is absolutely convergent for almost every $x = e^t$. Null sets for $\mathrm{d}t$, $\mathrm{d}x/x$, and $\mathrm{d}x$ correspond under the logarithmic change of variables. On bounded compactly supported data this agrees with the initial integral calculation, and therefore the whole-space operator is its canonical extension. □

Theorem 3 (Sharp Norm on the Mellin-Adapted Lorentz–Gamma Scale)**.** *Let* $1 < p < \infty$, $1 \le q \le \infty$, *and* $\boldsymbol{\alpha} \in \mathbb{R}^2$. *For every* $f \in \mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}_+)$, *the integral defining* $\mathbf{H}f$ *is absolutely convergent for almost every* $x > 0$, *and*

$$\|\mathbf{H}f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} \le \frac{\pi}{\sin(\pi/p)}\,\|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}}. \tag{28}$$

The constant $\pi/\sin(\pi/p)$ *is optimal for every* q *and every* $\boldsymbol{\alpha}$.

Proof. By Proposition 1, $Y = \Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R})$ is fully symmetric and has the Fatou property. Theorem 2, applied with this choice of Y, gives the almost-everywhere absolute convergence of the original Hilbert integral, the boundedness on $\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}$, and the exact operator norm. The construction in Theorem 2 is the truncation/Fatou construction from Theorem 1; hence it applies directly at $q = \infty$ and does not use density of compactly supported functions. The sets of measure zero for $\mathrm{d}t$, $\mathrm{d}x/x$, and $\mathrm{d}x$ correspond under $x = e^t$. Finally, (27) gives the value $\pi/\sin(\pi/p)$. □

In particular, the same exact operator norm holds at the two fine-index endpoints $q = 1$ and $q = \infty$.

Remark 4 (Parameter and Norm Dependence)**.** The logarithmic parameters affect the geometry of the space and the explicit normalization of the extremizing sequence, but not the exact Hilbert constant. The fixed-shift ratio used in the Følner lower bound is automatic by Lemma 2; no asymptotic hypothesis on the fundamental function is needed for the abstract mass principle. The numerical statement concerns precisely (10)–(11). Although customary f^*-based Lorentz–Zygmund norms may be equivalent to this norm, equivalence alone does not identify the corresponding numerical operator norm. We therefore make no exact-norm claim for an unspecified equivalent convention.

4.2. Classical Consistency Check

Taking $X = L^p(\mathbb{R})$ in Theorem 1 and using the isometry $U_p : L^p(\mathbb{R}_+, \mathrm{d}x) \to L^p(\mathbb{R}, \mathrm{d}t)$ gives a bounded operator satisfying

$$U_p\mathbf{H}f = k_p * U_p f,$$

first for bounded compactly supported functions and then, by density and the L^p convolution bound, for every $f \in L^p(\mathbb{R}_+)$. The construction in Theorem 1 also shows that the defining integral converges absolutely almost everywhere. Consequently,

$$\|\mathbf{H}\|_{L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)} = \frac{\pi}{\sin(\pi/p)}. \tag{29}$$

Consequently, for nonnegative $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$,

$$\int_0^\infty\int_0^\infty \frac{f(y)g(x)}{x+y}\,\mathrm{d}y\,\mathrm{d}x \le \frac{\pi}{\sin(\pi/p)}\,\|f\|_{L^p}\,\|g\|_{L^{p'}}, \tag{30}$$

and the constant is optimal. Also $C_p \sim p'$ as $p \downarrow 1$ and $C_p \sim p$ as $p \to \infty$; hence the fine-index endpoints $q = 1, \infty$ should not be confused with the power endpoints $p = 1, \infty$.

5. Extremizing Sequences and the Attainment Dichotomy

5.1. Multiplicative Plateaux

Let

$$\phi_{p,q,\boldsymbol{\alpha}}(L) = \left\| \mathbf{1}_{[0,L]} \right\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R})}.$$

For $R > 1$, define

$$F_R(t) = \frac{\mathbf{1}_{[-R,R]}(t)}{\phi_{p,q,\boldsymbol{\alpha}}(2R)}, \tag{31}$$

and

$$f_R(x) = \frac{x^{-1/p}\mathbf{1}_{(e^{-R},e^R)}(x)}{\phi_{p,q,\boldsymbol{\alpha}}(2R)}. \tag{32}$$

Then $U_p f_R = F_R$ and $\|f_R\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = 1$.

Proposition 2 (Explicit Extremizing Sequence)**.** *For every $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$, the sequence in (32) satisfies*

$$\lim_{R\to\infty} \|\mathbf{H} f_R\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = \frac{\pi}{\sin(\pi/p)}. \tag{33}$$

Proof. Fix $A > 0$ and put

$$m_A = \int_{-A}^{A} k_p(u)\, \mathrm{d}u.$$

If $R > A$ and $|t| \le R - A$, then $t - u \in [-R, R]$ whenever $|u| \le A$. Consequently,

$$(k_p * F_R)(t) \ge \frac{m_A}{\phi_{p,q,\boldsymbol{\alpha}}(2R)} \mathbf{1}_{[-R+A,R-A]}(t).$$

The lattice property and rearrangement invariance give

$$\left\| k_p * F_R \right\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \ge m_A \frac{\phi_{p,q,\boldsymbol{\alpha}}(2R - 2A)}{\phi_{p,q,\boldsymbol{\alpha}}(2R)}.$$

By Lemma 2, the ratio tends to one as $R \to \infty$. Thus the limit inferior is at least m_A. The upper bound in Theorem 3 is $C_p = \|k_p\|_1$, while $m_A \uparrow C_p$ as $A \to \infty$; hence the asserted limit follows. The normalization is completely explicit: by Lemma 1,

$$\phi_{p,q,\boldsymbol{\alpha}}(2R) \sim c_{p,q}(2R)^{1/p}(1 + \log(2R))^{\alpha_\infty}.$$

This calculation also covers $q = 1$ and $q = \infty$. □

5.2. The Fine-Index Attainment Dichotomy

Theorem 4 (Non-Attainment on the Principal Scale)**.** *Let $1 < p < \infty$, $1 \le q < \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$. If $f \in \mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}_+)$ satisfies*

$$\|\mathbf{H} f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}} = C_p \|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}}, \qquad C_p = \frac{\pi}{\sin(\pi/p)},$$

then $f = 0$ almost everywhere. Thus the sharp norm is not attained by a nonzero element of any space in the displayed scale with finite fine index.

Proof. Set $F = U_p f$, $H = |F|$, and $P = k_p / C_p$. Then P is a strictly positive probability density. Suppose equality holds. With $G = P * H$, positivity and the convolution submajorization used in Theorem 1 give

$$|P * F| \le G, \qquad G^{**}(s) \le H^{**}(s),$$

and hence

$$\|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} = \|P * F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \le \|G\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} \le \|H\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}} = \|F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,q}}.$$

Because $q < \infty$ and the weight in (10) is strictly positive, equality of the last two norms forces $G^{**} = H^{**}$ almost everywhere. Both functions of s are continuous on $(0, \infty)$, so equality holds for every $s > 0$. Differentiating $sG^{**}(s) = sH^{**}(s)$ almost everywhere gives $G^* = H^*$ almost everywhere. Thus G and H are equimeasurable.

For a rational $\lambda > 0$, let $\Psi_\lambda(r) = (r - \lambda)_+$. Membership of H in the Gamma space implies that $d_H(\lambda) = |\{H > \lambda\}| < \infty$: otherwise $H^*(s) \ge \lambda$ for every $s > 0$, contradicting the finite norm. If $d_H(\lambda) = 0$, equimeasurability gives $d_G(\lambda) = 0$, and both sides of the Jensen inequality below vanish almost everywhere. Suppose therefore that $d_H(\lambda) > 0$. Then

$$\int_{\mathbb{R}} \Psi_\lambda(H) = \int_0^{d_H(\lambda)} (H^*(s) - \lambda)\, \mathrm{d}s \le d_H(\lambda) H^{**}(d_H(\lambda)) < \infty.$$

Jensen's inequality and Tonelli's theorem give

$$\Psi_\lambda(G(t)) \le \int_{\mathbb{R}} P(u) \Psi_\lambda(H(t-u))\, \mathrm{d}u,$$

while equimeasurability makes the integrals of the two sides equal. Consequently equality holds almost everywhere, simultaneously for all positive rational λ. Equality for this countable family of hinge functions implies that, for almost every t, the function $u \mapsto H(t-u)$ is constant for $P(u)\,\mathrm{d}u$-almost every u. Indeed, if it were not essentially constant, there would be numbers $a < b$ such that both sets where $H(t-u) \le a$ and where $H(t-u) \ge b$ have positive $P(u)\,\mathrm{d}u$-measure. Choosing a rational $\lambda \in (a, b)$ places positive mass on both affine pieces of Ψ_λ and makes Jensen's inequality strict, a contradiction.

Choose one t for which the preceding conclusion holds. Since $P(u) > 0$ for every $u \in \mathbb{R}$, the measures $P(u)\,\mathrm{d}u$ and $\mathrm{d}u$ have the same null sets. Hence $u \mapsto H(t-u)$ is constant for Lebesgue-almost every u. The measure-preserving change of variable $r = t - u$ shows immediately that H is constant almost everywhere on $\mathbb{R}$. A nonzero constant does not belong to $\Gamma^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R})$, because its norm contains the divergent large-s integral $\int_1^\infty s^{q/p-1}(1 + \log s)^{q\alpha_\infty}\, \mathrm{d}s$.

Therefore $H = 0$, and hence $f = 0$ almost everywhere. □

The proof uses the strictness of the finite-q integral norm when $G^{**} \le H^{**}$. This mechanism disappears for the supremum norm. This distinction is genuine rather than an artifact of the proof: attainment does occur at the supremum fine-index endpoint.

Theorem 5 (Attainment at the Supremum Fine-Index Endpoint)**.** *Let $1 < p < \infty$ and $\boldsymbol{\alpha} \in \mathbb{R}^2$. There exists a nonnegative, nonzero function $f \in \mathfrak{X}^{\boldsymbol{\alpha}}_{p,\infty}(\mathbb{R}_+)$ such that*

$$\|f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,\infty}} = 1, \qquad \|\mathbf{H} f\|_{\mathfrak{X}^{\boldsymbol{\alpha}}_{p,\infty}} = C_p = \frac{\pi}{\sin(\pi/p)}. \tag{34}$$

Thus the sharp Hilbert norm is attained at $q = \infty$ for every pair of logarithmic parameters.

Proof. Write

$$a(s) = s^{1/p} \ell_{\boldsymbol{\alpha}}(s), \qquad J(s) = \frac{s}{a(s)} = s^{1/p'} (1 + \log s)^{-\alpha_\infty}, \qquad s > 1.$$

We construct a decreasing rearrangement whose endpoint norm is saturated at every sufficiently large distribution size. The function a is continuous, tends to zero at the origin, and tends to infinity at infinity. Moreover, a is eventually increasing. Also

$$J'(s) = s^{-1/p}(1+\log s)^{-\alpha_\infty}\left(\frac{1}{p'} - \frac{\alpha_\infty}{1+\log s}\right). \tag{35}$$

The expression in parentheses is eventually positive and smaller than one. Furthermore,

$$\frac{sJ''(s)}{J'(s)} = -\frac{1}{p} - \frac{\alpha_\infty}{1+\log s} + \frac{\alpha_\infty}{(1+\log s)^2\left(\frac{1}{p'} - \frac{\alpha_\infty}{1+\log s}\right)} \longrightarrow -\frac{1}{p}.$$

Consequently, one may choose $S > 1$ so large that

$$a(s) \le a(S) \quad (0 < s \le S), \qquad 0 < J'(s) \le J'(S) \le \frac{1}{a(S)} \quad (s \ge S), \tag{36}$$

and J' is decreasing on $[S, \infty)$.

Define F on $\mathbb{R}$ by

$$F(t) = \begin{cases} 0, & t \le 0, \\ a(S)^{-1}, & 0 < t \le S, \\ J'(t), & t > S. \end{cases} \tag{37}$$

By (36), the positive part of F is nonincreasing, and hence, up to values at the endpoints,

$$F^*(s) = \begin{cases} a(S)^{-1}, & 0 < s \le S, \\ J'(s), & s > S. \end{cases}$$

Since $J(S) = S/a(S)$, for $s \ge S$ we obtain

$$\int_0^s F^*(u)\,\mathrm{d}u = \frac{S}{a(S)} + J(s) - J(S) = J(s),$$

and therefore

$$F^{**}(s) = \begin{cases} a(S)^{-1}, & 0 < s \le S, \\ a(s)^{-1}, & s \ge S. \end{cases} \tag{38}$$

It follows from (36) that

$$\|F\|_{\Gamma^{\alpha}_{p,\infty}} = \sup_{s>0} a(s)F^{**}(s) = 1. \tag{39}$$

We now prove that convolution with the normalized Hilbert kernel preserves this norm. Put

$$P(u) = \frac{k_p(u)}{C_p}, \qquad \int_{\mathbb{R}} P(u)\,\mathrm{d}u = 1, \qquad G = P * F.$$

The kernel satisfies the two-sided exponential estimates

$$P(u) \le C \begin{cases} e^{u/p}, & u \le 0, \\ e^{-u/p'}, & u \ge 0, \end{cases} \tag{40}$$

where C depends only on p. On the other hand, (35) says that

$$F(t) = J'(t) \sim \frac{1}{p'} t^{-1/p}(1+\log t)^{-\alpha_\infty} \qquad (t \to \infty), \tag{41}$$

so F is regularly varying at infinity with index $-1/p$.

We claim that

$$G(t) \sim F(t) \qquad (t \to \infty). \tag{42}$$

Fix $M > 0$ and, for $t > 2M$, split

$$\frac{G(t)}{F(t)} = I_{0,M}(t) + I_{-,M}(t) + I_{+,M}(t) + I_{\text{far}}(t),$$

where the four terms are the integrals of $P(u)F(t-u)/F(t)$ over the respective regions $|u| \le M$, $u < -M$, $M < u \le t/2$, and $u > t/2$. The uniform convergence theorem for regularly varying functions [15, Theorem 1.2.1] gives

$$I_{0,M}(t) \longrightarrow \int_{-M}^{M} P(u)\ \mathrm{d}u.$$

For $u < -M$ and large t, eventual monotonicity gives $F(t-u) \le F(t)$, and hence

$$0 \le I_{-,M}(t) \le \int_{-\infty}^{-M} P(u)\ \mathrm{d}u.$$

For $M < u \le t/2$, one has $t/2 \le t-u < t$. Regular variation and eventual monotonicity therefore give, for all sufficiently large t,

$$\sup_{M<u\le t/2} \frac{F(t-u)}{F(t)} \le \frac{F(t/2)}{F(t)} \le C_0,$$

with C_0 independent of M. Thus

$$0 \le I_{+,M}(t) \le C_0 \int_{M}^{\infty} P(u)\ \mathrm{d}u.$$

Finally, since F is bounded,

$$0 \le I_{\text{far}}(t) \le \frac{\|F\|_\infty}{F(t)} \int_{t/2}^{\infty} P(u)\ \mathrm{d}u \longrightarrow 0,$$

by (40) and (41). The central part also gives

$$\liminf_{t\to\infty} \frac{G(t)}{F(t)} \ge \int_{-M}^{M} P(u)\ \mathrm{d}u.$$

Combining the four estimates, first letting $t \to \infty$ and then $M \to \infty$, proves (42).

To pass from pointwise tail equivalence to rearrangements, first note that for $t \le 0$ the substitution $v = t - u$ and (40) give

$$G(t) = \int_0^\infty P(t-v)F(v)\ \mathrm{d}v \le Ce^{t/p} \int_0^\infty e^{-v/p}F(v)\ \mathrm{d}v = O(e^{t/p}). \tag{43}$$

Let $r(\lambda) = |\{F > \lambda\}|$. To put asymptotic inversion in its standard positive-index form at infinity, define

$$h(s) = \frac{1}{F^*(s)}.$$

By (41), $h \in RV_{1/p}(\infty)$ and is eventually increasing. Up to endpoint conventions that do not affect asymptotics, $r(\lambda)$ is the generalized inverse of h evaluated at $1/\lambda$. The asymptotic inversion theorem [15, Theorem 1.5.12], applied to h, therefore gives $r \in RV_{-p}(0+)$. In particular, $r(\lambda)/|\log\lambda| \to \infty$ as $\lambda \downarrow 0$. For every $0 < \varepsilon < 1$, (42) yields, for all sufficiently large t,

$$(1-\varepsilon)F(t) \le G(t) \le (1+\varepsilon)F(t).$$

The portion of $\{G > \lambda\}$ lying to the left of that fixed threshold has measure $O(|\log\lambda|)$ by (43). Hence

$$r\left(\frac{\lambda}{1-\varepsilon}\right) - O(1) \le d_G(\lambda) \le r\left(\frac{\lambda}{1+\varepsilon}\right) + O(|\log\lambda|).$$

After division by $r(\lambda)$, regular variation gives

$$(1-\varepsilon)^p \le \liminf_{\lambda\downarrow 0} \frac{d_G(\lambda)}{r(\lambda)} \le \limsup_{\lambda\downarrow 0} \frac{d_G(\lambda)}{r(\lambda)} \le (1+\varepsilon)^p.$$

Letting $\varepsilon \downarrow 0$ shows that $d_G(\lambda) \sim r(\lambda)$ as $\lambda \downarrow 0$. Define the nondecreasing functions

$$\widetilde{d}_G(x) = d_G(1/x), \qquad \widetilde{r}(x) = r(1/x).$$

Then $\widetilde{d}_G \sim \widetilde{r}$ and both belong to $RV_p(\infty)$. Their generalized inverses are asymptotically equivalent by [15, Theorem 1.5.12]. Translating those inverses back to decreasing rearrangements gives

$$G^*(s) \sim F^*(s) \qquad (s \to \infty). \tag{44}$$

Because the index $-1/p$ is greater than -1, Karamata's integral theorem [15, Theorem 1.5.11], (41), and (44) imply

$$G^{**}(s) = \frac{1}{s}\int_0^s G^*(u)\,\mathrm{d}u \sim p' G^*(s) \sim p' F^*(s) \sim \frac{1}{a(s)}.$$

Consequently,

$$\|G\|_{\Gamma^{\alpha}_{p,\infty}} \ge \lim_{s\to\infty} a(s) G^{**}(s) = 1.$$

The reverse inequality follows from Corollary 1, since $\|P\|_1 = 1$ and (39) gives $\|F\| = 1$. Thus $\|P * F\|_{\Gamma^{\alpha}_{p,\infty}} = 1$.

Finally define $f = U_p^{-1}F$, that is, $f(x) = x^{-1/p}F(\log x)$. Then $f \ge 0$, $f \ne 0$, $\|f\|_{\mathfrak{X}^{\alpha}_{p,\infty}} = 1$, and

$$\|\mathbf{H}f\|_{\mathfrak{X}^{\alpha}_{p,\infty}} = \left\|k_p * F\right\|_{\Gamma^{\alpha}_{p,\infty}} = C_p \|P * F\|_{\Gamma^{\alpha}_{p,\infty}} = C_p.$$

□

Remark 5 (Scope of the Endpoint Mechanism)**.** The two-sided exponential bounds on the normalized Hilbert kernel are used above only for the stronger tail assertions (42)–(44). They are not needed for norm attainment itself. In fact, the same endpoint profile gives norm attainment for convolution by every nonnegative probability density $P \in L^1(\mathbb{R})$.

Indeed, keep the function F, the threshold S, and $J(s) = s^{1/p'}(1+\log s)^{-\alpha_\infty}$ from the proof of Theorem 5, and put $G = P * F$. The upper bound $\|G\|_{\Gamma^{\alpha}_{p,\infty}} \le 1$ follows from Corollary 1 and (39). For the reverse estimate, fix $A \ge S$, set $m_A = \int_{-A}^{A} P(u)\,\mathrm{d}u$, and take $s > 2A$. Since $sG^{**}(s) = \sup_{|E|=s}\int_E G \ge \int_0^s G(t)\,\mathrm{d}t$, Tonelli's theorem and nonnegativity give

$$\begin{aligned} a(s)G^{**}(s) &\ge \frac{1}{J(s)}\int_0^s G(t)\,\mathrm{d}t \\ &= \frac{1}{J(s)}\int_{\mathbb{R}} P(u)\left(\int_{-u}^{s-u} F(v)\,\mathrm{d}v\right)\mathrm{d}u \\ &\ge \frac{m_A}{J(s)}\int_A^{s-A} F(v)\,\mathrm{d}v = m_A\frac{J(s-A)-J(A)}{J(s)}. \end{aligned}$$

Here $[A, s-A] \subset [-u, s-u]$ for $|u| \le A$, and $F = J'$ on $[A,\infty)$. Since $J(s-A)/J(s) \to 1$ and $J(s) \to \infty$, taking the lower limit as $s \to \infty$ gives $\liminf_{s\to\infty} a(s)G^{**}(s) \ge m_A$. Letting $A \to \infty$ and using the upper bound

gives $\|P * F\|_{\Gamma^{\boldsymbol{\alpha}}_{p,\infty}} = 1$. Thus attainment of the norm does not require $P * F = F$, nor does it require the defining supremum of the output norm to be attained at a finite distribution size.

Remark 6 (An Explicit Endpoint Extremizer When $\boldsymbol{\alpha} = \mathbf{0}$)**.** When $\boldsymbol{\alpha} = \mathbf{0}$, one may take $S = 1$ in the preceding proof. The logarithmic profile and its pullback are then

$$F_0(t) = \mathbf{1}_{(0,1]}(t) + \frac{1}{p'} t^{-1/p} \mathbf{1}_{(1,\infty)}(t),$$
$$f_0(x) = x^{-1/p} \left[\mathbf{1}_{(1,e]}(x) + \frac{1}{p'} (\log x)^{-1/p} \mathbf{1}_{(e,\infty)}(x) \right].$$

A direct integration gives

$$F_0^{**}(s) = \begin{cases} 1, & 0 < s \le 1, \\ s^{-1/p}, & s > 1, \end{cases}$$

so $\|F_0\|_{\Gamma^{\mathbf{0}}_{p,\infty}} = 1$ and f_0 is an explicit nonnegative, nonzero extremizer.

Corollary 2 (Fine-Index Attainment Dichotomy)**.** *Let $1 < p < \infty$, $1 \le q \le \infty$, and $\boldsymbol{\alpha} \in \mathbb{R}^2$. The sharp norm of $\mathbf{H}$ on $\mathfrak{X}^{\boldsymbol{\alpha}}_{p,q}$ is attained by a nonzero function if and only if $q = \infty$.*

Proof. The finite-q assertion is Theorem 4; the $q = \infty$ assertion is Theorem 5. □

For the classical L^p norm, the normalized sequence is especially simple:

$$f_R^{(p)}(x) = (2R)^{-1/p} x^{-1/p} \mathbf{1}_{(e^{-R}, e^R)}(x). \tag{45}$$

Its logarithmic image is

$$F_R^{(p)}(t) := U_p f_R^{(p)}(t) = (2R)^{-1/p} \mathbf{1}_{[-R,R]}(t). \tag{46}$$

Thus $\left\| f_R^{(p)} \right\|_p = \left\| F_R^{(p)} \right\|_p = 1$ and $\left\| \mathbf{H} f_R^{(p)} \right\|_p \to C_p$.

5.3. Absence of Nonzero Extremizers in L^p

Theorem 6 (Non-Attainment)**.** *Let $1 < p < \infty$. If $f \in L^p(\mathbb{R}_+)$ satisfies*

$$\|\mathbf{H} f\|_{L^p} = \frac{\pi}{\sin(\pi/p)} \|f\|_{L^p}, \tag{47}$$

then $f = 0$ almost everywhere. Consequently, the sharp constant in (30) is not attained by any nonzero pair of nonnegative functions $(f, g) \in L^p \times L^{p'}$.

Proof. Let $F = U_p f$ and put

$$P_p(u) = \frac{k_p(u)}{C_p}, \qquad C_p = \|k_p\|_1. \tag{48}$$

Then P_p is a strictly positive probability density. Equality (47) is equivalent to

$$\|P_p * F\|_{L^p} = \|F\|_{L^p}. \tag{49}$$

Jensen's inequality gives, for almost every t,

$$|(P_p * F)(t)|^p \le \int_{\mathbb{R}} P_p(u) \, |F(t-u)|^p \, \mathrm{d}u. \tag{50}$$

The integrals of the two sides are equal under (49); hence equality holds in (50) for almost every t. Strict convexity of $z \mapsto |z|^p$ implies that, for almost every fixed t, the function $u \mapsto F(t-u)$ is essentially constant with respect to $P_p(u)\,\mathrm{d}u$. Choose one such t. Since $P_p(u) > 0$ for every $u \in \mathbb{R}$, the measures $P_p(u)\,\mathrm{d}u$ and $\mathrm{d}u$ have the same null sets. Hence $u \mapsto F(t-u)$ is constant for Lebesgue-almost every u. The change of variables

$r = t - u$ shows that F is constant almost everywhere on $\mathbb{R}$. Since $F \in L^p(\mathbb{R})$, that constant is zero. Therefore $F = 0$ almost everywhere and hence $f = 0$ almost everywhere. If a nonzero pair attained equality in (30), equality would also hold in the operator estimate for f, contradicting the first part. □

Remark 7 (Spreading of the Classical Plateau Sequence)**.** The functions $F_R^{(p)}$ in (46) become wider and lower as $R \to \infty$. For any choice of translations $a_R \in \mathbb{R}$ and every bounded interval I,

$$\left\| \tau_{a_R} F_R^{(p)} \right\|_{L^p(I)} \leq |I|^{1/p} (2R)^{-1/p} \longrightarrow 0.$$

Hence any possible strong $L^p(\mathbb{R})$ limit of a translated subsequence would vanish on every bounded interval and therefore be zero, contradicting the unit norms. Thus no sequence of dilations restores strong compactness for this family: it approaches the sharp constant by spreading in logarithmic coordinates. We do not claim a concentration–compactness or profile-decomposition theorem for arbitrary extremizing sequences.

6. Classical L^2 Consequences

The operator $\mathbf{H}$ on $L^2(\mathbb{R}_+)$ is the Carleman operator. Its Mellin diagonalization is classical [16]. We use it only as a short consistency check: the deficit identity and its elementary spectral-localization consequence are recorded in our normalization, with no claim of novelty for the classical diagonalization.

6.1. The Mellin Multiplier

We use the unitary Fourier transform

$$\widehat{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\xi} F(t)\, \mathrm{d}t. \tag{51}$$

For $f \in C_c^\infty(0, \infty)$, set

$$(\mathcal{M}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) x^{-1/2 - i\xi}\, \mathrm{d}x. \tag{52}$$

The change of variables $x = e^t$ gives $\mathcal{M}f = \widehat{U_2 f}$. Plancherel therefore extends $\mathcal{M}$ uniquely to a unitary map on $L^2(\mathbb{R}_+, \mathrm{d}x)$. For general L^2 data, (52) is understood in this extended sense.

For general p, define the convolution multiplier

$$m_p(\xi) := \sqrt{2\pi}\, \widehat{k_p}(\xi) = \int_{\mathbb{R}} e^{-iu\xi} k_p(u)\, \mathrm{d}u, \tag{53}$$

so that the unitary Fourier convention gives

$$\widehat{k_p * F}(\xi) = m_p(\xi) \widehat{F}(\xi). \tag{54}$$

A beta-integral calculation yields

$$m_p(\xi) = \int_0^\infty \frac{s^{1/p - i\xi - 1}}{1+s}\, \mathrm{d}s = \frac{\pi}{\sin(\pi/p - i\pi\xi)}. \tag{55}$$

Thus m_p is the convolution multiplier, not the unitary Fourier transform itself: $m_p = \sqrt{2\pi}\, \widehat{k_p}$. In particular,

$$|m_p(\xi)|^2 = \frac{\pi^2}{\sin^2(\pi/p) + \sinh^2(\pi\xi)}. \tag{56}$$

When $p = 2$,

$$m_2(\xi) = \frac{\pi}{\cosh(\pi\xi)} = \pi \operatorname{sech}(\pi\xi). \tag{57}$$

Proposition 3 (Exact Mellin Deficit). *For every $f \in L^2(\mathbb{R}_+)$,*

$$\pi^2 \|f\|_{L^2}^2 - \|\mathbf{H}f\|_{L^2}^2 = \pi^2 \int_{\mathbb{R}} \tanh^2(\pi\xi)\, |\mathcal{M}f(\xi)|^2 \, d\xi. \tag{58}$$

Consequently,

$$\pi^2 \tanh^2(\pi) \int_{\mathbb{R}} \min\{\xi^2, 1\}\, |\mathcal{M}f(\xi)|^2 \, d\xi \le \pi^2 \|f\|_2^2 - \|\mathbf{H}f\|_2^2 \le \pi^4 \int_{\mathbb{R}} \min\{\xi^2, 1\}\, |\mathcal{M}f(\xi)|^2 \, d\xi. \tag{59}$$

Proof. Set $F = U_2 f$. The classical construction above gives $U_2 \mathbf{H} f = k_2 * F$ in $L^2(\mathbb{R})$. By (54) and (57),

$$\mathcal{M}(\mathbf{H}f)(\xi) = m_2(\xi)\mathcal{M}f(\xi),$$

in $L^2(\mathbb{R})$. Plancherel's theorem therefore yields

$$\|\mathbf{H}f\|_2^2 = \int_{\mathbb{R}} \frac{\pi^2}{\cosh^2(\pi\xi)} |\mathcal{M}f(\xi)|^2 \, d\xi.$$

Subtracting this identity from $\pi^2 \|f\|_2^2 = \pi^2 \int |\mathcal{M}f|^2$ yields (58).

For $0 < r \le 1$, the function $r \mapsto \tanh(\pi r)/r$ is decreasing, and continuity of tanh extends the resulting inequality to $r = 0$, so

$$\tanh(\pi r) \ge \tanh(\pi) r.$$

For $r \ge 1$, $\tanh(\pi r) \ge \tanh(\pi)$. Thus

$$\tanh(\pi |\xi|) \ge \tanh(\pi) \min\{|\xi|, 1\}.$$

The upper bound follows from $\tanh(\pi |\xi|) \le \min\{\pi |\xi|, 1\}$. Squaring and inserting these bounds in (58) proves (59). □

Remark 8 (Spectral Meaning of the Deficit). The right-hand side of (58) vanishes only when the Mellin transform is supported at $\xi = 0$. Since $\mathcal{M}f \in L^2(\mathbb{R})$, support in the measure-zero set $\{0\}$ implies $\mathcal{M}f = 0$ almost everywhere, and hence $f = 0$. The formal homogeneous expression $x^{-1/2}$ is only a spectral heuristic for the endpoint $\xi = 0$, not a nonzero admissible equality case in $L^2(\mathbb{R}_+)$. Sharpness is produced instead by low-frequency approximation, as made explicit by Corollary 3 and by the plateau sequence. The comparison in (59) controls a first-order Mellin–Sobolev seminorm at low frequencies and the full L^2 mass at high frequencies; it is not a global homogeneous H^1 estimate.

6.2. Spectral Localization and Near-Extremizers

Here ess supp denotes essential support.

Corollary 3 (Spectral Localization). *Let $\delta > 0$.*

(a) *If* $\operatorname{ess\,supp}(\mathcal{M}f) \subset \{|\xi| \ge \delta\}$, *then*

$$\|\mathbf{H}f\|_2 \le \pi \operatorname{sech}(\pi\delta) \|f\|_2. \tag{60}$$

(b) *If* $\|f_n\|_2 = 1$, *then* $\|\mathbf{H}f_n\|_2 \to \pi$ *if and only if for every* $\delta > 0$,

$$\int_{|\xi| \ge \delta} |\mathcal{M}f_n(\xi)|^2 \, d\xi \longrightarrow 0. \tag{61}$$

Proof. Part (a) follows immediately from (57). For part (b), set $D_n = \pi^2 - \|\mathbf{H}f_n\|_2^2$. By (58), for every fixed $\delta > 0$,

$$\int_{|\xi| \ge \delta} |\mathcal{M}f_n(\xi)|^2 \, d\xi \le \frac{D_n}{\pi^2 \tanh^2(\pi\delta)}.$$

Thus $D_n \to 0$ implies (61). Conversely, for every $\delta > 0$,

$$\begin{aligned}\frac{D_n}{\pi^2} &\le \tanh^2(\pi\delta) \int_{|\xi|<\delta} |\mathcal{M}f_n(\xi)|^2 \, \mathrm{d}\xi + \int_{|\xi|\ge\delta} |\mathcal{M}f_n(\xi)|^2 \, \mathrm{d}\xi \\ &\le \tanh^2(\pi\delta) + \int_{|\xi|\ge\delta} |\mathcal{M}f_n(\xi)|^2 \, \mathrm{d}\xi.\end{aligned}$$

First take $n \to \infty$ and then $\delta \downarrow 0$ to obtain $D_n \to 0$. □

For the classical plateau at $p = 2$,

$$\mathcal{M}f_R^{(2)}(\xi) = \widehat{F_R^{(2)}}(\xi) = \frac{\sin(R\xi)}{\sqrt{\pi R}\,\xi}.$$

Hence, for every $\delta > 0$,

$$\int_{|\xi|\ge\delta} |\mathcal{M}f_R^{(2)}(\xi)|^2 \, \mathrm{d}\xi \le \frac{2}{\pi R\delta} \longrightarrow 0.$$

Thus the explicit plateaux satisfy the criterion. The criterion characterizes the frequency concentration mechanism without specifying a unique profile in physical space.

7. Conclusion

For the norm in (10)–(11),

$$\|\mathbf{H}\|_{\mathfrak{X}^{\alpha}_{p,q}\to\mathfrak{X}^{\alpha}_{p,q}} = \frac{\pi}{\sin(\pi/p)}, \qquad 1 < p < \infty, \quad 1 \le q \le \infty.$$

The norm identity follows from full symmetry for the upper estimate and from long-interval test functions for the matching lower estimate. The associated multiplicative plateaux are explicit. On the displayed Mellin-adapted scale, the norm is not attained by a nonzero function at any finite fine index, while nonnegative, nonzero extremizers exist when $q = \infty$. The classical L^p and L^2 statements serve as consistency checks within the same logarithmic model. Customary f^*-based equivalent norms lie outside the scope of the present result, and no numerical operator norm is asserted for them.

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