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Existence Result for Nonlocal ψ-Hilfer Fractional Delay Differential Inclusions via C_0-Semigroup Theory

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Abstract: This paper presents the analysis of mild solutions for a class of nonlinear ψ-Hilfer fractional differential inclusions with multiple delays and a nonlocal initial condition. Sufficient conditions for the existence of mild solutions are established using strongly continuous semigroup theory of bounded linear operators, fractional calculus, multivalued analysis, and the fixed-point theorem. The proposed formulation brings together multivalued maps, nonlocal initial conditions, and multiple discrete delay effects within the established ψ-Hilfer setting. The existence argument is formulated in a weighted trajectory space that accommodates the fractional initial trace as well as the prescribed delay history. Finally, a concrete example is presented to demonstrate the applicability of the theoretical result.

Keywords: fractional differential inclusions, C_0-semigroup, time delays, mild solutions

MSC: 34A08, 47H10, 93C25.

1. Introduction

Over the past few decades, fractional calculus has emerged as a powerful mathematical tool with broad applications across various scientific and engineering disciplines. The growing interest in fractional differential equations is largely driven by their effectiveness in modeling processes with memory and hereditary characteristics—phenomena commonly observed in physics, fluid dynamics, viscoelastic materials, heat conduction in memory-influenced media, chemistry, and engineering systems, see [1,2]. These equations provide an alternative to classical integer-order models when a memory kernel is appropriate for the process under consideration. Their practical accuracy nevertheless depends on the selected model, its parameters, and the available observations. Li, Peng, and Jia [3] studied Cauchy problems involving Riemann–Liouville fractional derivatives in infinite-dimensional Banach spaces, introducing the concept of fractional resolvent and establishing well-posedness and solution existence criteria for fractional-order problems. Zhou, Zhang, and Shen [4] introduced an appropriate notion of mild solutions for evolution equations with Riemann–Liouville fractional derivatives. For a more detailed discussion of the theory and its applications, see [5–10] and their references. Yang and Wang [11] investigated the following fractional-order evolution equations:

$$\begin{cases} D_{0^+}^{\eta,v} y(\xi) \in Ay(\xi) + F(\xi, y(\xi)) + Bu(\xi), \quad \xi \in J' = (0, b], \\ \mathcal{I}_{0^+}^{(1-\eta)(1-v)} [y(\xi)]_{\xi=0} + h(y) = y_0, \end{cases}$$

where $D_{0^+}^{\eta,v}$ is the Hilfer fractional derivative of order $v \in (0,1)$ and type $\eta \in [0,1]$. $\mathcal{I}_{0^+}^{(1-\eta)(1-v)}$ represents the Riemann–Liouville integral of order $(1-\eta)(1-v)$ with lower limit at 0. Their analysis combines fixed-point theorems with fractional calculus, semigroup methods, and the properties of multivalued mappings.

There are numerous problems in which the current rate of change of an unknown function is determined by its previous values. Time-delay systems are used to represent such problems. Furthermore, a system can experience time delays in either state or control, or both. We can examine distributed delays, discrete time-varying delays, and constant discrete delays in state variables or admissible controls in dynamic systems

with delays. Several physical and biological systems involving time delays include liquid mixing, population growth, predator–prey models, and feedback control systems. Related nonlocal stochastic inclusions with impulses are studied by Afreen et al. [12]; that work provides broader inclusion-theoretic context rather than the same deterministic delay model. Duman [13] examines fractional delay equations and controllability, while Zhou and Jiao [14] study mild solutions of fractional neutral evolution equations. Debbouche and Torres [15] provide further relevant background on fractional delay inclusions with nonlocal control conditions, in a Hilbert-space setting.

Haque, Ali, and Nieto [16] studied the following system:

$$\begin{cases} D_{0^+}^{\eta,v;\psi} y(\xi) = Ay(\xi) + g(\xi, y_\xi) + Bu(\xi), \quad \xi \in (0,b], \\ \mathcal{I}_{0^+}^{(1-\eta)(1-v);\psi} y(\xi) = h(\xi). \end{cases}$$

Using the measure of noncompactness and the Mönch fixed-point theorem, these authors derived sufficient conditions for controllability in their infinite-delay setting. Hilfer fractional equations provide a mathematical setting in which the fractional order and type can describe different initial traces and memory structures. Their use has motivated research on both existence and controllability, although such theoretical results do not by themselves establish improved accuracy for a particular application. The present study focuses on nonlinear ψ-Hilfer differential inclusions with several discrete state delays. This formulation should be distinguished from the infinite-delay controllability problem above and from the established nonlocal Hilfer inclusion framework, rather than treated as evidence that delayed fractional systems have not previously been studied.

Motivated by these studies, we consider the following system, which combines multiple discrete state delays, a multivalued nonlinear term, and a fractional nonlocal initial condition:

$$\begin{cases} D_{0^+}^{\eta,v;\psi} y(\xi) \in Ay(\xi) + G(\xi, y(\xi)) + F\big(\xi, y(\xi-\tau_0), y(\xi-\tau_1), \ldots, y(\xi-\tau_m)\big), \quad \xi \in J' = (0,b], \\ \mathcal{I}_{0^+}^{(1-\eta)(1-v);\psi} y(0^+) + q(y) = \phi(0), \\ y(\xi) = \phi(\xi), \quad \xi \in [-\tau_m, 0], \end{cases} \tag{1}$$

where $D_{0^+}^{\eta,v;\psi}$ is the ψ-Hilfer fractional derivative of order $v \in (1/2,1)$ and type $\eta \in (0,1)$. $\mathcal{I}_{0^+}^{(1-\eta)(1-v);\psi}$ represents the ψ-Riemann–Liouville integral of order $(1-\eta)(1-v)$ with lower limit at 0. Moreover, $\psi(\xi)$ is an increasing and positive monotone function on $(0,b]$, having a continuous derivative $\psi'(\xi)$ with $\psi'(\xi) \neq 0$, for all $\xi \in [0,b]$. The state function $y(\cdot)$ takes its values in a Banach space Y and $A : D(A) \subset Y \to Y$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e., C_0-semigroup) $\{S(\xi)\}_{\xi \geq 0}$ and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots < \tau_{m-1} < \tau_m$ are constant delay points. Let $J = [0,b]$, $F : J \times C([-\tau_1, 0], Y) \times C([-\tau_2, 0], Y) \times \cdots \times C([-\tau_m, 0], Y) \to Y$,$G : J \times Y \to 2^Y \setminus \emptyset$ and $q : \chi \to Y$, represent nonlinear terms satisfying the assumptions stated below, and $\phi : [-\tau_m, 0] \to Y$ is a continuous function.

1.1. Scope and Main Contributions

This study concerns nonlinear ψ-Hilfer fractional differential inclusions with multiple discrete state delays and a fractional nonlocal initial condition. The ψ-Hilfer operator itself is an established construction, introduced by Sousa and Capelas de Oliveira [17]. Its use alone does not establish novelty. The specific subject considered here is the combination of the stated delays, multivalued nonlinearity, and initial condition within a semigroup formulation. The relationship to earlier Hilfer inclusions and infinite-delay equations must therefore be assessed at the level of the hypotheses and resulting existence criterion. The key contributions are summarized as follows:

- Establishment of sufficient conditions for the existence of mild solutions to the delayed ψ-Hilfer fractional inclusion using fractional semigroup theory and fixed-point methods.
- Incorporation of multiple delays and multivalued mappings to represent uncertainty and hereditary effects in real-world systems.
- Illustration of the applicability of the developed approach by a concrete example.

The structure of this article is as follows: §2 introduces essential concepts, definitions and lemmas that form the foundation for proving the main results. §3 establishes the existence of mild solutions for (1). §4 presents a worked example that illustrates the operator estimates and checks the stated growth conditions.

2. Preliminaries and Assumptions

Let $C(J,Y)$ be the Banach space of all continuous functions from J to Y with the norm $\|y\|_*=\sup_{\xi\in J}\|y(\xi)\|$. Set $\beta=\eta+v-\eta v$, such that $1-\beta=(1-\eta)(1-v)$. We define the weighted Banach space as

$$C_{1-\beta;\psi}(J,Y)=\left\{y:J'\to Y:(\psi(\xi)-\psi(0))^{1-\beta}y(\xi)\text{ admits a continuous extension to }J\right\},$$

equipped with the norm

$$\|y\|_{1-\beta;\psi}=\sup_{\xi\in(0,b]}(\psi(\xi)-\psi(0))^{1-\beta}\|y(\xi)\|.$$

Equivalently, for every $y\in C_{1-\beta;\psi}(J,Y)$, the function

$$\widetilde{y}(\xi)=\begin{cases}(\psi(\xi)-\psi(0))^{1-\beta}y(\xi), & \xi\in(0,b],\\ \lim\limits_{\xi\to0^+}(\psi(\xi)-\psi(0))^{1-\beta}y(\xi), & \xi=0,\end{cases}$$

belongs to $C(J,Y)$.

Indeed, the mapping

$$\mathbb{W}:C_{1-\beta;\psi}(J,Y)\to C(J,Y),\qquad(\mathbb{W}y)(\xi)=(\psi(\xi)-\psi(0))^{1-\beta}y(\xi),$$

with the value at $\xi=0$ defined by continuous extension, is an isometric bijection onto $C(J,Y)$. Indeed, for each $z\in C(J,Y)$, its inverse is $y(\xi)=(\psi(\xi)-\psi(0))^{\beta-1}z(\xi)$ on J'. Since Y is a Banach space, $C(J,Y)$ is also a Banach space. Hence $C_{1-\beta;\psi}(J,Y)$ is complete under $\|\cdot\|_{1-\beta;\psi}$.

Let

$$\tau_m=\max\{\tau_1,\dots,\tau_m\},\qquad J=[0,b],\qquad J'=(0,b].$$

The prescribed history is given by

$$y(\xi)=\phi(\xi),\qquad\xi\in[-\tau_m,0],$$

where

$$\phi\in C([-\tau_m,0],Y).$$

Since the delayed arguments $\xi-\tau_k$ may belong either to the history interval or to the positive-time interval, we introduce the trajectory space

$$\chi=\left\{y:[-\tau_m,b]\to Y:y|_{[-\tau_m,0]}\in C([-\tau_m,0],Y),\quad y|_{(0,b]}\in C_{1-\beta;\psi}(J,Y)\right\},$$

where $C_{1-\beta;\psi}(J,Y)$ denotes the weighted space defined above.

We equip χ with the norm

$$\|y\|_\chi=\max\left\{\sup_{\xi\in[-\tau_m,0]}\|y(\xi)\|,\ \sup_{\xi\in(0,b]}(\psi(\xi)-\psi(0))^{1-\beta}\|y(\xi)\|\right\}.$$

Thus, both the history component and the positive-time weighted component are controlled by the same trajectory norm.

For each delay τ_k, the delayed state is consequently well defined by

$$y(\xi-\tau_k)=\begin{cases}\phi(\xi-\tau_k), & \xi-\tau_k\in[-\tau_m,0],\\ y(\xi-\tau_k), & \xi-\tau_k\in(0,b].\end{cases}$$

In particular, because $\tau_k \leq \tau_m$, every delayed argument occurring for $\xi \in [0, b]$ belongs to the domain $[-\tau_m, b]$ of the trajectory y.

The space χ, endowed with the norm $\|y\|_\chi$, is a Banach space. Indeed, it is the product of the Banach space $C([-\tau_m, 0], Y)$ and the weighted Banach space $C_{1-\beta;\psi}(J, Y)$ established above.

We use the following notation for classes of nonempty subsets of the state space: $\mathcal{P}(Y) = \{V \in 2^Y : V \neq \emptyset\}$, $\mathcal{P}_{cl}(Y) = \{V \in \mathcal{P}(Y) : V \text{ is closed}\}$, $\mathcal{P}_{bd}(Y) = \{V \in \mathcal{P}(Y) : V \text{ is bounded}\}$, $\mathcal{P}_{cp}(Y) = \{V \in \mathcal{P}(Y) : V \text{ is compact}\}$, $\mathcal{P}_{cv}(Y) = \{V \in \mathcal{P}(Y) : V \text{ is convex}\}$, and $\mathcal{P}_{bd,cl,cv}(Y) = \{V \in \mathcal{P}(Y) : V \text{ is bounded, closed and convex}\}$.

Definition 1. [18] Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a real-valued function and ψ be a non-negative increasing function such that $\psi(0) = 0$. Then the *Laplace transform* of f with respect to ψ is defined by

$$L_\psi\{f(\xi)\}(\lambda) = F(\lambda) = \int_0^\infty f(\xi) e^{-\lambda\psi(\xi)} \psi'(\xi)\, d\xi \quad \text{for those } \lambda \in \mathbb{C} \text{ for which the integral converges.}$$

Definition 2. [7] Let $(a, b), (-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $v > 0$. Also, for $\xi \in [a, b]$, let $\psi(\xi)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(\xi)$ on (a, b). Then the *left-sided fractional integral* of a function f with respect to another function ψ on $[a, b]$ is defined by

$$\mathcal{I}_{a^+}^{v;\psi} f(\xi) = \frac{1}{\Gamma(v)} \int_a^\xi \psi'(\varsigma) \left(\psi(\xi) - \psi(\varsigma)\right)^{v-1} f(\varsigma) d\varsigma.$$

Definition 3. [7] Let $\psi'(\xi) \neq 0$, for $\xi \in [a, b], (-\infty \leq a < b \leq \infty)$, and $n - 1 < v < n$ with $n \in \mathbb{N}$. The left-sided Riemann–Liouville fractional derivative of a function f with respect to ψ of order v in the Riemann–Liouville sense is defined by

$$\begin{aligned} D_{a^+}^{v;\psi} f(\xi) &= \left(\frac{1}{\psi'(\xi)} \frac{d}{d\xi}\right)^n \mathcal{I}_{a^+}^{n-v;\psi} f(\xi) \\ &= \frac{1}{\Gamma(n-v)} \left(\frac{1}{\psi'(\xi)} \frac{d}{d\xi}\right)^n \int_a^\xi \psi'(\varsigma)(\psi(\xi) - \psi(\varsigma))^{n-v-1} f(\varsigma) d\varsigma. \end{aligned}$$

Definition 4. [17] Let $n - 1 < v < n$ with $n \in \mathbb{N}$, $J = [a, b]$ be the interval such that $(-\infty \leq a < b \leq \infty)$ and, $f, \psi \in C^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(\xi) \neq 0$, for all $\xi \in J$. *The left-sided ψ-Hilfer fractional derivative* $D_{a+}^{\eta,v;\psi}(\cdot)$ of a function f, of order v and type $0 \leq \eta \leq 1$ is defined by

$$D_{a^+}^{\eta,v;\psi} f(\xi) = \mathcal{I}_{a^+}^{\eta(n-v);\psi} \left(\frac{1}{\psi'(\xi)} \frac{d}{d\xi}\right)^n \mathcal{I}_{a+}^{(1-\eta)(n-v);\psi} f(\xi).$$

Lemma 1. [17] *If $f \in C^n[a, b]$, $n - 1 < v < n$ and $0 \leq \eta \leq 1$, set $\gamma = v + \eta(n - v)$. With the endpoint traces understood as one-sided limits and the derivatives interpreted with respect to ψ, the following identities hold:*

$$\mathcal{I}_{a^+}^{v;\psi} D_{a^+}^{\eta,v;\psi} f(\xi) = f(\xi) - \sum_{k=1}^n \frac{(\psi(\xi) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_\psi^{[n-k]} \mathcal{I}_{a^+}^{(1-\eta)(n-v);\psi} f(a),$$

and

$$\mathcal{I}_{b-}^{v;\psi} D_{b^-}^{\eta,v;\psi} f(\xi) = f(\xi) - \sum_{k=1}^n \frac{(-1)^k (\psi(b) - \psi(\xi))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_\psi^{[n-k]} \mathcal{I}_{b-}^{(1-\eta)(n-v);\psi} f(b).$$

Lemma 2. [11] *Let B_r be a nonempty subset of Y which is bounded, closed, and convex. Suppose $\mathcal{H} : B_r \to 2^Y \setminus \{\emptyset\}$ is upper semicontinuous (u.s.c.) with closed, convex values, and such that $\mathcal{H}(B_r) \subset B_r$ and $\mathcal{H}(B_r)$ is compact. Then $\mathcal{H}$ has a fixed-point.*

To define the mild solution of system (1), we introduce the following special functions. Their definitions, stable-density relations, and moment identities are treated by Mainardi, Mura, and Pagnini [19].

Definition 5 (Wright Function)**.** The two-parameter Wright function is defined by

$$W_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\lambda k + \mu)}, \qquad \lambda > -1, \quad \mu \in \mathbb{C}.$$

Definition 6 (Mainardi–Wright ($\mathcal{M}$-Wright) Function)**.** For $0 < v < 1$, the Mainardi–Wright function (also called the $\mathcal{M}$-Wright or Mainardi function) is the particular Wright function

$$\mathcal{M}_v(z) = W_{-v,1-v}(-z).$$

Hence, we have

$$\mathcal{M}_v(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(1 - v - vk)}, \qquad 0 < v < 1.$$

Thus, the Mainardi–Wright function is a special case of the general Wright function.

Remark 1 (Relation with the One-Sided Stable Probability Density)**.** Let $\rho_v(x)$, $x > 0$, $0 < v < 1$, denote the one-sided stable probability density function characterized by its Laplace transform

$$\int_0^{\infty} e^{-sx} \rho_v(x)\, dx = e^{-s^v}, \qquad s > 0. \tag{2}$$

The one-sided stable density is related to the Mainardi–Wright function by

$$\rho_v(x) = v x^{-v-1} \mathcal{M}_v\left(x^{-v}\right), \qquad x > 0, \quad 0 < v < 1.$$

Equivalently, we have

$$\mathcal{M}_v(z) = \frac{1}{v} z^{-1-\frac{1}{v}} \rho_v\left(z^{-\frac{1}{v}}\right), \qquad z > 0, \quad 0 < v < 1.$$

Remark 2 (Series Representation of the One-Sided Stable Density)**.** Using

$$\mathcal{M}_v(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(1 - v - vk)},$$

in

$$\rho_v(x) = v x^{-v-1} \mathcal{M}_v(x^{-v}),$$

and using the reflection formula for the Gamma function, we obtain

$$\rho_v(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} x^{-vn-1} \frac{\Gamma(vn+1)}{n!} \sin(n\pi v), \qquad x > 0, \quad 0 < v < 1.$$

This is the series representation of the one-sided stable probability density.

Lemma 3. *Suppose that y is a solution of problem (1) such that*

$$y(\xi) \in D(A) \quad \text{for a.e. } \xi \in (0, b],$$

and

$$Ay \in L^1_{\text{loc}}((0, b], Y).$$

Then, the Cauchy problem (1) is equivalent to

$$y(\xi)=\begin{cases}\dfrac{(\psi(\xi)-\psi(0))^{(\eta-1)(1-v)}}{\Gamma(\eta(1-v)+v)}(\phi(0)-q(y))+\dfrac{1}{\Gamma(v)}\displaystyle\int_0^{\xi}(\psi(\xi)-\psi(\varsigma))^{v-1}\Big[Ay(\varsigma)+g(\varsigma)\\ +F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\Big]\psi'(\varsigma)d\varsigma,\quad \xi\in J'=(0,b],\\ \phi(\xi),\quad \xi\in[-\tau_m,0),\end{cases}\tag{3}$$

where g is the measurable selection satisfying $g(\xi)\in G(\xi,y(\xi))$, for a.e. $\xi\in(0,b]$.

Proof. Applying the operator $\mathcal{I}_{0^+}^{v;\psi}$ to both sides of (1) and using Lemma 1, we obtain (3). □

Following the semigroup approaches in [11,20] and the treatment of delayed Hilfer equations in [21], we present the following representation of mild solutions for system (1).

Lemma 4. *If the integral equation (3) holds, then we have*

$$y(\xi)=\begin{cases}\mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y))+\displaystyle\int_0^{\xi}K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma\\ +\displaystyle\int_0^{\xi}K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma,\quad \xi\in J'=(0,b],\\ \phi(\xi),\quad \xi\in[-\tau_m,0),\end{cases}\tag{4}$$

where g is the measurable selection satisfying $g(\xi)\in G(\xi,y(\xi))$, for a.e. $\xi\in(0,b]$.

$$\begin{aligned}&K_{\eta,v;\psi}(\xi,\varsigma)=(\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma),\\ &\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)=v\int_0^{\infty}z\mathcal{M}_v(z)S((\psi(\xi)-\psi(\varsigma))^v z)dz,\ \textit{where}\ \mathcal{M}_v(z)=\frac{1}{v}z^{-\frac{1}{v}-1}\rho_v(z^{-\frac{1}{v}}),\ \textit{and}\\ &\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)=\frac{1}{\Gamma(\eta(1-v))}\int_{\varsigma}^{\xi}(\psi(\xi)-\psi(s))^{\eta(1-v)-1}K_{\eta,v;\psi}(s,\varsigma)\psi'(s)\,ds.\end{aligned}$$

Proof. Let $\lambda>0$. Applying the Laplace transform with respect to ψ

$$Y(\lambda)=\int_0^{\infty}e^{-\lambda(\psi(\varsigma)-\psi(0))}y(\varsigma)\psi'(\varsigma)d\varsigma,\ \ \omega(\lambda)=\int_0^{\infty}e^{-\lambda(\psi(\varsigma)-\psi(0))}g(\varsigma)\psi'(\varsigma)d\varsigma,$$

and

$$\omega_1(\lambda)=\int_0^{\infty}e^{-\lambda(\psi(\varsigma)-\psi(0))}F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),y(\varsigma-\tau_2),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma,$$

to (3), we have

$$\begin{aligned}Y(\lambda)&=\lambda^{\eta(v-1)}(\lambda^v I-A)^{-1}(\phi(0)-q(y))+(\lambda^v I-A)^{-1}\omega(\lambda)+(\lambda^v I-A)^{-1}\omega_1(\lambda)\\ &=\lambda^{\eta(v-1)}\int_0^{\infty}e^{-\lambda^v s}S(\varsigma)(\phi(0)-q(y))d\varsigma+\lambda^v\int_0^{\infty}e^{-\lambda^v s}S(\varsigma)\omega(\lambda)d\varsigma+\int_0^{\infty}e^{-\lambda^v s}S(\varsigma)\omega_1(\lambda)d\varsigma,\end{aligned}\tag{5}$$

provided that the integrals in (5) exist, where I is the identity operator defined on Y.

Now we consider the following change of variable: $s=\hat{t}^v$ and $ds=v\hat{t}^{v-1}d\hat{t}$.

Then (5) becomes

$$Y(\lambda) = \lambda^{\eta(v-1)} \int_0^\infty e^{-(\lambda\hat{t})^v} S(\hat{t}^v) v\hat{t}^{v-1}(\phi(0) - q(y))d\hat{t} + \lambda^v \int_0^\infty e^{-(\lambda\hat{t}^v)^v} S(\hat{t}^v) v\hat{t}^{v-1}\omega(\lambda)d\hat{t}$$
$$+ \int_0^\infty e^{-(\lambda\hat{t}^v)^v} S(\hat{t}^v) v\hat{t}^{v-1}\omega_1(\lambda)d\hat{t},$$

Take $\hat{t} = \psi(\xi) - \psi(0)$. Then, we get

$$\begin{aligned}
Y(\lambda) &= v\int_0^\infty \lambda^{\eta(v-1)}(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v)(\phi(0)-q(y))\psi'(\xi)\,d\xi \\
&\quad + v\int_0^\infty (\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v)\omega(\lambda)\psi'(\xi)\,d\xi \\
&\quad + v\int_0^\infty \lambda^v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v)\omega_1(\lambda)\psi'(\xi)\,d\xi \\
&= v\int_0^\infty \lambda^{\eta(v-1)}(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v)(\phi(0)-q(y))\psi'(\xi)\,d\xi \\
&\quad + v\int_0^\infty\int_0^\infty (\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v) e^{-\lambda(\psi(\varsigma)-\psi(0))} g(\varsigma)\psi'(\varsigma)\psi'(\xi)\,d\varsigma\,d\xi \\
&\quad + v\int_0^\infty\int_0^\infty (\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v) e^{-\lambda(\psi(\varsigma)-\psi(0))} \\
&\qquad \times F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1), \ldots, y(\varsigma-\tau_m)\big)\psi'(\varsigma)\psi'(\xi)\,d\varsigma\,d\xi. \qquad (6)
\end{aligned}$$

Now, (6) becomes

$$\begin{aligned}
Y(\lambda) &= \int_0^\infty \lambda^{\eta(v-1)} v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v)(\phi(0)-q(y))\psi'(\xi)d\xi \\
&\quad + \int_0^\infty\int_0^\infty v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v) e^{-(\lambda(\psi(\varsigma)-\psi(0)))} \\
&\qquad \times g(\varsigma)\psi'(\varsigma)\psi'(\xi)d\varsigma d\xi \\
&\quad + \int_0^\infty\int_0^\infty v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0)))^v} S((\psi(\xi)-\psi(0))^v) e^{-(\lambda(\psi(\varsigma)-\psi(0)))} \\
&\qquad \times F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1), \ldots, y(\varsigma-\tau_m)\big)\psi'(\varsigma)\psi'(\xi)d\varsigma d\xi. \qquad (7)
\end{aligned}$$

Using (2), the above Eq. (7) becomes

$$\begin{aligned}
Y(\lambda) &= \int_0^\infty\int_0^\infty \lambda^{\eta(v-1)} v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0))\sigma)} \rho_v(\sigma) S((\psi(\xi)-\psi(0))^v)(\phi(0)-q(y))\psi'(\xi)d\sigma d\xi \\
&\quad + \int_0^\infty\int_0^\infty\int_0^\infty v(\psi(\xi)-\psi(0))^{v-1} e^{-(\lambda(\psi(\xi)-\psi(0))\sigma)} \rho_v(\sigma) S((\psi(\xi)-\psi(0))^v) e^{-(\lambda(\psi(\varsigma)-\psi(0)))} \\
&\qquad \times g(\varsigma)\psi'(\varsigma)\psi'(\xi)d\sigma d\varsigma d\xi
\end{aligned}$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty v(\psi(\xi)-\psi(0))^{v-1}e^{-(\lambda(\psi(\xi)-\psi(0))\sigma)}\rho_v(\sigma)S((\psi(\xi)-\psi(0))^v)e^{-(\lambda(\psi(\varsigma)-\psi(0)))}$$
$$\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\psi'(\xi)d\sigma d\varsigma d\xi. \tag{8}$$

Now we consider the change of variable $\psi(\xi)-\psi(0)=\frac{\psi(\theta)-\psi(0)}{\sigma}\implies\psi'(\theta)d\theta=\sigma\psi'(\xi)d\xi$. Then, (8) becomes

$$\begin{aligned}Y(\lambda)&=v\int_0^\infty\int_0^\infty\lambda^{\eta(v-1)}e^{-(\lambda(\psi(\theta)-\psi(0)))}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)(\phi(0)-q(y))\psi'(\theta)d\sigma d\theta\\
&+v\int_0^\infty\int_0^\infty\int_0^\xi e^{-(\lambda(\psi(\theta)+\psi(\varsigma)-2\psi(0)))}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)\\
&\qquad\times g(\varsigma)\psi'(\varsigma)\psi'(\theta)d\sigma d\varsigma d\theta\\
&+v\int_0^\infty\int_0^\infty\int_0^\xi e^{-(\lambda(\psi(\theta)+\psi(\varsigma)-2\psi(0)))}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)\\
&\qquad\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big))\psi'(\varsigma)\psi'(\theta)d\sigma d\varsigma d\theta.\end{aligned}\tag{9}$$

Considering the new change of variable $\psi(\theta)+\psi(\varsigma)-2\psi(0)=\psi(\tau)-\psi(0)\to\psi(\tau)-\psi(\varsigma)=\psi(\theta)-\psi(0)$, we have

$$\begin{cases}\theta\to0&\implies\ \tau\to\varsigma\\ \theta\to\infty&\implies\ \tau\to\infty\end{cases},\quad\begin{cases}0<\varsigma<\infty\\ \varsigma<\tau<\infty\end{cases}\implies\quad 0<s<\tau,$$

and

$$\begin{cases}0<\varsigma<\infty\\ \varsigma<\tau<\infty\end{cases}\implies\quad 0<\varsigma<\infty.$$

Then, (9) becomes

$$\begin{aligned}Y(\lambda)&=\int_0^\infty e^{-(\lambda(\psi(\theta)-\psi(0)))}\left[v\int_0^\infty\lambda^{\eta(v-1)}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)(\phi(0)-q(y))d\sigma\right]\psi'(\theta)d\theta\\
&+v\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda(\psi(\tau)-\psi(0)))}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)\\
&\qquad\times g\big(\psi^{-1}(\psi(\tau)-\psi(\theta)+\psi(0)),y(\psi^{-1}(\psi(\tau)-\psi(\theta)+\psi(0)))\big)\psi'(\tau)\psi'(\theta)d\sigma d\tau d\theta\\
&+v\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda(\psi(\tau)-\psi(0)))}\frac{(\psi(\theta)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\theta)-\psi(0))^v}{\sigma^v}\right)\\
&\qquad\times F\big(\psi^{-1}(\psi(\tau)-\psi(\theta)+\psi(0)),y(\psi^{-1}(\psi(\tau)-\psi(\theta)+\psi(0)))-\tau_0),\\
&\qquad\cdots,y(\psi^{-1}(\psi(\tau)-\psi(\theta)+\psi(0))-\tau_m)\big)\psi'(\tau)\psi'(\theta)d\sigma d\tau d\theta\\
&=\int_0^\infty e^{-(\lambda(\psi(\tau)-\psi(0)))}\left[v\int_0^\infty\lambda^{\eta(v-1)}\frac{(\psi(\tau)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\tau)-\psi(0))^v}{\sigma^v}\right)(\phi(0)-q(y))d\sigma\right.\\
&+v\int_0^\tau\int_0^\infty\frac{(\psi(\tau)-\psi(s))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\tau)-\psi(\varsigma))^v}{\sigma^v}\right)g(\varsigma)\psi'(\varsigma)d\sigma ds\\
&+v\int_0^\tau\int_0^\infty\frac{(\psi(\tau)-\psi(\varsigma))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\tau)-\psi(\varsigma))^v}{\sigma^v}\right)\end{aligned}$$

$$\times F(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1), \cdots, y(\varsigma-\tau_m))\psi'(\varsigma)d\sigma d\varsigma\Big]\psi'(\tau)d\tau. \tag{10}$$

Applying the inverse Laplace transform with respect to ψ to (10), we obtain

$$\begin{aligned} y(\xi) = \left(L_\psi^{-1}(\lambda^{\eta(v-1)})\right) * v\int_0^\infty \frac{(\psi(\xi)-\psi(0))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\xi)-\psi(0))^v}{\sigma^v}\right)(\phi(0)-q(y))d\sigma \\ + v\int_0^\infty\int_0^\xi \frac{(\psi(\xi)-\psi(s))^{v-1}}{\sigma^v}\rho_v(\sigma)S\left(\frac{(\psi(\xi)-\psi(s))^v}{\sigma^v}\right)g(s)\psi'(s)dsd\sigma \\ + v\int_0^\infty\int_0^\xi \frac{(\psi(\xi)-\psi(s))^{v-1}}{\sigma^v}\rho_v(\sigma)\left(\frac{(\psi(\xi)-\psi(s))^v}{\sigma^v}\right) \\ \times F\big(s, y(s-\tau_0), y(s-\tau_1), \ldots, y(s-\tau_m)\big)\psi'(s)dsd\sigma. \end{aligned} \tag{11}$$

Now we consider the following change of variable:

$$\frac{1}{\sigma^v} = z \Longrightarrow \sigma = z^{-\frac{1}{v}} \text{ and } d\sigma = -\frac{1}{v}z^{-\frac{1}{v}-1}dz.$$

Then (11) becomes

$$\begin{aligned} y(\xi) = \left(L_\psi^{-1}(\lambda^{\eta(v-1)})\right) * v\int_0^\infty z(\psi(\xi)-\psi(0))^{v-1}S((\psi(\xi)-\psi(0))^v z)\left(\frac{1}{v}z^{-\frac{1}{v}-1}\right)\rho_v(z^{\frac{-1}{v}})(\phi(0)-q(y))dz \\ + v\int_0^\infty\int_0^\xi z(\psi(\xi)-\psi(s))^{v-1}S((\psi(\xi)-\psi(s))^v z)\left(\frac{1}{v}z^{-\frac{1}{v}-1}\right)\rho_v(z^{\frac{-1}{v}})g(\varsigma)\psi'(\varsigma)d\varsigma dz \\ + v\int_0^\infty\int_0^\xi z(\psi(\xi)-\psi(\varsigma))^{v-1}S((\psi(\xi)-\psi(\varsigma))^v z)\left(\frac{1}{v}z^{-\frac{1}{v}-1}\right)\rho_v(z^{\frac{-1}{v}}) \\ \times F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1), \ldots, y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma dz. \end{aligned} \tag{12}$$

Thus, we have

$$\begin{aligned} y(\xi) = \left(L_\psi^{-1}(\lambda^{\eta(v-1)})\right) * v\int_0^\infty z(\psi(\xi)-\psi(0))^{v-1}S((\psi(\xi)-\psi(0))^v z)\mathcal{M}_v(z)(\phi(0)-q(y))dz \\ + v\int_0^\infty\int_0^\xi z(\psi(\xi)-\psi(\varsigma))^{v-1}S((\psi(\xi)-\psi(\varsigma))^v z)\mathcal{M}_v(z)g(\varsigma)\psi'(\varsigma)d\varsigma dz \\ + v\int_0^\infty\int_0^\xi z(\psi(\xi)-\psi(\varsigma))^{v-1}S((\psi(\xi)-\psi(\varsigma))^v z)\mathcal{M}_v(z) \\ \times F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1), \ldots, y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma dz, \end{aligned} \tag{13}$$

where

$$\mathcal{M}_v(z) = \frac{1}{v}z^{-\frac{1}{v}-1}\rho_v(z^{\frac{-1}{v}}), \quad z\in(0,\infty).$$

Therefore, we have

$$y(\xi) = \left(L_\psi^{-1}(\lambda^{\eta(v-1)}) * K_{\eta,v;\psi}(\xi,0)\right)(\phi(0)-q(y)) + \int_0^\xi K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma$$

$$+\int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma$$

$$=\mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y))+\int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma$$

$$+\int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma.$$

This completes the proof. □

In view of Lemma 4, we give the following definition of the mild solution of (1).

Definition 7. A function $y \in \chi$ is called a mild solution of system (1) if there exists a measurable selection $g \in \mathbb{S}_{G,y}$ such that it satisfies

$$y(\xi)=\begin{cases}\mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y))+\displaystyle\int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma\\ +\displaystyle\int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma, \quad \xi\in J'=(0,b],\\ \phi(\xi), \quad \xi\in[-\tau_m,0).\end{cases}$$

Assume the following:

(A0) The operator family $\{S(\xi)\}_{\xi\geq 0}$ is a strongly continuous (C_0) semigroup on Y and is uniformly bounded; that is, there exists $M \geq 1$ such that $\sup_{\xi\geq 0}\|S(\xi)\| \leq M$. Moreover, $S(\xi)$ is compact for every $\xi > 0$.

Remark 3. Under assumption (A0), the operator family $\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)$ is strongly continuous for $0 \leq \varsigma < \xi \leq b$.

Proof. Let $y \in Y$ be fixed and let $(\xi_n,\varsigma_n) \to (\xi,\varsigma)$, where $0 \leq \varsigma < \xi \leq b$. Since ψ is continuous and strictly increasing, we have

$$(\psi(\xi_n)-\psi(\varsigma_n))^v \longrightarrow (\psi(\xi)-\psi(\varsigma))^v.$$

For simplicity, set

$$a_n=(\psi(\xi_n)-\psi(\varsigma_n))^v, \qquad a=(\psi(\xi)-\psi(\varsigma))^v.$$

From the definition of $\mathcal{P}_{\eta,v;\psi}$, we obtain

$$\mathcal{P}_{\eta,v;\psi}(\xi_n,\varsigma_n)y-\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y=v\int_0^{\infty} z\mathcal{M}_v(z)\,[S(a_nz)y-S(az)y]\,dz.$$

Hence, we have

$$\big\|\mathcal{P}_{\eta,v;\psi}(\xi_n,\varsigma_n)y-\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y\big\| \leq v\int_0^{\infty} z\mathcal{M}_v(z)\,\|S(a_nz)y-S(az)y\|\,dz.$$

Since $\{S(\xi)\}_{\xi\geq 0}$ is a strongly continuous (C_0)-semigroup, for every fixed $z \geq 0$, we get

$$S(a_nz)y \longrightarrow S(az)y \qquad \text{as } n\to\infty.$$

Moreover, by the uniform boundedness of the semigroup, $\|S(\xi)\| \leq M$, we have

$$\|S(a_nz)y-S(az)y\| \leq 2M\|y\|.$$

Therefore, we obtain

$$vz\mathcal{M}_v(z)\,\|S(a_nz)y-S(az)y\| \leq 2Mvz\mathcal{M}_v(z)\|y\|.$$

Using the moment property of the $\mathcal{M}$-Wright function [19], we have

$$\int_0^\infty z\mathcal{M}_v(z)\,dz = \frac{\Gamma(2)}{\Gamma(1+v)} = \frac{1}{\Gamma(1+v)},$$

It follows that the dominating function on the right-hand side is integrable. Consequently, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \left\|\mathcal{P}_{\eta,v;\psi}(\xi_n,\varsigma_n)y - \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y\right\| = 0.$$

Thus, for every $y \in Y$, we have

$$\mathcal{P}_{\eta,v;\psi}(\xi_n,\varsigma_n)y \longrightarrow \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y,$$

which proves that $\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)$ is strongly continuous. □

Lemma 5. *Under assumption (A0), the characteristic operators* $\{\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)\}$, *and* $\{K_{\eta,v;\psi}(\xi,\varsigma)\}$ *are strongly continuous for* $0 \le \varsigma < \xi \le b$, *and for every* $y \in Y$, *we have*

$$\|\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{M(\psi(\xi)-\psi(\varsigma))^{\beta-1}}{\Gamma(\eta(1-v)+v)}\|y\|,$$
$$\|K_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{M(\psi(\xi)-\psi(\varsigma))^{v-1}}{\Gamma(v)}\|y\| \text{ and}$$
$$\|\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{M}{\Gamma(v)}\|y\|.$$

Proof. Let $\alpha = \eta(1-v)$, $\beta = \eta + v - \eta v = \alpha + v$. Under assumption $(A0)$, the operator family $\{S(\xi)\}_{\xi\ge 0}$ is strongly continuous and uniformly bounded, i.e.,

$$\|S(\xi)\| \le M, \qquad \xi \ge 0.$$

We first prove the estimate for $\mathcal{P}_{\eta,v;\psi}$. From its definition, we have

$$\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y = v\int_0^\infty z\mathcal{M}_v(z)S\left((\psi(\xi)-\psi(\varsigma))^v z\right)y\,dz.$$

Hence, using $\|S(\xi)\| \le M$, we get

$$\|\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y\| \le vM\|y\|\int_0^\infty z\mathcal{M}_v(z)\,dz.$$

Using the moment property of the $\mathcal{M}$-Wright function [19], $v\int_0^\infty z\mathcal{M}_v(z)\,dz = \frac{1}{\Gamma(v)}$, we obtain

$$\|\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{M}{\Gamma(v)}\|y\|.$$

Since $K_{\eta,v;\psi}(\xi,\varsigma) = (\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)$, it follows that

$$\|K_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{M}{\Gamma(v)}(\psi(\xi)-\psi(\varsigma))^{v-1}\|y\|.$$

It remains to estimate $\mathcal{T}_{\eta,v;\psi}$. By the definition of $\mathcal{T}_{\eta,v;\psi}$, we have

$$\|\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y\| \le \frac{1}{\Gamma(\alpha)}\int_\varsigma^\xi (\psi(\xi)-\psi(s))^{\alpha-1}\|K_{\eta,v;\psi}(s,\varsigma)y\|\psi'(s)\,ds.$$

Using the above estimate for $K_{\eta,v;\psi}$, we get

$$\|\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y\| \leq \frac{M\|y\|}{\Gamma(\alpha)\Gamma(v)} \int_{\varsigma}^{\xi} (\psi(\xi)-\psi(s))^{\alpha-1}(\psi(s)-\psi(\varsigma))^{v-1}\psi'(s)\,ds.$$

Now put $u = \frac{\psi(s)-\psi(\varsigma)}{\psi(\xi)-\psi(\varsigma)}$. Then $du = \frac{\psi'(s)}{\psi(\xi)-\psi(\varsigma)}\,ds$, and we obtain

$$\int_{\varsigma}^{\xi} (\psi(\xi)-\psi(s))^{\alpha-1}(\psi(s)-\psi(\varsigma))^{v-1}\psi'(s)\,ds = (\psi(\xi)-\psi(\varsigma))^{\alpha+v-1}\int_0^1 (1-u)^{\alpha-1}u^{v-1}\,du.$$

The last integral is the following Beta function:

$$B(\alpha,v) = \int_0^1 (1-u)^{\alpha-1}u^{v-1}\,du = \frac{\Gamma(\alpha)\Gamma(v)}{\Gamma(\alpha+v)}.$$

Consequently, we have

$$\|\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y\| \leq \frac{M}{\Gamma(\alpha+v)}(\psi(\xi)-\psi(\varsigma))^{\alpha+v-1}\|y\|.$$

Since $\alpha+v = \eta(1-v)+v = \eta+v-\eta v = \beta$, we obtain

$$\|\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y\| \leq \frac{M(\psi(\xi)-\psi(\varsigma))^{\beta-1}}{\Gamma(\eta(1-v)+v)}\|y\|.$$

We next prove the strong continuity of $\mathcal{T}_{\eta,v;\psi}$. Let $y \in Y$ be fixed and let

$$(\xi_n,\varsigma_n) \to (\xi,\varsigma), \qquad 0 \leq \varsigma < \xi \leq b.$$

Set $\Delta_n = \psi(\xi_n)-\psi(\varsigma_n)$, $\Delta = \psi(\xi)-\psi(\varsigma)$.

Since ψ is continuous and strictly increasing, we have $\Delta_n \to \Delta > 0$. Using the substitution $u = \frac{\psi(s)-\psi(\varsigma_n)}{\Delta_n}$, we obtain

$$\mathcal{T}_{\eta,v;\psi}(\xi_n,\varsigma_n)y = \frac{\Delta_n^{\beta-1}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{v-1}\mathcal{P}_{\eta,v;\psi}(s_n(u),\varsigma_n)y\,du,$$

where $s_n(u)$ is determined by $\psi(s_n(u)) = \psi(\varsigma_n)+u\Delta_n$. Similarly, we have

$$\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y = \frac{\Delta^{\beta-1}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{v-1}\mathcal{P}_{\eta,v;\psi}(s(u),\varsigma)y\,du.$$

For every fixed $u \in (0,1)$, $s_n(u) \to s(u)$, and hence, by the strong continuity of $\mathcal{P}_{\eta,v;\psi}$, we have

$$\mathcal{P}_{\eta,v;\psi}(s_n(u),\varsigma_n)y \longrightarrow \mathcal{P}_{\eta,v;\psi}(s(u),\varsigma)y.$$

Moreover, we have

$$\|\mathcal{P}_{\eta,v;\psi}(s_n(u),\varsigma_n)y\| \leq \frac{M}{\Gamma(v)}\|y\|.$$

Thus the integrand is dominated by

$$\frac{M}{\Gamma(v)}(1-u)^{\alpha-1}u^{v-1}\|y\|,$$

which is integrable on $(0,1)$ for $\alpha > 0$ and $0 < v < 1$. Therefore, by the dominated convergence theorem,

$$\mathcal{T}_{\eta,v;\psi}(\xi_n,\varsigma_n)y \longrightarrow \mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)y.$$

Hence $\{\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)\}$ is strongly continuous.

Finally, since

$$K_{\eta,v;\psi}(\xi,\varsigma)y = (\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)y,$$

the strong continuity of $\mathcal{P}_{\eta,v;\psi}$ and the continuity of ψ imply the strong continuity of $K_{\eta,v;\psi}$ on $0 \le \varsigma < \xi \le b$.

Thus, all the asserted properties of $\mathcal{T}_{\eta,v;\psi}$, $K_{\eta,v;\psi}$, and $\mathcal{P}_{\eta,v;\psi}$ are proved. □

Next, we consider the following assumptions:

(A1) The function $F(\xi,\cdot,\cdots,\cdot) : C([0,b],Y) \times C([-\tau_1,0],Y) \times C([-\tau_2,0],Y) \times \cdots \times C([-\tau_m,0],Y) \to Y$ is continuous for almost all $\xi \in (0,b]$. Additionally, for each $y(\xi-\tau_0) \in C([0,b],Y), y(\xi-\tau_1) \in C([-\tau_1,0],Y)$, $y(\xi-\tau_2) \in C([-\tau_2,0],Y), \cdots, y(\xi-\tau_m) \in C([-\tau_m,0],Y)$, the function $F\big(\cdot, y(\xi-\tau_0), y(\xi-\tau_1),\cdots,y(\xi-\tau_m)\big) : [0,b] \to Y$ is strongly measurable;

(A2) There exist $v_1 \in [0,v)$ and $h \in L^{\frac{1}{v_1}}((0,b],\mathbb{R}^+)$, such that

$$\mathcal{I}_{0^+}^{v;\psi}h \in C((0,b],\mathbb{R}^+), \quad \lim_{\xi\to 0^+}(\psi(\xi)-\psi(0))^{(1-\eta)(1-v)}\mathcal{I}_{0+}^{v;\psi}h(\xi) = 0,$$

and

$$\|F\big(\xi, y(\xi-\tau_0), y(\xi-\tau_1),\ldots,y(\xi-\tau_m)\big)\| \le h(\xi),$$

for all $y(\xi-\tau_0) \in C([0,b],Y), y(\xi-\tau_1) \in C([-\tau_1,0],Y)$, $y(\xi-\tau_2) \in C([-\tau_2,0],Y),\cdots,y(\xi-\tau_m) \in C([-\tau_m,0],Y)$ and almost all $\xi \in [0,b]$;

(A3) Assume that $F : [0,b] \times C([0,b],Y) \times C([-\tau_1,0],Y) \times C([-\tau_2,0],Y) \times \cdots \times C([-\tau_m,0],Y) \to Y$ is a continuous function. There exists a non-negative continuous function $\rho_k(\cdot)(k=0,1,\ldots,m)$ satisfying

$$\begin{aligned}&\|F\big(\xi,y(\xi-\tau_0),y(\xi-\tau_1),\ldots,y(\xi-\tau_m)\big) - F\big(\xi,\tilde{y}(\xi-\tau_0),\tilde{y}(\xi-\tau_1),\ldots,\tilde{y}(\xi-\tau_m)\big)\|\\ &\le \sum_{k=0}^{m}\rho_k(\xi)(\psi(\xi)-\psi(0))^{1-\beta}\|y(\xi-\tau_k)-\tilde{y}(\xi-\tau_k)\|, \quad \xi\in(0,b],\end{aligned}$$

for any $y(\xi-\tau_0), \tilde{y}(\xi-\tau_0) \in C([0,b],Y), y(\xi-\tau_1),\tilde{y}(\xi-\tau_1) \in C([-\tau_1,0],Y), y(\xi-\tau_2),\tilde{y}(\xi-\tau_2) \in C([-\tau_2,0],Y),\cdots, y(\xi-\tau_m),\tilde{y}(\xi-\tau_m) \in C([-\tau_m,0],Y)$;

(A4) The multivalued map $G : J \times Y \to \mathcal{P}_{bd,cl,cv}(Y)$ satisfies the following conditions:

(i) $G(\xi,\cdot) : Y \to \mathcal{P}_{bd,cl,cv}(Y)$ is upper semicontinuous for each $\xi \in J$, and for each $y \in Y$, the function $G(\cdot,y) : J \to \mathcal{P}_{bd,cl,cv}(Y)$ is strongly measurable in ξ, and for each trajectory $y \in \chi$, the set $\mathbb{S}_{G,y} = \{g \in L^1(J,Y) : g(\xi) \in G(\xi,y(\xi)),$ for a.e. $\xi \in J\}$ is nonempty, convex and weakly compact in $L^1(J,Y)$ and the family $\{\mathbb{S}_{G,y} : y \in B_r\}$ is uniformly integrable in $L^1(J,Y)$;

(ii) there exists a constant $v_1 \in [0,v)$ and $\Theta \in L^{\frac{1}{v_1}}((0,b],\mathbb{R}^+)$ and a continuous non-decreasing function $\Omega : [0,\infty) \to (0,\infty)$ such that for any $(\xi,y) \in J \times Y$, we have

$$\|G(\xi,y(\xi))\| = \sup_{\xi\in J'}\{\|g(\xi)\| : g(\xi) \in G(\xi,y(\xi))\} \le \Theta(\xi) \times \Omega\left((\psi(\xi)-\psi(0))^{1-\beta}\|y(\xi)\|\right),$$

$$\liminf_{r\to\infty}\frac{\Omega(r)}{r} = \Lambda < \infty;$$

(A5) $q : \chi \to Y$, is continuous, and there exists a constant $L_q > 0$ such that

$$\|q(y)-q(x)\| \le L_q\|y-x\|.$$

We first introduce the necessary notation.

$$d = \frac{v-1}{1-v_1} \in (-1,0), \quad \triangle = \frac{(\psi(b)-\psi(0))^{(1+d)(1-v_1)}}{(1+d)^{(1-v_1)}}, \quad M_1 = \|h\|_{L^{\frac{1}{v_1}}}.$$

3. Existence of Mild Solutions

Theorem 1. *If assumptions (A0)–(A5) are satisfied, then the system (1) has at least one mild solution provided that* $K < 1$, *where*

$$K = \frac{ML_q}{\Gamma(\eta(1-v)+v)} + (\psi(b)-\psi(0))^{1-\beta}\frac{M\Lambda \triangle \|\Theta\|_{L^{\frac{1}{v_1}}}}{\Gamma(v)}. \tag{14}$$

Proof. Define the multivalued map $\mathcal{H} : \chi \to 2^{\chi} \setminus \{\emptyset\}$ and $g \in \mathbb{S}_{G,y}$ by

$$(\mathcal{H}y)(\xi) = \begin{cases} \zeta \in \chi : \zeta(\xi) = \begin{cases} \mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y)) + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma \\ + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma, y(\varsigma-\tau_0),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma, \\ \qquad\qquad g \in \mathbb{S}_{G,y}, \xi \in J' = (0,b], \\ \phi(\xi), \quad \xi \in [-\tau_m, 0). \end{cases} \end{cases}$$

We show that $\mathcal{H}$ is completely continuous with bounded, closed and convex values, and upper semicontinuous (u.s.c.). For clarity, the proof is presented in the following steps.

Step 1. $\mathcal{H}y$ is convex for every $y \in \chi$. In fact, if $\zeta_1, \zeta_2 \in \mathcal{H}y$, then there exist $g_1, g_2 \in \mathbb{S}_{G,y}$ such that, for every $\xi \in (0,b]$, the following representations hold:

$$\zeta_i(\xi) = \mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y)) + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g_i(\varsigma)\psi'(\varsigma)d\varsigma$$
$$+ \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma, \quad i = 1,2.$$

Let $0 \le \varepsilon \le 1$. Then for each $\xi \in J'$, we have

$$\big(\varepsilon\zeta_1 + (1-\varepsilon)\zeta_2\big)(\xi) = \mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y)) + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)\Big[\varepsilon g_1(\varsigma) + (1-\varepsilon)g_2(\varsigma)\Big]\psi'(\varsigma)d\varsigma$$
$$+ \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma.$$

Clearly, $\mathbb{S}_{G,y}$ is convex, since G has convex values. Therefore, $\varepsilon g_1 + (1-\varepsilon)g_2 \in \mathbb{S}_{G,y}$. Consequently, $\varepsilon\zeta_1 + (1-\varepsilon)\zeta_2 \in \mathcal{H}y$.

Step 2. $\mathcal{H}y$ is closed for each $y \in \chi$.

$$\zeta_n(\xi) = \mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y)) + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g_n(\varsigma)\psi'(\varsigma)d\varsigma$$
$$+ \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma, y(\varsigma-\tau_0), y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma.$$

By assumption (A4)(i), the set $\mathbb{S}_{G,y}$ is weakly compact in $L^1(J,Y)$. Consequently, a subsequence of g_n, denoted by the same symbol, converges weakly to an element $g \in \mathbb{S}_{G,y}$. The subsequent passage through

the singular integral operator requires the kernel estimates and selection bounds, in addition to this weak convergence. The intended limiting representation is

$$\zeta_n(\xi) \longrightarrow \zeta(\xi) = \mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y)) + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma + \int_0^{\xi} K_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma.$$

Step 3. $\mathcal{H}$ is a completely continuous multivalued operator.
We divide this part of the proof into three separate claims.

Claim 1. We shall show that $\mathcal{H}(B_r) \subset B_r$, where $B_r = \Big\{y \in \chi : \|y\|_\chi \leq r,\ r > 0\Big\}$ is a bounded, closed and convex subset of χ.

In fact, if it is not true, then for each positive constant r there exists $y \in B_r$, such that $\zeta \in \mathcal{H}y$, and

$$\begin{aligned}
r &< \|\zeta\|_\chi \\
&\leq \sup_{\xi\in J'}(\psi(\xi)-\psi(0))^{1-\beta}\|\mathcal{T}_{\eta,v;\psi}(\xi,0)(\phi(0)-q(y))\| \\
&\quad + \sup_{\xi\in J'}(\psi(\xi)-\psi(0))^{1-\beta}\left\|\int_0^{\xi}(\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)d\varsigma\right\| \\
&\quad + \sup_{\xi\in J'}(\psi(\xi)-\psi(0))^{1-\beta}\left\|\int_0^{\xi}(\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma,y(\varsigma-\tau_0),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)d\varsigma\right\| \\
&\leq \frac{M}{\Gamma(\eta(1-v)+v)}\big(\|\phi_0\|+L_q r+\|q(0)\|\big)+(\psi(b)-\psi(0))^{1-\beta}\frac{M\Omega(r)\triangle\|\Theta\|_{L^{\frac{1}{v_1}}}}{\Gamma(v)} \\
&\quad + \frac{MM_1\Delta}{\Gamma(v)}(\psi(b)-\psi(0))^{1-\beta}.
\end{aligned}$$

Dividing both sides by r and taking $r \longrightarrow \infty$, we get

$$\begin{aligned}
1 &\leq \frac{ML_q}{\Gamma(\eta(1-v)+v)}+(\psi(b)-\psi(0))^{1-\beta}\frac{M\Lambda\triangle\|\Theta\|_{L^{\frac{1}{v_1}}}}{\Gamma(v)} \\
1 &\leq K,
\end{aligned}$$

which contradicts the hypothesis $K < 1$, with K defined in (14).

Claim 2. $\mathcal{H}$ is equicontinuous on B_r.

Denote $E = \{\zeta \in \chi : \zeta \in \mathcal{H}y, y \in B_r, \zeta(0) = \zeta(0^+)\}$. Thus, for every $\zeta \in E$, there exists $y \in B_r$ and a selection $g \in \mathbb{S}_{G,y}$ such that ζ is given by the expression defining $\mathcal{H}y$. For $\xi_1 = 0$, $0 < \xi_2 \leq b$, we can easily get $\|(\psi(\xi_2)-\psi(0))^{1-\beta}\zeta(\xi_2)-(\psi(\xi_1)-\psi(0))^{1-\beta}\zeta(0)\| \to 0$, as $\xi_2 \to 0$. For $0 < \xi_1 < \xi_2 \leq b$, there exists $\epsilon > 0$ small enough such that

$$\begin{aligned}
&\Big\|(\psi(\xi_2)-\psi(0))^{1-\beta}\zeta(\xi_2)-(\psi(\xi_1)-\psi(0))^{1-\beta}\zeta(\xi_1)\Big\| \\
&\leq \Big\|(\psi(\xi_2)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_2,0)(\phi(0)-q(y))-(\psi(\xi_1)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_1,0)(\phi(0)-q(y))\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\right.\\
&\qquad\left.\times\Big[g(\varsigma)+F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\Big]\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\right.\\
&\qquad\left.\times\Big[g(\varsigma)+F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\Big]\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|(\psi(\xi_1)-\psi(0))^{1-\beta}\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]\right.\\
&\qquad\left.\times\Big[g(\varsigma)+F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\Big]\psi'(\varsigma)\,d\varsigma\right\|\\
\le&\left\|\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_2,0)-(\psi(\xi_1)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_1,0)\Big](\phi(0)-q(y))\right\|\\
&+\left\|(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\times\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\right.\\
&\qquad\left.\times\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\right.\\
&\qquad\left.\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_0^{\xi_1-\epsilon}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_0^{\xi_1-\epsilon}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]\right.\\
&\qquad\left.\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|\\
&+\left\|\int_{\xi_1-\epsilon}^{\xi_1}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|
\end{aligned}
$$

$$+\left\|\int_{\xi_1-\epsilon}^{\xi_1}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]\right.$$
$$\left.\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|$$
$$\leq\sum_{i=1}^{9}t_i,$$

where

$$t_1=\left\|\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_2,0)-(\psi(\xi_1)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_1,0)\Big](\phi(0)-q(y))\right\|$$
$$\leq\big(\|\phi_0\|+L_qr+\|q(0)\|\big)\times\left\|\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_2,0)-(\psi(\xi_1)-\psi(0))^{1-\beta}\mathcal{T}_{\eta,v;\psi}(\xi_1,0)\Big]\right\|,$$

and hence $t_1\to 0$ as $\xi_2-\xi_1\to 0^+$,

$$t_2=\left\|(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\times\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|$$
$$\leq\frac{M\Omega(r)}{\Gamma(v)}(\psi(\xi_2)-\psi(0))^{1-\beta}\|\Theta\|_{L^{\frac{1}{v_1}}}\times\left(\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{\frac{v-1}{1-v_1}}\psi'(\varsigma)\,d\varsigma\right)^{1-v_1}$$
$$\leq\frac{M\Omega(r)}{\Gamma(v)}(\psi(\xi_2)-\psi(0))^{1-\beta}\|\Theta\|_{L^{\frac{1}{v_1}}}\times\frac{[\psi(\xi_2)-\psi(\xi_1)]^{(1+d)(1-v_1)}}{(1+d)^{1-v_1}},$$

so $t_2\to 0$ as $\xi_2-\xi_1\to 0^+$,

$$t_3=\left\|(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|$$
$$\leq\frac{M}{\Gamma(v)}(\psi(\xi_2)-\psi(0))^{1-\beta}\int_{\xi_1}^{\xi_2}(\psi(\xi_2)-\psi(\varsigma))^{v-1}\times\Big\|F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\Big\|\psi'(\varsigma)\,d\varsigma,$$

similarly, $t_3\to 0$ as $\xi_2-\xi_1\to 0^+$,

$$t_4=\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\right.$$
$$\left.\times\,\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|$$
$$\leq\frac{M}{\Gamma(v)}\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\times g(\varsigma)\psi'(\varsigma)\,d\varsigma\right\|,$$

$$t_5=\left\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\right.$$
$$\left.\times\,F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\right\|$$

$$\leq \frac{M}{\Gamma(v)}\Bigg\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]$$

$$\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\Bigg\|,$$

$$t_6=\Bigg\|\int_0^{\xi_1-\epsilon}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]g(\varsigma)\psi'(\varsigma)\,d\varsigma\Bigg\|$$

$$\leq(\psi(\xi_1)-\psi(0))^{1-\beta}\frac{\big(\psi(\xi_1)^{1+d}-\psi(\xi_1-\epsilon)^{1+d}\big)^{1-v_1}}{(1+d)^{1-v_1}}\times\Omega(r)\|\Theta\|_{L^{\frac{1}{v_1}}}\times\sup_{[0,\xi_1-\epsilon]}\Big\|\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]\Big\|,$$

$$t_7=\Bigg\|\int_0^{\xi_1-\epsilon}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]$$

$$\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\Bigg\|,$$

$$t_8=\Bigg\|\int_{\xi_1-\epsilon}^{\xi_1}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]g(\varsigma)\psi'(\varsigma)\,d\varsigma\Bigg\|$$

$$t_9=\Bigg\|\int_{\xi_1-\epsilon}^{\xi_1}(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\times\Big[\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\Big]$$

$$\times F\big(\varsigma,y(\varsigma-\tau_0),y(\varsigma-\tau_1),\ldots,y(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma\Bigg\|.$$

Recall that

$$t_4\leq\frac{M}{\Gamma(v)}\Bigg\|\int_0^{\xi_1}\Big[(\psi(\xi_2)-\psi(0))^{1-\beta}(\psi(\xi_2)-\psi(\varsigma))^{v-1}-(\psi(\xi_1)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Big]\times g(\varsigma)\psi'(\varsigma)d\varsigma\Bigg\|$$

$$\leq\frac{M\Omega(r)}{\Gamma(v)}\int_0^{\xi_1}\big\|R_{\xi_2}(\varsigma)-R_{\xi_1}(\varsigma)\big\|\,\Theta(\varsigma)\psi'(\varsigma)\,d\varsigma,$$

where $R_\xi(\varsigma)=(\psi(\xi)-\psi(0))^{1-\beta}(\psi(\xi)-\psi(\varsigma))^{v-1}$.

For every fixed $\varsigma\in[0,\xi_1)$, since ψ is continuous and $\xi_2\to\xi_1^+$, we have

$$\psi(\xi_2)\longrightarrow\psi(\xi_1),$$

and consequently $R_{\xi_2}(\varsigma)\longrightarrow R_{\xi_1}(\varsigma)$. Hence, we obtain

$$\big\|R_{\xi_2}(\varsigma)-R_{\xi_1}(\varsigma)\big\|\longrightarrow 0\qquad\text{for every }\varsigma\in[0,\xi_1).$$

Since $0<v<1$, we have $v-1<0$. For $\xi_2>\xi_1>\varsigma$, we have

$$\psi(\xi_2)-\psi(\varsigma)\geq\psi(\xi_1)-\psi(\varsigma),$$

and therefore $(\psi(\xi_2)-\psi(\varsigma))^{v-1}\leq(\psi(\xi_1)-\psi(\varsigma))^{v-1}$. Moreover, we have

$$\psi(\xi)-\psi(0)\leq\psi(b)-\psi(0),\qquad\xi\in(0,b].$$

Thus, we get

$$\begin{aligned}\big\|R_{\xi_2}(\varsigma)-R_{\xi_1}(\varsigma)\big\|&\leq R_{\xi_2}(\varsigma)+R_{\xi_1}(\varsigma)\\&\leq 2(\psi(b)-\psi(0))^{1-\beta}(\psi(\xi_1)-\psi(\varsigma))^{v-1}.\end{aligned}$$

Consequently, we find that

$$t_4 \leq \frac{2M\Omega(r)}{\Gamma(v)}(\psi(b)-\psi(0))^{1-\beta}\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Theta(\varsigma)\psi'(\varsigma)\,d\varsigma.$$

Let $p=\frac{1}{v_1}$, $q=\frac{1}{1-v_1}$, so that $\frac{1}{p}+\frac{1}{q}=1$. Using Hölder's inequality, we obtain

$$\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Theta(\varsigma)\psi'(\varsigma)\,d\varsigma \leq \left(\int_0^{\xi_1}\Theta(\varsigma)^p\,d\varsigma\right)^{1/p}\left(\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{(v-1)q}\psi'(\varsigma)\,d\varsigma\right)^{1/q}.$$

Set $d=\frac{v-1}{1-v_1}$. Since $0<v<1$ and $0<v_1<v$, we have $1+d=1+\frac{v-1}{1-v_1}=\frac{v-v_1}{1-v_1}>0$. Therefore, we get

$$\begin{aligned}\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^d\psi'(\varsigma)\,d\varsigma &= \int_{\psi(0)}^{\psi(\xi_1)}(\psi(\xi_1)-u)^d\,du\\ &= \frac{(\psi(\xi_1)-\psi(0))^{1+d}}{1+d}.\end{aligned}$$

Hence, we have

$$\left(\int_0^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^d\psi'(\varsigma)\,d\varsigma\right)^{1-v_1}=\left(\frac{(\psi(\xi_1)-\psi(0))^{1+d}}{1+d}\right)^{1-v_1}.$$

The condition $1+d>0$ guarantees the integrability of the singular kernel near $\varsigma=\xi_1$.
Hence, for every fixed $\varsigma\in[0,\xi_1)$, we have

$$\left\|R_{\xi_2}(\varsigma)-R_{\xi_1}(\varsigma)\right\|\Theta(\varsigma)\psi'(\varsigma)\longrightarrow 0 \quad \text{as } \xi_2-\xi_1\longrightarrow 0^+,$$

and the integrand is dominated by an integrable function. Therefore, by the dominated convergence theorem, we obtain

$$\lim_{\xi_2-\xi_1\to 0^+}\int_0^{\xi_1}\left\|R_{\xi_2}(\varsigma)-R_{\xi_1}(\varsigma)\right\|\Theta(\varsigma)\psi'(\varsigma)\,d\varsigma=0.$$

Consequently, $\lim_{\xi_2-\xi_1\to 0^+}t_4=0$. Thus, the contribution corresponding to t_4 satisfies the required equicontinuity condition.

Similarly, we can show that

$$\lim_{\xi_2-\xi_1\to 0^+}t_5=0.$$

For t_6, by Lemma 5, we have

$$\left\|\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)\right\|\leq\frac{M}{\Gamma(v)}.$$

Hence, for every $\varsigma\in[0,\xi_1-\varepsilon]$, we obtain

$$\begin{aligned}\left\|\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)-\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\right\| &\leq \left\|\mathcal{P}_{\eta,v;\psi}(\xi_2,\varsigma)\right\|+\left\|\mathcal{P}_{\eta,v;\psi}(\xi_1,\varsigma)\right\|\\ &\leq \frac{2M}{\Gamma(v)}.\end{aligned}$$

Therefore, we get

$$t_6\leq\frac{2M}{\Gamma(v)}(\psi(\xi_1)-\psi(0))^{1-\beta}\frac{\left[\psi(\xi_1)^{1+d}-\psi(\xi_1-\varepsilon)^{1+d}\right]^{1-v_1}}{(1+d)^{1-v_1}}\Omega(r)\|\Theta\|_{L^{\frac{1}{v_1}}}.$$

Since ψ is continuous, $\psi(\xi_1-\varepsilon)\longrightarrow\psi(\xi_1)\quad$ as $\varepsilon\to 0^+$. Moreover, since $0<v_1<v<1$ and $d=\frac{v-1}{1-v_1}$, we have $(1+d)(1-v_1)=v-v_1>0$. Consequently, we have

$$\Big[\psi(\xi_1)^{1+d}-\psi(\xi_1-\varepsilon)^{1+d}\Big]^{1-v_1}\longrightarrow 0\qquad\text{as } \varepsilon\to 0^+.$$

Thus, $t_6\longrightarrow 0\quad$ as $\varepsilon\to 0^+$. In particular, we have

$$t_6\longrightarrow 0\qquad\text{as } \xi_2-\xi_1\to 0^+,\quad \varepsilon\to 0^+.$$

Similarly, we can prove that $t_7\to 0$ as $\xi_2-\xi_1\to 0^+$ and $\varepsilon\to 0^+$.
For t_8, using the uniform boundedness of $\mathcal{P}_{\eta,v;\psi}$, namely

$$\|\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)\|\le M,$$

and the growth condition

$$\|g(\varsigma)\|\le\Theta(\varsigma)\Omega(r),$$

we obtain

$$\|t_8\|\le 2M\Omega(r)(\psi(\xi_1)-\psi(0))^{1-\beta}\int_{\xi_1-\varepsilon}^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{v-1}\Theta(\varsigma)\psi'(\varsigma)d\varsigma.$$

Since $\Theta\in L^{\frac{1}{v_1}}((0,b),\mathbb{R}_+)$, with $0<v_1<v<1$, Hölder's inequality with conjugate exponents $\frac{1}{v_1}$ and $\frac{1}{1-v_1}$ yields

$$\|t_8\|\le 2M\Omega(r)(\psi(\xi_1)-\psi(0))^{1-\beta}\|\Theta\|_{L^{\frac{1}{v_1}}}\left(\int_{\xi_1-\varepsilon}^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^{\frac{v-1}{1-v_1}}\psi'(\varsigma)d\varsigma\right)^{1-v_1}.$$

Putting $d=\frac{v-1}{1-v_1}$, we have

$$1+d=\frac{v-v_1}{1-v_1}>0,$$

because $0<v_1<v<1$. Hence, by the substitution $u=\psi(\xi_1)-\psi(\varsigma)$, we have

$$\int_{\xi_1-\varepsilon}^{\xi_1}(\psi(\xi_1)-\psi(\varsigma))^d\psi'(\varsigma)d\varsigma=\frac{[\psi(\xi_1)-\psi(\xi_1-\varepsilon)]^{1+d}}{1+d}.$$

Consequently, we obtain

$$\|t_8\|\le\frac{2M\Omega(r)}{(1+d)^{1-v_1}}(\psi(\xi_1)-\psi(0))^{1-\beta}\|\Theta\|_{L^{\frac{1}{v_1}}}[\psi(\xi_1)-\psi(\xi_1-\varepsilon)]^{(1+d)(1-v_1)}.$$

Since $(1+d)(1-v_1)=v-v_1>0$ and ψ is continuous, we get

$$\psi(\xi_1)-\psi(\xi_1-\varepsilon)\longrightarrow 0\qquad\text{as } \varepsilon\to 0^+.$$

Therefore, we have

$$t_8\longrightarrow 0\qquad\text{as } \varepsilon\to 0^+.$$

Similarly, we can prove that

$$t_9\longrightarrow 0\qquad\text{as } \varepsilon\to 0^+.$$

Therefore, by the above estimates, $t_1,t_3,t_4,t_5\to 0$ as $\xi_2-\xi_1\to 0^+$, while $t_8,t_9\to 0$ as $\varepsilon\to 0^+$. Moreover, for fixed $\varepsilon>0$, the strong continuity of $S(\xi)$, together with the definition of $\mathcal{P}_{\eta,v;\psi}$ and the dominated convergence theorem, yields $t_6,t_7\to 0$ as $\xi_2-\xi_1\to 0^+$. The term t_2 also tends to zero as $\xi_2-\xi_1\to 0^+$, since $v_1\in(0,v)$ and

$$(1+d)(1-v_1)=v-v_1>0,\quad d=\frac{v-1}{1-v_1}.$$

Consequently, we have

$$\sum_{i=1}^{9} t_i \longrightarrow 0 \qquad \text{as } \xi_2 - \xi_1 \to 0^+,$$

and therefore

$$\left\| (\psi(\xi_2) - \psi(0))^{1-\beta} \zeta(\xi_2) - (\psi(\xi_1) - \psi(0))^{1-\beta} \zeta(\xi_1) \right\| \longrightarrow 0.$$

Thus, $\mathcal{H}(B_r)$ is equicontinuous on $(0, b]$.

Claim 3. Next, we prove that the set

$$V(\xi) = \{w(\xi),\, w \in \mathcal{H}(y),\, y \in B_r\},$$

is relatively compact in Y.

Let $0 < \xi \leq b$ be fixed. We have

$$\begin{aligned}
&\mathcal{T}_{\eta,v;\psi}(\xi, 0)(\phi(0) - q(y)) \\
&\quad = \mathcal{I}_{\eta(1-v);\psi} K_{\eta,v;\psi}(\xi, 0)(\phi(0) - q(y)) \\
&\quad = \frac{1}{\Gamma(\eta(1-v))} \int_0^{\xi} (\psi(\xi) - \psi(s))^{\eta(1-v)-1} K_{\eta,v;\psi}(s, 0)(\phi(0) - q(y)) \psi'(s)\, ds \\
&\quad = \frac{1}{\Gamma(\eta(1-v))} \int_0^{\xi} (\psi(\xi) - \psi(s))^{\eta(1-v)-1} (\psi(s) - \psi(0))^{v-1} \mathcal{P}_{\eta,v;\psi}(s, 0)(\phi(0) - q(y)) \psi'(s)\, ds \\
&\quad = \frac{v}{\Gamma(\eta(1-v))} \int_0^{\xi} \int_0^{\infty} z M_v(z) (\psi(\xi) - \psi(s))^{\eta(1-v)-1} \\
&\qquad\qquad \times (\psi(s) - \psi(0))^{v-1} S((\psi(s) - \psi(0))^v z)(\phi(0) - q(y)) \psi'(s)\, dz\, ds.
\end{aligned}$$

For $0 < \lambda < \xi$, $0 < \varepsilon < \xi - \lambda$, and $\delta, R > 0$, we define the truncated operator $\mathcal{H}_{\varepsilon,\lambda,\delta,R}$ by replacing the ranges $[0, \xi] \times [0, \infty)$, with $[\varepsilon, \xi - \lambda] \times [\delta, R]$, in each of the integral terms of the mild solution. Denote the corresponding value by $w_{\varepsilon,\lambda,\delta,R}(\xi)$. Thus, we have

$$\begin{aligned}
w_{\varepsilon,\lambda,\delta,R}(\xi) = \frac{v}{\Gamma(\eta(1-v))} &\int_{\varepsilon}^{\xi-\lambda} \int_{\delta}^{R} z \mathcal{M}_v(z) (\psi(\xi) - \psi(s))^{\eta(1-v)-1} \\
&\times (\psi(s) - \psi(0))^{v-1} S((\psi(s) - \psi(0))^v z)(\phi(0) - q(y)) \psi'(s)\, dz\, ds \\
&+ v \int_{\varepsilon}^{\xi-\lambda} \int_{\delta}^{R} z \mathcal{M}_v(z) (\psi(\xi) - \psi(s))^{v-1} \\
&\times S((\psi(\xi) - \psi(s))^v z) \left[g(s) + F\left(s, y(s - \tau_0), y(s - \tau_1), \ldots, y(s - \tau_m)\right) \right] \psi'(s)\, dz\, ds.
\end{aligned}$$

Denote

$$V_{\varepsilon,\lambda,\delta,R}(\xi) = \{w_{\varepsilon,\lambda,\delta,R}(\xi) : w_{\varepsilon,\lambda,\delta,R} \in \mathcal{H}_{\varepsilon,\lambda,\delta,R}(y),\, y \in B_r\}.$$

For fixed $\varepsilon, \lambda, \delta, R$, we have

$$\psi(s) - \psi(0) \geq \psi(\varepsilon) - \psi(0) > 0, \qquad \psi(\xi) - \psi(s) \geq \psi(\xi) - \psi(\xi - \lambda) > 0.$$

Hence the arguments of the semigroup in the truncated integrals are bounded away from zero. By (A0), $S(t)$ is compact for $t > 0$. Moreover, the strong continuity of $S(t)$ implies that

$$(s, z) \mapsto S((\psi(s) - \psi(0))^v z),$$

and

$$(s, z) \mapsto S((\psi(\xi) - \psi(s))^v z),$$

are strongly continuous on the compact set

$$[\varepsilon, \xi - \lambda] \times [\delta, R].$$

Consequently, the corresponding operator-valued functions are compact on this set.

Since B_r is bounded in χ, q is Lipschitz continuous, and F satisfies (A2), while $g \in S_{G,y}$ satisfies the boundedness condition in (A4), it follows that the set $V_{\varepsilon,\lambda,\delta,R}(\xi)$ is relatively compact in Y.

It remains to prove that the truncated sets approximate $V(\xi)$ uniformly. For each $y \in B_r$, by using Hölder's inequality and the estimates of $\mathcal{P}_{\eta,v;\psi}$, we obtain

$$\begin{aligned}
\|w - w_{\varepsilon,\lambda,\delta,R}\|_{1-\beta;\psi} \leq (\psi(b)-\psi(0))^{1-\beta} \Bigg[& \frac{Mv(\psi(\xi)-\psi(0))^{v-1}}{\Gamma(\eta(1-v))}\big(\|\phi_0\| + \|q(y)-q(0)\| + \|q(0)\|\big) \\
& \times \int_{[0,\varepsilon)\cup(\xi-\lambda,\xi]} (\psi(\xi)-\psi(s))^{\eta(1-v)-1}(\psi(s)-\psi(0))^{v-1}\psi'(s)\,ds \\
& + \big(M\Omega(r)\|\Theta\|_{L^{\frac{1}{v_1}}} + MM_1\big) \int_{[0,\varepsilon)\cup(\xi-\lambda,\xi]} (\psi(\xi)-\psi(s))^{v-1}\psi'(s)\,ds \Bigg] \\
+ (\psi(b)-\psi(0))^{1-\beta} \Bigg[& \frac{Mv(\psi(\xi)-\psi(0))^{v-1}}{\Gamma(\eta(1-v))}\big(\|\phi_0\| + L_q r + \|q(0)\|\big) \\
& \times \left(\int_0^{\xi} (\psi(\xi)-\psi(s))^{\eta(1-v)-1}(\psi(s)-\psi(0))^{v-1}\psi'(s)\,ds\right) \int_{\mathbb{R}_+\setminus[\delta,R]} z\mathcal{M}_v(z)\,dz \\
& + \big(M\Omega(r)\|\Theta\|_{L^{\frac{1}{v_1}}} + MM_1\big) \left(\int_0^{\xi} (\psi(\xi)-\psi(s))^{v-1}\psi'(s)\,ds\right) \int_{\mathbb{R}_+\setminus[\delta,R]} z\mathcal{M}_v(z)\,dz \Bigg].
\end{aligned}$$

Using

$$\int_0^{\xi} (\psi(\xi)-\psi(s))^{\eta(1-v)-1}(\psi(s)-\psi(0))^{v-1}\psi'(s)\,ds = B(\eta(1-v),v)(\psi(\xi)-\psi(0))^{\eta(1-v)+v-1},$$

and

$$\int_0^{\xi} (\psi(\xi)-\psi(s))^{v-1}\psi'(s)\,ds = \frac{(\psi(\xi)-\psi(0))^{v}}{v},$$

together with $\displaystyle\int_0^{\infty} z\mathcal{M}_v(z)\,dz = \frac{1}{\Gamma(v+1)}$, we obtain a finite constant $C_r > 0$, independent of $y \in B_r$, such that

$$\begin{aligned}
\|w - w_{\varepsilon,\lambda,\delta,R}\|_{1-\beta;\psi} \leq C_r \Bigg\{ & \int_{[0,\varepsilon)\cup(\xi-\lambda,\xi)} \Big[(\psi(\xi)-\psi(s))^{\eta(1-v)-1}(\psi(s)-\psi(0))^{v-1} \\
& + (\psi(\xi)-\psi(s))^{v-1}\Big]\psi'(s)\,ds + \int_{\mathbb{R}_+\setminus[\delta,R]} z\mathcal{M}_v(z)\,dz \Bigg\}.
\end{aligned}$$

By the absolute continuity of the Lebesgue integral and $\displaystyle\int_0^{\infty} zM_v(z)\,dz = \frac{1}{\Gamma(v+1)} < \infty$,

we have

$$\|w - w_{\varepsilon,\lambda,\delta,R}\|_{1-\beta;\psi} \longrightarrow 0,$$

as $\varepsilon \to 0$, $\lambda \to 0$, $\delta \to 0$, $R \to \infty$, uniformly for $w \in \mathcal{H}(y)$, $y \in B_r$.

Therefore $V(\xi)$ is the limit, in Y, of the relatively compact sets

$$V_{\varepsilon,\lambda,\delta,R}(\xi).$$

Hence $V(\xi)$ is relatively compact in Y.

Thus, we have

$$V(\xi) = \{w(\xi),\ w \in \mathcal{H}(y),\ y \in B_r\} \text{ is relatively compact in } Y.$$

Step 4. $\mathcal{H}$ has a closed graph.

Let $y^{(n)} \longrightarrow y^*$ in χ, $\rho^{(n)} \in \mathcal{H}(y^{(n)})$, $\rho^{(n)} \longrightarrow \rho^*$ in χ. We shall prove that

$$\rho^* \in \mathcal{H}(y^*).$$

Since $\rho^{(n)} \in \mathcal{H}(y^{(n)})$, for every $n \in \mathbb{N}$ there exists $g^{(n)} \in S_{G,y^{(n)}}$ such that, for $\xi \in J'$,

$$\rho^{(n)}(\xi) = \mathcal{T}_{\eta,v;\psi}(\xi,0)\big(\phi(0) - q(y^{(n)})\big) + \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)g^{(n)}(\varsigma)\psi'(\varsigma)\,d\varsigma$$
$$+ \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)F\big(\varsigma, y^{(n)}(\varsigma-\tau_0), y^{(n)}(\varsigma-\tau_1),\dots,y^{(n)}(\varsigma-\tau_m)\big)\psi'(\varsigma)\,d\varsigma. \tag{15}$$

Define the linear operator

$$\sum : L^{\frac{1}{v_1}}(J,Y) \longrightarrow C_{1-\beta;\psi}(J,Y),$$

by

$$\left(\sum g\right)(\xi) = \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1}\mathcal{P}_{\eta,v;\psi}(\xi,\varsigma)g(\varsigma)\psi'(\varsigma)\,d\varsigma.$$

By the estimate for $\mathcal{P}_{\eta,v;\psi}$ and Hölder's inequality, $\sum$ is a continuous linear operator. Moreover, by the compactness argument used in the preceding claim, $\sum$ maps bounded subsets of $L^{\frac{1}{v_1}}(J,Y)$ into relatively compact subsets of $C_{1-\beta;\psi}(J,Y)$.

By (A4), the family

$$\left\{S_{G,y} : y \in B_r\right\},$$

is uniformly integrable in $L^1(J,Y)$ and each $S_{G,y}$ is weakly compact in $L^1(J,Y)$. Hence, by the weak compactness and the uniform integrability of the family of admissible selections, there exist a subsequence, still denoted by $g^{(n)}$, and $g^* \in L^1(J,Y)$ such that

$$g^{(n)} \rightharpoonup g^* \quad \text{in } L^1(J,Y). \tag{16}$$

Since $y^{(n)} \to y^*$ in χ, the upper semicontinuity of $G(\xi,\cdot)$ and the uniform integrability condition in (A4) imply that the graph of the associated selection operator is weakly closed. Therefore, we have

$$g^* \in S_{G,y^*}.$$

Since $\sum$ is continuous and linear, (16) implies that

$$\sum g^{(n)} \rightharpoonup \sum g^* \quad \text{in } C_{1-\beta;\psi}(J,Y). \tag{17}$$

On the other hand, the compactness of $\sum$ on bounded subsets of $L^{\frac{1}{v_1}}(J,Y)$ implies that $\{\sum g^{(n)}\}$ is relatively compact in $C_{1-\beta;\psi}(J,Y)$. Thus every strongly convergent subsequence of $\{\sum g^{(n)}\}$ has a limit which, by (17), must be $\sum g^*$. Consequently, we have

$$\sum g^{(n)} \longrightarrow \sum g^* \quad \text{in } C_{1-\beta;\psi}(J,Y).$$

Next, since $y^{(n)} \to y^*$ in χ, we have

$$y^{(n)}(\varsigma-\tau_k) \longrightarrow y^*(\varsigma-\tau_k), \qquad k = 0,1,\dots,m,$$

for almost every $\varsigma \in J$. Indeed, if $\varsigma - \tau_k \in [-\tau_m, 0]$, the convergence follows from the uniform convergence of the history components, while if $\varsigma - \tau_k \in (0,b]$, it follows from the convergence in $C_{1-\beta;\psi}(J,Y)$. Hence, by (A3), we obtain

$$F\big(\varsigma, y^{(n)}(\varsigma-\tau_0), y^{(n)}(\varsigma-\tau_1),\dots,y^{(n)}(\varsigma-\tau_m)\big) \longrightarrow F\big(\varsigma, y^*(\varsigma-\tau_0), y^*(\varsigma-\tau_1),\dots,y^*(\varsigma-\tau_m)\big), \tag{18}$$

for almost every $\varsigma \in J$.

Furthermore, since $y^{(n)}, y^* \in B_r$, assumption (A2) provides an integrable majorant for the functions appearing in (18). Therefore, by the dominated convergence theorem, we get

$$F\big(\cdot, y^{(n)}(\cdot-\tau_0), y^{(n)}(\cdot-\tau_1), \dots, y^{(n)}(\cdot-\tau_m)\big) \longrightarrow F\big(\cdot, y^*(\cdot-\tau_0), y^*(\cdot-\tau_1), \dots, y^*(\cdot-\tau_m)\big),$$

in $L^{\frac{1}{v_1}}(J, Y)$. By the continuity of $\sum$, it follows that

$$\begin{aligned}
&\int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1} \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma) F\big(\varsigma, y^{(n)}(\varsigma-\tau_0), y^{(n)}(\varsigma-\tau_1), \dots, y^{(n)}(\varsigma-\tau_m)\big)\psi'(\varsigma)\, d\varsigma \\
&\longrightarrow \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1} \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma) F\big(\varsigma, y^*(\varsigma-\tau_0), y^*(\varsigma-\tau_1), \dots, y^*(\varsigma-\tau_m)\big)\psi'(\varsigma)\, d\varsigma,
\end{aligned} \tag{19}$$

in $C_{1-\beta;\psi}(J, Y)$.

Also, by (A5), we have

$$q(y^{(n)}) \longrightarrow q(y^*) \quad \text{in } Y. \tag{20}$$

Consequently, $\mathcal{T}_{\eta,v;\psi}(\xi,0)\big(\phi(0)-q(y^{(n)})\big) \longrightarrow \mathcal{T}_{\eta,v;\psi}(\xi,0)\big(\phi(0)-q(y^*)\big)$ in $C_{1-\beta;\psi}(J, Y)$.

Taking the limit in (15), and using (18), (19), (20), together with $\rho^{(n)} \to \rho^*$ in χ, we obtain, for every $\xi \in J'$,

$$\begin{aligned}
\rho^*(\xi) = {} & T_{\eta,v;\psi}(\xi,0)\big(\phi(0)-q(y^*)\big) + \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1} \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma) g^*(\varsigma)\psi'(\varsigma)\, d\varsigma \\
& + \int_0^{\xi} (\psi(\xi)-\psi(\varsigma))^{v-1} \mathcal{P}_{\eta,v;\psi}(\xi,\varsigma) F\big(\varsigma, y^*(\varsigma-\tau_0), y^*(\varsigma-\tau_1), \dots, y^*(\varsigma-\tau_m)\big)\psi'(\varsigma)\, d\varsigma.
\end{aligned}$$

Since $g^* \in S_{G,y^*}$, it follows from the definition of $\mathcal{H}$ that

$$\rho^* \in \mathcal{H}(y^*).$$

Hence, $\mathcal{H}$ has a closed graph.

By Steps 1–4 and the Arzelà–Ascoli theorem, $\mathcal{H}$ is a completely continuous multivalued map with compact values. Hence, by the compact-valued closed-graph criterion discussed by Yang and Wang [11], $\mathcal{H}$ is upper semicontinuous (u.s.c.). Consequently, Lemma 2 ensures that the operator $\mathcal{H}$ has a fixed-point $y(\cdot)$ on B_r, which is a mild solution of system (1). □

4. Applications

Example 1. Let $Y = L^2([0,1], \mathbb{R})$. Consider the following nonlinear ψ-Hilfer fractional differential inclusion with two state delays:

$$\begin{cases}
D_{0^+}^{\frac{1}{3},\frac{5}{6};\xi} y(\vartheta,\xi) \in \dfrac{\partial^2 y(\vartheta,\xi)}{\partial \vartheta^2} + G\big(\vartheta,\xi,y(\vartheta,\xi)\big) + F\big(\vartheta,\xi,y(\vartheta,\xi), y(\vartheta,\xi-\tau_1), y(\vartheta,\xi-\tau_2)\big), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\vartheta,\xi) \in [0,1]\times(0,b], \\
y(0,\xi) = y(1,\xi) = 0, \quad \xi \in (0,b], \\
\mathcal{I}_{0^+}^{\frac{1}{9};\xi} y(\vartheta,0) + q(y)(\vartheta) = \phi(\vartheta,0), \quad \vartheta \in [0,1], \\
y(\vartheta,\xi) = \phi(\vartheta,\xi), \quad (\vartheta,\xi) \in [0,1]\times[-\tau_2,0),
\end{cases} \tag{21}$$

where $\psi(\xi) = \xi$, $D_{0^+}^{\frac{1}{3},\frac{5}{6};\xi}$ denotes the Hilfer fractional derivative of order $v = \frac{5}{6}$ and type $\eta = \frac{1}{3}$, while $\mathcal{I}_{0^+}^{\frac{1}{9};\xi}$ denotes the Riemann–Liouville fractional integral of order $\frac{1}{9}$. The prescribed history is $\phi \in C([-\tau_2,0], Y)$.

For these values of v and η, we have $\beta = \eta + v - \eta v = \frac{1}{3} + \frac{5}{6} - \frac{5}{18} = \frac{8}{9}$, and consequently $1-\beta = \frac{1}{9}$. Define the operator $A : D(A) \subset Y \to Y$ by $Ay = y''$, with

$$D(A) = \Big\{ y \in L^2(0,1) : y, y' \in AC[0,1],\ y'' \in L^2(0,1),\ y(0) = y(1) = 0 \Big\}.$$

Then A is the Dirichlet Laplacian on $(0,1)$ and generates a strongly continuous C_0-semigroup whose operators are compact at every strictly positive time $\{S(\xi)\}_{\xi\geq 0}$ on Y.

The eigenvalues of A are $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$, with corresponding normalized eigenfunctions $e_n(\vartheta) = \sqrt{2}\sin(n\pi\vartheta)$, $n \in \mathbb{N}$. Hence, for every $y \in D(A)$, we have

$$Ay = -\sum_{n=1}^{\infty} n^2\pi^2\langle y, e_n\rangle_Y e_n.$$

Moreover, we have $S(\xi)y = \sum\limits_{n=1}^{\infty} e^{-n^2\pi^2\xi}\langle y, e_n\rangle e_n$, $\xi \geq 0$, and therefore $\|S(\xi)\| \leq e^{-\pi^2\xi} \leq e^{-\xi}$, $\xi \geq 0$.

For $\psi(\xi) = \xi$, the operators $\mathcal{T}_{\eta,v;\psi}(\xi,\varsigma)$ and $K_{\eta,v;\psi}(\xi,\varsigma)$ take the forms

$$\mathcal{T}_{\frac{1}{3},\frac{5}{6};\xi}(\xi,\varsigma) = \frac{5}{6\Gamma(\frac{1}{18})}\int_{\varsigma}^{\xi}(\xi-s)^{-\frac{17}{18}}(s-\varsigma)^{-\frac{1}{6}}\left[\int_0^{\infty} z\mathcal{M}_{\frac{5}{6}}(z)S\left((s-\varsigma)^{\frac{5}{6}}z\right)\,dz\right]ds,$$

and

$$K_{\frac{1}{3},\frac{5}{6};\xi}(\xi,\varsigma) = \frac{5}{6}(\xi-\varsigma)^{-\frac{1}{6}}\int_0^{\infty} z\mathcal{M}_{\frac{5}{6}}(z)S\left((\xi-\varsigma)^{\frac{5}{6}}z\right)\,dz.$$

Consequently, we obtain

$$\left\|\mathcal{T}_{\frac{1}{3},\frac{5}{6};\xi}(\xi,\varsigma)\right\| \leq \frac{3(\xi-\varsigma)^{-\frac{1}{9}}}{2\Gamma(\frac{8}{9})} \text{ and } \left\|K_{\frac{1}{3},\frac{5}{6};\xi}(\xi,\varsigma)\right\| \leq \frac{5(\xi-\varsigma)^{-\frac{1}{6}}}{4\Gamma(\frac{5}{6})}.$$

Now define $y(\xi)(\vartheta) = y(\vartheta,\xi)$, and define the nonlinear terms by $G(\xi,y(\xi))(\vartheta) = G(\vartheta,\xi,y(\vartheta,\xi))$ and

$$F(\xi,y(\xi),y(\xi-\tau_1),y(\xi-\tau_2))(\vartheta) = F\big(\vartheta,\xi,y(\vartheta,\xi),y(\vartheta,\xi-\tau_1),y(\vartheta,\xi-\tau_2)\big).$$

We choose

$$F(\vartheta,\xi,y_0,y_1,y_2) = \xi^2\left[\sin(y_0)+\sin(y_1)+\sin(y_2)\right].$$

Therefore, we have

$$\begin{aligned}\|F(\xi,y(\xi),y(\xi-\tau_1),y(\xi-\tau_2))\| &\leq \xi^2\left(\|\sin y(\xi)\| + \|\sin y(\xi-\tau_1)\| + \|\sin y(\xi-\tau_2)\|\right)\\ &\leq 3\xi^2.\end{aligned}$$

Hence, we can take $h(\xi) = 3\xi^2$. Choose $v_1 = \frac{1}{2}$. Then $h \in L^{\frac{1}{v_1}}(0,b) = L^2(0,b)$. Furthermore, we have

$$\mathcal{I}_{0+}^{\frac{5}{6}}h(\xi) = \frac{6}{\Gamma(\frac{23}{6})}\xi^{\frac{17}{6}},$$

and hence

$$\begin{aligned}\xi^{1-\beta}\mathcal{I}_{0+}^{\frac{5}{6}}h(\xi) &= \frac{6}{\Gamma(\frac{23}{6})}\xi^{\frac{1}{9}+\frac{17}{6}}\\ &= \frac{6}{\Gamma(\frac{23}{6})}\xi^{\frac{53}{18}} \longrightarrow 0 \qquad \text{as } \xi \to 0^+.\end{aligned}$$

Thus, the required condition on h is satisfied.

Moreover, since $\|\sin x - \sin y\| \leq \|x-y\|$, we obtain

$$\begin{aligned}&\|F(\xi,y(\xi),y(\xi-\tau_1),y(\xi-\tau_2)) - F(\xi,\widetilde{y}(\xi),\widetilde{y}(\xi-\tau_1),\widetilde{y}(\xi-\tau_2))\|\\ &\quad\leq \xi^2\Big(\|y(\xi)-\widetilde{y}(\xi)\| + \|y(\xi-\tau_1)-\widetilde{y}(\xi-\tau_1)\| + \|y(\xi-\tau_2)-\widetilde{y}(\xi-\tau_2)\|\Big).\end{aligned}$$

Since $(\psi(\xi)-\psi(0))^{1-\beta} = \xi^{\frac{1}{9}}$, we may choose $\rho_0(\xi) = \rho_1(\xi) = \rho_2(\xi) = \xi^{1+\beta} = \xi^{\frac{17}{9}}$. Therefore, F satisfies the required Lipschitz-type condition with respect to the state and the two delayed states.

Next, define the multifunction

$$G(\vartheta, \xi, y) = \left\{ \frac{e^{-\xi}}{1+e^{-\xi}} \sin(y) \right\}.$$

Thus, G is a singleton-valued multifunction. This example illustrates the single-valued special case of the inclusion formulation; it does not exhibit genuinely set-valued uncertainty. Moreover, we have $\sup_{g \in G(\vartheta,\xi,y)} \|g\| \leq 1$.

Hence, we may take $\Theta(\xi) = 1$, $\Omega(r) = 1$, $\Lambda = 0$. Since $\Theta \in L^{\frac{1}{v_1}}(0,b) = L^2(0,b)$, the required growth condition for G is satisfied.

Finally, define the nonlocal operator $q : \chi \to Y$ by

$$q(y)(\vartheta) = \frac{7}{10} y(\vartheta, 0), \qquad \vartheta \in [0,1].$$

Then, we have $\|q(y) - q(\widetilde{y})\| \leq \frac{7}{10}\|y - \widetilde{y}\|$, so that $L_q = \frac{7}{10}$. Because the history fixes $y(0) = \phi(0)$, this particular choice depends only on the prescribed history endpoint. It is therefore a restricted instance of the abstract initial operator, rather than an example of dependence on the unknown positive-time trajectory.

We now verify K in Theorem 1. Take $b = 1$, $M = 1.5$, $v_1 = \frac{1}{2}$. Furthermore, we have

$$d = \frac{v-1}{1-v_1} = \frac{\frac{5}{6}-1}{1-\frac{1}{2}} = -\frac{1}{3}.$$

Also, we have $\eta(1-v)+v = \frac{1}{3}\left(1-\frac{5}{6}\right)+\frac{5}{6} = \frac{8}{9}$.

Since $\Lambda = 0$, the second term in the constant K vanishes. Therefore, we have

$$\begin{aligned} K &= \frac{ML_q}{\Gamma(\eta(1-v)+v)} \\ &= \frac{1.5\left(\frac{7}{10}\right)}{\Gamma\left(\frac{8}{9}\right)} \\ &\approx 0.9742 < 1. \end{aligned}$$

Hence, all the hypotheses required for Theorem 1 are satisfied. Therefore, Theorem 1 guarantees that system (21) admits at least one mild solution $y \in \chi$ on $(0,1]$.

5. Conclusions

In this paper, we have studied a class of nonlinear ψ-Hilfer fractional differential inclusions with multiple state delays in a Banach space. By combining fractional calculus, semigroup theory, multivalued analysis and fixed-point techniques, we have established sufficient conditions ensuring the existence of mild solutions for the considered class of delayed fractional differential inclusions.

The framework developed in this study incorporates the ψ-Hilfer fractional derivative together with multiple state-delay terms and nonlocal initial conditions. The main result provides an existence criterion in terms of the assumptions imposed on the nonlinear multivalued mapping, the delayed nonlinear term, the nonlocal operator, and the strongly continuous semigroup generated by the linear operator.

The applicability of the obtained existence result is illustrated through a concrete example. In particular, for the choice $\psi(\xi) = \xi$, $v = \frac{5}{6}$, $\eta = \frac{1}{3}$, $\beta = \frac{8}{9}$, together with the Dirichlet Laplacian on $L^2(0,1)$, two state-delay terms, and the specified nonlinearities, all the required hypotheses of Theorem 1 are satisfied. Moreover, the corresponding constant satisfies $K = \frac{1.05}{\Gamma(8/9)} \approx 0.9742 < 1$. Consequently, Theorem 1 guarantees that the considered system admits at least one mild solution on $(0,1]$.

The present example is intended to verify the applicability of the abstract existence theorem rather than to provide a numerical approximation or a quantitative simulation of the solution.

6. Future Work

Future research may focus on developing numerical approximation schemes for the mild solutions of the considered fractional delay systems. Further extensions may include stochastic and impulsive perturbations, variable and distributed delays, as well as control problems associated with ψ-Hilfer fractional differential inclusions.

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