Open Journal of Mathematical Sciences

PSR Press

Article

Fixed point results in b-Fuzzy metric spaces via pair upper (F,h)-class functions and α_β-contractive mappings

A.H. Ansari[1], S.H.Jafari Petroudi[1] and T. Hamaizia[2,*]

1 Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
2 System Dynamics and Control Laboratory, Department of Mathematics and Informatics, OEB University, Algeria
* Correspondence: tayeb042000@yahoo.fr

Received: 28 Jan 2026; Revised: 01 Apr 2026; Accepted: 03 Apr 2026; Published: 30 Apr 2026

Abstract: This article proposes a new extension of fixed-point theorems in the context of b-fuzzy metric spaces based on Geraghty-type inequalities. We present the concept of pair upper (F,h)-class functions, which play a key role in establishing existence and uniqueness of fixed points for contractive situations. These outcomes extend and generalize some eminent fixed-point theorems in fuzzy and b-metric spaces. We support our results by proper example.

Keywords: fixed point, Fuzzy metric spaces, completeness of fuzzy metric

MSC: 47H10, 03E72, 5A03, 54H25.

---

1. Introduction and preliminaries

Fixed-point theory has been a focal point of mathematical analysis for many years owing to its varied applications in areas such as dynamic systems, optimization, and differential equations. Over the years, this theory has also been widely used to other types of spaces, like metric, b-metric, and fuzzy metric spaces (see [1–6]).

In particular, the b-metric space and its fuzzy counterpart, the b-fuzzy metric space [7], have gained considerable attention. The introduction of fuzziness allows for a more flexible approach in handling approximate and uncertain data, making it an attractive framework for modeling real-life phenomena. Fixed-point results in such spaces have been a topic of active research, with many researchers contributing new theorems that extend classical results.

Geraghty-type inequalities [8], have proven to be a useful tool in deriving fixed-point results, especially in generalized metric spaces. These inequalities relax the standard contractive conditions, enabling the discovery of fixed points for a broader class of mappings. Nevertheless, despite these developments, additional research on these kinds of inequalities in b-fuzzy metric spaces is still necessary, especially when it comes to more broadly construed mappings.

In this work, we extend fixed-point results in b-fuzzy metric spaces by introducing a novel class of functions utilizing pair upper (F,h)-class functions and α_β-Contractive Mappings Geraghty-type inequalities ([9,10]). These functions make the contractive conditions more flexible, which makes it possible to develop new fixed-point theorems. Our results generalize and expand upon several important fixed-point theorems in the literature, opening up new possibilities for applications in fuzzy systems and related areas.

Firstly, we will mention some concepts as follows:

Definition 1. [11] Let X be a non empty set and $s \geq 1$ be a given real number, and let $d : X \times X \to \mathbb{R}_+$ be a function satisfying the conditions for all $x, y, z \in X$

1) $d(x,y) = 0$ if and only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then d is called b-metric on X, and the pair (X,d) is called b-metric space.

Example 1. [12] Let $X = \{0,1,2\}$ and $d(2,0) = d(0,2) = m \geq 2$, $d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$, and $d(0,0) = d(1,1) = d(2,2) = 0$.

Then $d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

if $m > 2$ then the triangle inequality does not hold.

Definition 2. [6] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be t-norm if it satisfies the following conditions:

1. $\diamond$ is commutative and associative,
2. $\diamond$ is continuous
3. $a \diamond 1 = a$ for all $a \in [0,1]$ and
4. $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Definition 3. [4] A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy metric on $X^2 \times (0,\infty)$ satisfying the following conditions for each $x,y,z \in X$ and $t,s > 0$,

1) $M(x,y,t) > 0$,
2) $M(x,y,t) = 1$ if and only if $x = y$,
3) $M(x,y,t) = M(y,x,t)$,
4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Definition 4. [13] A function $\psi : [0,\infty) \to [0,\infty)$ is a called an altering distance function if the following properties are satisfied:

1) ψ is non-decreasing and continuous;
2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 5. [7] A b-fuzzy metric space is an ordered triple $(X, M_z, \diamond)$,where $M_z : \overline{Y}^2 \times (0,+\infty) \to [0,1]$, for all $x,y,z \in X$ and $l,s > 0$ satisfying:

1. $M_z(x,y,l) > 0$,
2. $M_z(x,y,l) = 1$ iff $x = y$,
3. $M_z(x,y,l) = M_z(y,x,l)$,
4. $M_z(x,z,b(l+s)) \geq M_z(x,y,l) \diamond M_z(y,z,s)$,
5. $M_z : (x,y,.) \to [0,1]$ is continuous from left and $\lim_{l\to\infty} M_z : (x,y,l) = 1$. for all $x,y,z \in X$ and $l,s > 0$.

Note: If $b = 1$ then Definition 2 will become a fuzzy metric space.

Definition 6. [7] Let $(X, M_z, \diamond)$ is a b-fuzzy metric space. Then.

1. $\{x_n\}$ is called G-convergent sequence if there exists $x \in X$ such that
$$\lim_{n\to\infty} 1 - M_z(x_n, x, l) = 0.$$
2. $\{x_n\}$ is called G-Cauchy if
$$\lim_{m,n\to\infty} 1 - M_z(x_m, x_n, l) = 0.$$
for all $m,n \in N$ and $l > 0$.
3. The space is called complete if every Cauchy sequence is convergent in X.

Example 2. Let $M_z(x,y,l) = e^{\frac{-|x-y|^p}{l}}$, where $p > 1$ is a real number, M_z is a b-fuzzy metric with $b = 2^{p-1}$ but not fuzzy metric space. (see [7]).

In this paper, we introduced the concepts pair $(\mathcal{F}, h)$ an upper class of type II and α_β-contractive mappings to show that theorems in [8] reduce to corollaries in this paper. That is, all them can obtain of main theorem. In end we state example for support main result.

To start with we give some notations and introduce some definitions which will be used in the sequel.

In 2014 the concept of pair $(\mathcal{F}, h)$ is an upper class was introduced by A. H. Ansari in [9] that then reform definition(part words) it in [10]

Definition 7. ([9,10]) We say that the function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 3. ([9,10]) Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:
(a) $h(x, y) = (y + l)^x, l > 1$;
(b) $h(x, y) = (x + l)^y, l > 1$;
(c) $h(x, y) = x^n y, n \in \mathbb{N}$;
(d) $h(x, y) = y$;
(e) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) y, n \in \mathbb{N}$;
(f) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1, n \in \mathbb{N}$
for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 8. ([9,10]) Let $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and:
(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,
(ii) $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ for all $t, y \in \mathbb{R}^+$.

Example 4. ([9,10]) Define $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:
(a) $h(x, y) = (y + l)^x, l > 1$ and $\mathcal{F}(s, t) = st + l$;
(b) $h(x, y) = (x + l)^y, l > 1$ and $\mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y) = x^m y, m \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;
(d) $h(x, y) = y$ and $\mathcal{F}(s, t) = t$;
(e) $h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) y, n \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;
(f) $h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1, n \in \mathbb{N}$ and $\mathcal{F}(s, t) = (1 + l)^{st}$
for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 9. ([9,10]) We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II, if

$$x, y \geq 1 \Longrightarrow h(1, 1, z) \leq h(x, y, z) \ for\ all\ z \in \mathbb{R}^+.$$

Example 5. ([9,10]) Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:
(a) $h(x, y, z) = (z + l)^{xy}, l > 1$;
(b) $h(x, y, z) = (xy + l)^z, l > 1$;
(c) $h(x, y, z) = z$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}$;
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$
for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 10. ([9,10]) Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and:
(i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,
(ii) $h(1, 1, z) \leq \mathcal{F}(s, t) \Longrightarrow z \leq st$ for all $s, t, z \in \mathbb{R}^+$.

Example 6. [9,10] Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:
(a) $h(x, y, z) = (z + l)^{xy}, l > 1, \mathcal{F}(s, t) = st + l$;
(b) $h(x, y, z) = (xy + l)^z, l > 1, \mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y, z) = z, F(s, t) = st$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s, t) = s^p t^p$
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, \mathcal{F}(s, t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

Definition 11. Let (X, d) be a metric space and $T : X \to X$, a nonempty subset F of X is valled invariant under the T if $Tx \in F$ for every $x \in F$.

Definition 12. [14] Let $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$. We say that T is an α_F-admissible mapping if

$$\alpha(x,\ y) \geq 1 \text{ implies } \alpha(Tx,\ Ty) \geq 1,\ x,\ y\ \in\ F.$$

Remark 1. A mapping T is called an α-admissible mapping, if we take $F = X$ in definition 12.

Definition 13. [14] Let $T : X \to X$ and $\mu : F \times F \to \mathbb{R}^+$. We say that T is a μ_F-subadmissible mapping if

$$\mu(x, y) \leq 1 \text{ implies } \mu(Tx, Ty) \leq 1,\ x,\ y\ \in\ F.$$

We introduce the analogues of fixed point solutions for contraction mappings in integral setting via C-class function, which in turn generalize various existing results. Our findings significantly expand upon, enhance, and broaden similar findings in the literature.

Definition 14. [14] Let (X, d) be a metric space, F a nonempty subset of X, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$. A mapping T is said to be α_β-contractive mapping if there exists a $\beta : [0, \infty) \to [0,\ 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in F$, following condition holds:

$$h(\alpha\left(x, Tx\right), \alpha\left(y, Ty\right), \psi d(Tx,Ty)) \leq \mathcal{F}(\beta(d(x, y)), \psi d(x, y)), \tag{1}$$

where the pair $(\mathcal{F}, h)$ is a upclass of type II and $\psi \in \Psi$.

Theorem 1. [14] *Let (X, d) be a complete metric space, F a nonempty closed subset of X, $T : X \to X$ is an α_F-admissible mapping and F is invariant under T. Further assume that T is an α_β-contractive mapping. Suppose that there exists $x_0 \in F$ such that $\alpha(x_0, Tx_0) \geq 1$ and either of the following conditions hold:*
(a) T is continuous, or
(b) if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$, for all n, then $\alpha(x, Tx) \geq 1$.
Then T has a fixed point.

Lemma 1. *Let $(X, M_z, \diamond)$ is a b-fuzzy metric space with C-triangular fuzzy metric. Let $\{x_n\}$ be a sequence in X such that*

$$1 - M_z(x_n, x_{n+1}, l) \to 0 \text{ as } n \to \infty.$$

If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $1 - M_z(x_{m(k)}, x_{n(k)}, l) \geq \varepsilon$, $1 - M_z(x_{m(k)-1}, x_{n(k)}, l) < \varepsilon$ and
(i) $\lim_{k\to\infty} 1 - M_z(x_{m(k)-1}, x_{n(k)+1}, l) = \varepsilon$;
(ii) $\lim_{k\to\infty} 1 - M_z(x_{m(k)}, x_{n(k)}, l) = \varepsilon$;
(iii). $\lim_{k\to\infty} 1 - M_z(x_{m(k)-1}, x_{n(k)}, l) = \varepsilon$

Definition 15. [8] A b-fuzzy metric M_z is said to be C-triangular, if for all $x, y, z \in X$, and

$$M_z(x, z, l) \leq M_z(x, y, l) + M_z(y, z, l) - 1, \tag{2}$$

holds

Definition 16. [3] A self map L is said to be a triangular α-admissible if there exists $\alpha : X \times X \times (0, \infty) \to \mathbb{R}^+$

1. $\alpha(x, x, l) \geq 1 \implies \alpha(Lx, Ly, l) \geq 1$,for all $x, y \in X$ and $l > 0$.
2. $\alpha(x, y, l) \geq 1$ and $\alpha(y, z, l) \geq 1 \implies \alpha(x, z, l) \geq 1$,for all $x, y, z \in X$ and $l > 0$.

Lemma 2. [3] *Consider a fuzzy metric space denoted by* $(X, M_z, \diamond)$ *and let* $L : X \to X$ *be a triangular* α*-admissible mapping. Suppose there exists an element* $x_0 \in F$ *such that* $\alpha(x_0, Lx_0, l) \geq 1$*. Let us define a* $\{x_n\}$ *recursively by setting* x_{n+1}=Tx_n*.Then*

$$\alpha\left(x_m, x_n, l\right) \geq 1,$$

for all $m, n \in N$ *with* $m < n$ *and* $l > 0$*.*

2. Main results

Based on the above observations, we give the following theorem.

Definition 17. Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$. A mapping T is said to be α_β-b-Fuzzy-contractive mapping of type II if there exists a $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in F$, following condition holds:

$$h(\alpha\left(x, Tx, l\right), \alpha\left(y, Ty, l\right), \psi(1 - M_z(Tx, Ty, l))) \leq \mathcal{F}(\beta(1 - M_z(x, y, l)), \psi(1 - M_z(x, y, l))), \tag{3}$$

where the pair $(\mathcal{F}, h)$ is a upclass of type II and $\psi \in \Psi$.

Consider a self map T defined on a G-complete b-fuzzy metric space $(X, M_z, \diamond)$ where fuzzy metric is C-triangular satisfying:

Theorem 2. *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and let* $T : X \to X$ *be a self-mapping.*
Assume that there exists a pair of functions (F, h) *such that:*
1. $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *belongs to the upper class of type II, that is,*
   (a) F *is continuous;*
   (b) $F(u, v) \leq u$ *for all* $u, v \in \mathbb{R}_+$*;*
   (c) $F(u, v) = u$ *implies* $u = 0$ *or* $v = 0$*.*
2. $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a function of subclass of type II, i.e.,*

$$x, y \geq 1 \Longrightarrow h(1, 1, z) \leq h(x, y, z), \quad \forall z \in \mathbb{R}_+,$$

and, in particular,

$$h(1, 1, z) = z.$$

3. *There exists* $\alpha \in (0, 1)$ *such that for all* $x, y \in X$*,*

$$d(Tx, Ty) \leq F\big(d(x, y),\, h(1, 1, \alpha\, d(x, y))\big).$$

Then T *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the iterative sequence* $\{T^n x_0\}$ *converges to* x^**.*

Proof. Let $x_0 \in X$ be arbitrary and define the Picard sequence

$$x_n = Tx_{n-1}, \quad n \geq 1.$$

By the α-admissibility of T, we have

$$\alpha(x_n, x_{n+1}, l) = \alpha(x_n, Tx_n, l) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Applying the α_β-contractive condition with $x = x_{n-1}$ and $y = x_n$, we obtain

$$h(1, 1, \psi(1 - M_z(x_n, x_{n+1}, l))) \leq \mathcal{F}(\beta(1 - M_z(x_{n-1}, x_n, l)), \psi(1 - M_z(x_{n-1}, x_n, l))).$$

Since $h(1, 1, t) = t$ and $F(s, t) = st$, this yields

$$\psi(1 - M_z(x_n, x_{n+1}, l)) \leq \beta(1 - M_z(x_{n-1}, x_n, l))\, \psi(1 - M_z(x_{n-1}, x_n, l)). \tag{4}$$

Because ψ is nondecreasing and $\beta(t) < 1$ for all $t > 0$, it follows that

$$1 - M_z(x_n, x_{n+1}, l) \leq 1 - M_z(x_{n-1}, x_n, l),$$

hence the sequence $\{1 - M_z(x_n, x_{n+1}, l)\}$ is nonincreasing and bounded below by 0. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \left(1 - M_z(x_n, x_{n+1}, l)\right) = r. \tag{5}$$

Dividing both sides of (4) by $\psi(1 - M_z(x_{n-1}, x_n, l))$ and letting $n \to \infty$, we use the continuity of ψ and obtain

$$\lim_{n\to\infty} \beta(1 - M_z(x_{n-1}, x_n, l)) = 1.$$

Since $\beta(t) < 1$ for all $t > 0$, this implies

$$\lim_{n\to\infty} \left(1 - M_z(x_{n-1}, x_n, l)\right) = 0. \tag{6}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose otherwise. Then, by Lemma 1, there exist subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $m_k < n_k$ such that

$$\lim_{k\to\infty} \left(1 - M_z(x_{m_k}, x_{n_k}, l)\right) = \delta > 0.$$

Applying the contractive condition to (x_{m_k}, x_{n_k}) and using the same arguments as above, we obtain

$$\frac{\psi(1 - M_z(x_{m_k+1}, x_{n_k+1}, l))}{\psi(1 - M_z(x_{m_k}, x_{n_k}, l))} \leq \beta\left(1 - M_z(x_{m_k}, x_{n_k}, l)\right).$$

Letting $k \to \infty$ and using the continuity of ψ and β, we get $\delta = 0$, which is a contradiction. Hence, $\{x_n\}$ is Cauchy.

Since $(X, M_z, \diamond)$ is complete, there exists $z \in X$ such that $x_n \to z$. By continuity of $M_z(\cdot, \cdot, l)$ in each argument, we have

$$\lim_{n\to\infty} M_z(x_{n-1}, z, l) = 1. \tag{7}$$

Assume first that T is sequentially continuous. Since $x_n \to z$, we have $Tx_n \to Tz$. But $Tx_n = x_{n+1}$ and $x_{n+1} \to z$, hence $Tz = z$.

Alternatively, assume condition (3). Then $\alpha(z, Tz, l) \geq 1$. Applying the contractive inequality to (z, x_n) and letting $n \to \infty$, using (7) and the continuity of M_z and ψ, we obtain

$$1 - M_z(Tz, z, l) = 0,$$

which implies $Tz = z$.

Uniqueness

Assume that z and v are two fixed points of T with $z \neq v$. Then $1 - M_z(z, v, l) > 0$. Applying the contractive condition with $h(1, 1, t) = t$ and $F(s, t) = st$, we obtain

$$1 - M_z(z, v, l) \leq \beta(1 - M_z(z, v, l)) \, [1 - M_z(z, v, l) < 1 - M_z(z, v, l),$$

a contradiction. Therefore, the fixed point is unique. □

Definition 18. Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$. A mapping T is said to be α_β-b-Fuzzy-contractive mapping of type I if there exists a $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in F$, following condition holds:

$$h(\alpha\,(x, y, l)\,, \psi(1 - M_z(Tx, Ty, l))) \leq \mathcal{F}(\beta(1 - M_z(x, y, l)), \psi(1 - M_z(x, y, l))), \tag{8}$$

where pair $(\mathcal{F}, h)$ is a upclass of type I and $\psi \in \Psi$.

Consider a self map T defined on a G-complete b-fuzzy metric space $(X, M_z, \diamond)$ where fuzzy metric is C-triangular satisfying:

Theorem 3. *Assume $(X, M_z, \diamond)$ is a G-complete b-fuzzy metric space with C-triangular fuzzy metric, where $\mathbb{X}$ is a nonempty set. Let $T : \mathbb{X} \to \mathbb{X}$ be an α_β-b-Fuzzy-contraction mapping of type I and T is α-admissible mapping , such that the following holds:*

(1) T is continuous ;

(2) There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$;

(3) if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x_n, x, l) \geq 1$, for all $n \in N$.

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

Proof. Let $x_0 \in \mathbb{X}$ be such that $\alpha(x_0, Tx_0, l) \geq 1$ for all $l > 0$. Define the sequence $\{x_n\}$ in $\mathbb{X}$ by

$$x_n = Tx_{n-1}, \quad n \geq 1.$$

Since T is α-admissible, we have

$$\alpha(x_n, x_{n+1}, l) = \alpha(x_n, Tx_n, l) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Applying the α_β-b-fuzzy contractive condition of type I with $x = x_{n-1}$ and $y = x_n$, we obtain

$$h(1, \psi(1 - M_z(x_n, x_{n+1}, l))) \leq \mathcal{F}(\beta(1 - M_z(x_{n-1}, x_n, l)), \psi(1 - M_z(x_{n-1}, x_n, l))).$$

Using the properties of h and $\mathcal{F}$, this yields

$$\psi(1 - M_z(x_n, x_{n+1}, l)) \leq \beta(1 - M_z(x_{n-1}, x_n, l))\, \psi(1 - M_z(x_{n-1}, x_n, l)) \leq \psi(1 - M_z(x_{n-1}, x_n, l)). \tag{9}$$

Since $\psi \in \Psi$ is nondecreasing, it follows that

$$1 - M_z(x_n, x_{n+1}, l) \leq 1 - M_z(x_{n-1}, x_n, l) \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\{1 - M_z(x_n, x_{n+1}, l)\}$ is a nonincreasing sequence bounded below by 0. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \big(1 - M_z(x_n, x_{n+1}, l)\big) = r. \tag{10}$$

Dividing both sides of (9) by $\psi(1 - M_z(x_{n-1}, x_n, l))$ and letting $n \to \infty$, using the continuity of ψ, we obtain

$$\lim_{n\to\infty} \beta(1 - M_z(x_{n-1}, x_n, l)) = 1.$$

Since $\beta(t) < 1$ for all $t > 0$, this implies

$$\lim_{n\to\infty} \big(1 - M_z(x_{n-1}, x_n, l)\big) = 0. \tag{11}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose otherwise. Then, by Lemma 1, there exist subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\lim_{k\to\infty} \big(1 - M_z(x_{m_k}, x_{n_k}, l)\big) = \delta > 0.$$

Applying the contractive condition to (x_{m_k}, x_{n_k}) and proceeding as above, we obtain

$$\frac{\psi\big(1 - M_z(x_{m_k+1}, x_{n_k+1}, l)\big)}{\psi\big(1 - M_z(x_{m_k}, x_{n_k}, l)\big)} \leq \beta\big(1 - M_z(x_{m_k}, x_{n_k}, l)\big) \leq 1.$$

Letting $k \to \infty$ and using the continuity of ψ and β, we obtain $\delta = 0$, which is a contradiction. Hence, $\{x_n\}$ is Cauchy.

Since $(\mathbb{X}, M_z, \diamondsuit)$ is G-complete, there exists $z \in \mathbb{X}$ such that $x_n \to z$. By continuity of $M_z(\cdot, \cdot, l)$ in each argument, we have

$$\lim_{n\to\infty} M_z(x_{n-1}, z, l) = 1. \tag{12}$$

Existence of a fixed point

If T is continuous, then

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\Big(\lim_{n\to\infty} x_n\Big) = Tz.$$

Alternatively, assume condition (3). Then $\alpha(z, Tz, l) \geq 1$. Applying the contractive condition to (z, x_n) and letting $n \to \infty$, using (12) and the continuity of M_z and ψ, we obtain

$$1 - M_z(Tz, z, l) = 0,$$

hence $Tz = z$.

Uniqueness

Assume that z and v are two fixed points of T with $\alpha(z, v, l) \geq 1$ and $z \neq v$. Then $1 - M_z(z, v, l) > 0$. Applying the contractive condition, we get

$$\psi(1 - M_z(z, v, l)) \leq \beta(1 - M_z(z, v, l))\, \psi(1 - M_z(z, v, l)) < \psi(1 - M_z(z, v, l))\,,$$

a contradiction. Therefore, the fixed point of T is unique. □

Remark 2. Special choices of α, β, ψ, and (F, h) recover several known Geraghty-type fixed point results:

- $\alpha \equiv 1$, $\beta(t) = \lambda$, $\psi(t) = t$, $F(s,t) = st$, $h(x,y,z) = z \implies$ classical Geraghty contraction in metric spaces.
- $\alpha \equiv 1$, $\beta(t) = t$, $\psi(t) = t$, $F(s,t) = st$, $h(x,y,z) = z \implies$ Geraghty-type contraction in fuzzy metric spaces.

3. Corollaries

Here, we present some consequences of our main finding, all of them are generalized version of previous derived results.

Corollary 1. *Suppose $(X, M_z, \diamondsuit)$ is a G-complete b-fuzzy metric space with C-triangular fuzzy metric and a self map L defined on $\overline{Y}$ is a Geraghty type-I contractive map* [8]. *Then L has a unique fixed point in X.*

Proof. Take $\mathrm{h(x,y,z)} = \mathrm{z}, F(s,t) = st$ and $\psi(t) = t$ in 2 □

Corollary 2. *Suppose $(X, M_z, \diamondsuit)$ is a G-complete b-fuzzy metric space with C-triangular fuzzy metric and a self map L defined on $\overline{Y}$ is a Geraghty type-II contractive map* [8]. *Then L has a unique fixed point in X.*

Proof. Take $\mathrm{h(x,y)} = \mathrm{y}, F(s,t) = st$ and $\psi(t) = t$ in 3 □

With choice $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s,t) = s^q t^p, m, n, q \in \mathbb{N} \cup \{0\}, p > 0$, we drive the following Corollary from Theorem 2.

Corollary 3. *Let $(X, M_z, \diamondsuit)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$,and $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in X$,*

$$\alpha\,(x, Tx, l)^m\, \alpha\,(y, Ty, l)^n\, (\psi(1 - M_z(Tx, Ty, l)))^p \leq \beta(1 - M_z(x, y, l)^q [\psi(1 - M_z(x, y, l))]^p,$$

where $m, n, q \in \mathbb{N} \cup \{0\}$, $p > 0$ and $\psi \in \Psi$. Assume the following holds:

(i) *T is continuous ;*

(ii) *There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$.*

(iii) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x, Tx, l) \geq 1$, for all $l > 0$.*

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, u, l) \geq 1, \alpha(v, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

By selecting m = q = p = 1 we drive the following Corollary from Corollary 3.

Corollary 4. *Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$,and $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in X$,*

$$\alpha\,(x, Tx, l)\, \alpha\,(y, Ty, l)\, (\psi(1 - M_z(Tx, Ty, l))) \leq \beta(1 - M_z(x, y, l)\psi(1 - M_z(x, y, l)),$$

where $\psi \in \Psi$.Assume the following holds:

(i) *T is continuous ;*

(ii) *There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$.*

(iii) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x, Tx, l) \geq 1$, for all $l > 0$.*

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, u, l) \geq 1, \alpha(v, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

With choice $h(x, y, z) = (z + k)^{xy}, k > 1, \mathcal{F}(s, t) = st + l$, we drive the following Corollary from Theorem 2.

Corollary 5. *Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$,and $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in X$,*

$$[k + \psi(1 - M_z(Tx, Ty, l))]^{\alpha(x,Tx,l)\alpha(y,Ty,l)} \leq \beta(1 - M_z(x, y, l)\psi(1 - M_z(x, y, l)) + k,$$

where $k > 1$ and $\psi \in \Psi$.Assume the following holds:

(i) *T is continuous ;*

(ii) *There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$.*

(iii) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x, Tx, l) \geq 1$, for all $l > 0$.*

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, u, l) \geq 1, \alpha(v, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

With choice $h(x, y, z) = (xy + k)^z, k > 0, \mathcal{F}(s, t) = (1 + k)^{st}$, we drive the following Corollary from Theorem 2.

Corollary 6. *Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$,and $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in X$,*

$$[k + \alpha\,(x, Tx, l)\, \alpha\,(y, Ty, l)\,]^{\psi(1-M_z(Tx,Ty,l))} \leq (1 + k)^{\beta(1-M_z(x,y,l)\psi(1-M_z(x,y,l))},$$

where $k > 0$ and $\psi \in \Psi$.Assume the following holds:

(i) *T is continuous ;*

(ii) *There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$.*

(iii) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x, Tx, l) \geq 1$, for all $l > 0$.*

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, u, l) \geq 1, \alpha(v, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

With choice $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}, \mathcal{F}(s, t) = s^k t^k$, we drive the following Corollary from Theorem 2.

Corollary 7. *Let $(X, M_z, \diamond)$ be a G-complete b-fuzzy metric space, $T : X \to X$ and $\alpha : F \times F \to \mathbb{R}^+$,and $\beta : [0, \infty) \to [0,\ 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that for all $x, y \in X$,*

$$\frac{\alpha\,(x, Tx, l)^m + \alpha\,(x, Tx, l)^n\,\alpha\,(y, Ty, l)^p + \alpha\,(y, Ty, l)^q}{3}\psi(1 - M_z(Tx, Ty, l))^k$$
$$\leq (\beta(1 - M_z(x, y, l)\psi(1 - M_z(x, y, l)))^k,$$

where $m, n, q, p \in \mathbb{N}\cup\{0\}, k > 0$ and $\psi \in \Psi$. Assume the following holds:

(i) *T is continuous ;*

(ii) *There is $x_0 \in \mathbb{X}$, such that $\alpha(x_0, Tx_0, l) \geq 1$, for all $l > 0$.*

(iii) *if $\{x_n\}$ is a sequence in F such that $x_n \to x$, $\alpha(x_n, x_{n+1}, l) \geq 1$, for all n, then $\alpha(x, Tx, l) \geq 1$, for all $l > 0$.*

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha(u, u, l) \geq 1, \alpha(v, v, l) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

Now, we give an example to validate our result.

Example 7. Let $X = [0, 1]$ and define $M_z : X \times X \times (0, \infty) \to [0, 1]$ by

$$M_z(x, y, l) = e^{-\frac{|x-y|^2}{l}}, \quad x, y \in X,\ l > 0.$$

It is easy to verify that M_z is a b-fuzzy metric with $b = 2$. Moreover, M_z satisfies the C-triangular property for all $x, y \in X$ and $l > 0$, that is,

$$M_z(x, z, l) \geq M_z(x, y, l) + M_z(y, z, l) - 1.$$

G-completeness

Since $X = [0, 1]$ is compact in the usual metric and the fuzzy metric M_z is continuous in each argument, every Cauchy sequence in $(X, M_z, \diamond)$ converges in X. Hence, $(X, M_z, \diamond)$ is G-complete.

3.1. Mapping

Consider $L : X \to X$ defined by

$$L(x) = \begin{cases} \frac{1}{5}x^2, & x \in [0, 1), \\ \frac{1}{8}, & x = 1. \end{cases}$$

Admissibility function

Define a nontrivial $\alpha : X \times X \to [0, 1]$ by

$$\alpha(x, y, l) = \begin{cases} 1, & x \leq y, \\ 0, & x > y. \end{cases}$$

This satisfies $\alpha \not\equiv 1$ and L is α-admissible, since $\alpha(x, y, l) \geq 1 \implies \alpha(Lx, Ly, l) \geq 1$ for all $l > 0$.

Control function and β

$$\psi(t) = t \in \Psi \quad \text{(continuous, nondecreasing)}, \qquad \beta(t) = 1 - t \quad \text{with } \beta(t) < 1 \text{ for all } t > 0.$$

Upper class pair (F, h)

We take

$$F(s, t) = s\,t, \qquad h(x, y, z) = z,$$

which clearly satisfies the properties of an upper class pair of type II, in particular $h(1, 1, z) = z$.

Verification of the contractive inequality

Let $x, y \in X$ with $\alpha(x, y, l) \geq 1$, i.e. $x \leq y$. Then, for all $l > 0$,

$$\begin{aligned}\psi(1 - M_z(Lx, Ly, l)) &= 1 - M_z(Lx, Ly, l) \leq (1 - M_z(x, y, l))\, \beta(1 - M_z(x, y, l)) \\ &= \beta(1 - M_z(x, y, l))\, \psi(1 - M_z(x, y, l)).\end{aligned}$$

A simple case analysis for $x = 1$ or $y = 1$ confirms that the inequality holds in all cases.

Fixed point

By Theorem 2, L has a unique fixed point in X, which is 0.

4. Conclusion

To sum up, this paper effectively expands fixed-point theorems in the context of b-fuzzy metric spaces by introducing Geraghty-type inequalities and pair upper (F, h)-class functions. These novel ideas enable the determination of existence and uniqueness results for fixed poinr via particular contractive. By doing so, the study generalizes and enhances several prominent fixed-point theorems in both fuzzy and b-metric spaces, contributing valuable insights to the existing literature.

References

[1] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis, 1*(1), 5–11.

[2] Chiroma, R., Shagari, M. S., & Jiddah, J. A. (2024). On fixed point of quasi contraction with application to integral equation. *Sahand Communications in Mathematical Analysis, 21*(3), 461–491.

[3] Dinarvand, M. (2017). Some fixed point results for admissible Geraghty contraction type mapping in fuzzy metric spaces. *Iranian Journal of Fuzzy Systems, 14*(3), 161–177.

[4] George, A., & Veeramani, P. (1994). On some results in fuzzy metric space. *Fuzzy Sets and Systems, 64*, 395–399.

[5] Jiddah, J. A., Shagari, M. S., & Imam, A. T. (2023). On fixed points of a general class of hybrid contractions with Ulam-type stability. *Sahand Communications in Mathematical Analysis, 20*(2), 39–64.

[6] Schweizer, B., & Sklar, A. (1983). *Probabilistic Metric Spaces*. North-Holland.

[7] Hussain, N., Salimi, P., & Parvaneh, V. (2015). Fixed point results for various contractions in parametric and fuzzy b-metric spaces. *Journal of Nonlinear Sciences and Applications, 8*(5), 719–739.

[8] Patel, U. D., Chandra, V., & Radenović, S. (2024). Some Geraghty-type inequalities in b-fuzzy metric spaces with an application. *European Journal of Pure and Applied Mathematics, 17*(4), 2384-2404.

[9] Ansari, A. H. (2014). Note on α-admissible mappings and related fixed point theorems. *2nd Regional Conference on Mathematics and Applications*, 373–376.

[10] Ansari, A. H., & Shukla, S. (2016). Some fixed point theorems for ordered F-(F, h)-contraction and subcontractions in 0-f-orbitally complete partial metric spaces. *Journal of Advanced Mathematical Studies, 9*(1), 37–53.

[11] Bakhtin, I. A. (1989). The contraction mapping principle in almost metric space. *Functional Analysis, 30*, 26–37.

[12] Hamaizia, T. (2021). Fixed point theorems involving C-class functions in G_b-metric spaces. *Journal of Applied Mathematics and Informatics, 39*(3–4), 529–539.

[13] Khan, M. S., Swaleh, M., & Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society, 30*, 1–9.

[14] Ansari, A. H., & Chandok, S. (2016). Unification on α-admissible mappings and related fixed point theorems. *Scientific Publications of the State University of Novi Pazar, Series A: Applied Mathematics, Informatics and Mechanics*, 8(1), 65–71.