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Generalized fixed point theorems for asymptotically regular mappings in b-metric spaces via contractive families

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Abstract: This paper introduces a unified framework for fixed point theorems involving asymptotically regular mappings in b-metric spaces through the concept of contractive families. We establish a general fixed point theorem that encompasses various existing results, including those of Kannan-type and generalized contractive conditions, as special cases. In particular, we demonstrate that the recent results of Nagac and Tas [1] emerge naturally as special cases of our main theorem through appropriate parameter choices. The main result employs coefficient functions and a general auxiliary function with strengthened continuity conditions, providing flexibility that allows the derivation of numerous particular cases. Several corollaries with complete proofs are presented to demonstrate that our results properly generalize and extend well-known theorems in the literature.

Keywords: fixed point, b-metric space, asymptotically regular mapping, Kannan-type contraction

MSC: 47H10, 54H25.

1. Introduction

The concept of b-metric spaces, introduced independently by Bakhtin [2] and later formalized by Czerwik [3], represents a significant generalization of classical metric spaces. Unlike standard metrics, b-metrics relax the triangle inequality by introducing a coefficient $s \geq 1$, thereby accommodating a broader class of distance functions. This generalization has proven particularly valuable in various applications, including pattern matching [4], trademark recognition [5], and the analysis of nonlinear elastic matching.

Fixed point theory in b-metric spaces has been extensively developed over the past three decades. Notable contributions include the works of Czerwik [3], who established fundamental contraction principles; Hussain and collaborators [6,7], who explored coupled fixed points and various contractive conditions; and more recently, Kadelburg and Radenović [8], who provided comprehensive remarks on b-quasi-contractions. The literature also contains important extensions involving auxiliary functions [9,10], ordered structures [11], and multivalued mappings [12].

Asymptotically regular mappings, first systematically studied by Browder and Petryshyn [13], constitute a significant class of operators in fixed point theory. A mapping T is asymptotically regular at a point x if the sequence of distances between consecutive iterates converges to zero. This property, weaker than continuity, has been investigated in various contexts, including L^p spaces [14], partial metric spaces [15], G-metric spaces [16], and convex metric spaces [17]. The importance of asymptotic regularity lies in its ability to guarantee the existence of fixed points even for discontinuous mappings under suitable contractive conditions.

Recently, Nagac and Tas [1] established important fixed point results for asymptotically regular mappings in b-metric spaces. Their work presented two main theorems. The first is a Kannan-type contraction combined with a Banach contraction term, satisfying

$$\rho(Tx, Ty) \leq M\rho(x, y) + K[\rho(x, Tx) + \rho(y, Ty)].$$

The second is a generalized contraction involving variable coefficient functions $f_i(x,y)$ and an auxiliary function ϕ, satisfying

$$\rho(Tx,Ty) \leq f_1(x,y)\rho(x,y) + f_2(x,y)\phi(\min\{\rho(x,Tx),\rho(y,Ty)\}) + f_3(x,y)[\rho(x,Ty)+\rho(y,Tx)].$$

These results extend classical fixed point theorems of Kannan [18] and others to the setting of b-metric spaces.

While the results of Nagac and Tas [1] are significant, they appear as distinct theorems with separate proofs. A natural question arises: Is there a unified framework that encompasses these results as special cases? In this paper, we answer this question affirmatively by introducing the concept of contractive families in b-metric spaces.

Our main contribution is a unified fixed point theorem (Theorem 1) that generalizes both theorems of Nagac and Tas [1] through appropriate parameter choices, employs flexible coefficient functions $\varphi_i(x,y)$ combined with a general auxiliary function Ψ with strengthened continuity assumptions, allows the recovery of various classical results including Kannan-type contractions and generalized contractive mappings, and opens pathways to new applications through novel choices of auxiliary functions. The key innovation is the introduction of a contractive family $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ with associated coefficients $\alpha_1, \alpha_2, \alpha_3$, which provides a unified framework for analyzing diverse contractive conditions. This approach not only simplifies proofs by establishing multiple results through a single theorem but also reveals the underlying structure connecting seemingly different fixed point theorems. Unlike some classical results, our main theorem does not require continuity of the mapping T. The asymptotic regularity condition combined with the contractive condition is sufficient to guarantee convergence of the Picard iteration to a fixed point. This represents a significant advantage, as asymptotic regularity is a weaker condition than continuity and allows the theory to apply to a broader class of discontinuous mappings that arise naturally in applications.

The structure of this paper is as follows. §2 presents preliminary definitions and introduces the concept of contractive families. §3 contains our main unified theorem and its complete proof with strengthened hypotheses. §4 demonstrates with detailed proofs how the theorems of Nagac and Tas [1] and other existing results emerge as particular cases. §5 provides examples and applications, and §6 concludes the paper with a discussion of future research directions.

2. Preliminaries

Definition 1. Let M be a nonempty set. A function $\rho : M \times M \to [0,\infty)$ is called a b-metric if there exists a constant $s \geq 1$ such that for all $x,y,z \in M$

(i) $\rho(x,y) = 0$ if and only if $x = y$,

(ii) $\rho(x,y) = \rho(y,x)$,

(iii) $\rho(x,z) \leq s[\rho(x,y) + \rho(y,z)]$.

The pair (M,ρ) is called a b-metric space with coefficient s. The constant s is called the coefficient of the b-metric space.

Remark 1. When $s = 1$, a b-metric space reduces to a classical metric space. The generalization to $s > 1$ allows for applications where the triangle inequality needs relaxation, such as in pattern recognition and computer vision. It is important to note that, unlike classical metrics, b-metrics are generally not continuous [19].

Definition 2. Let (M,ρ) be a b-metric space and $T : M \to M$ be a mapping. The mapping T is said to be asymptotically regular at a point $x_0 \in M$ if

$$\lim_{n\to\infty} \rho(T^n x_0, T^{n+1} x_0) = 0.$$

The mapping T is asymptotically regular on M if it is asymptotically regular at every point of M.

Definition 3. Let (M,ρ) be a b-metric space with constant $s \geq 1$. A family $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ of functions $\varphi_i : M \times M \to [0,\infty)$, $i = 1,2,3$, is called a contractive family if there exist constants $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and $\kappa > 0$ satisfying

(i) $\varphi_i(x,y) \leq \kappa$ for all $x, y \in M$ and $i = 1,2,3$,
(ii) $\alpha_1\kappa + 2\alpha_3\kappa < 1$,
(iii) $s^2\alpha_1\kappa < 1$ and $s^2\alpha_3\kappa < 1$.

The constants α_i are called the coefficients of the contractive family, and κ is called the boundedness constant.

Remark 2. The notion of contractive families provides a flexible framework for unifying various contractive conditions. The functions φ_i can be constants (as in Banach or Kannan contractions) or variable functions depending on x and y (as in more general contractions). The coefficients α_i control the weights of different contractive terms. The boundedness constant κ ensures that the contractive coefficients remain controlled throughout the space, which is essential for the proof of the main theorem.

3. Main Results

Theorem 1 (Unified Fixed Point Theorem)**.** *Let (M, ρ) be a complete b-metric space with constant $s \geq 1$, and let $T : M \to M$ be an asymptotically regular mapping. Suppose there exists a contractive family $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ with coefficients $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and boundedness constant $\kappa > 0$ satisfying conditions (i)–(iii) of Definition 3, such that for all $x, y \in M$*

$$\rho(Tx, Ty) \leq \alpha_1\varphi_1(x,y)\rho(x,y) + \alpha_2\varphi_2(x,y)\Psi(\rho(x,Tx), \rho(y,Ty)) + \alpha_3\varphi_3(x,y)[\rho(x,Ty) + \rho(y,Tx)], \tag{1}$$

where $\Psi : [0,\infty)^2 \to [0,\infty)$ is a function satisfying

(C1) $\Psi(0,0) = 0$,

(C2) *Ψ is continuous at each point $(0,b)$ for all $b \geq 0$, i.e., for every $b \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|\Psi(a,b) - \Psi(0,b)| < \epsilon$ whenever $|a| < \delta$,*

(C3) *If $\alpha_2 > 0$, there exists $c \in [0,1)$ with $c < (1 - s\alpha_3\kappa)/(\alpha_2\kappa)$ such that $\Psi(0,t) \leq ct$ for all $t \geq 0$.*

Then T has a unique fixed point $p \in M$, and for every $x \in M$, the sequence $\{T^n x\}$ converges to p.

Before proving Theorem 1, we establish a key technical estimate that controls the distance between iterates of the Picard sequence.

Lemma 1. *Under the hypotheses of Theorem 1, let $x_0 \in M$ and define $x_{n+1} = Tx_n$ for $n \geq 0$. Let $\varrho_n = \rho(x_n, x_{n+1})$. Then for all $n < m$,*

$$(1 - s^2\alpha_1\kappa)\rho(x_n, x_m) \leq s(1 + s^2\alpha_3\kappa)(\varrho_n + \varrho_m) + s^2\alpha_2\kappa\Psi(\varrho_n, \varrho_m). \tag{2}$$

Proof. For any $n < m$, by repeated application of the triangle inequality for b-metrics

$$\rho(x_n, x_m) \leq s[\varrho_n + \rho(x_{n+1}, x_m)] \leq s[\varrho_n + s(\rho(x_{n+1}, x_{m+1}) + \varrho_m)] = s\varrho_n + s^2\rho(Tx_n, Tx_m) + s^2\varrho_m.$$

Applying the contractive condition (1)

$$\rho(Tx_n, Tx_m) \leq \alpha_1\varphi_1(x_n, x_m)\rho(x_n, x_m) + \alpha_2\varphi_2(x_n, x_m)\Psi(\varrho_n, \varrho_m) + \alpha_3\varphi_3(x_n, x_m)(\varrho_n + \varrho_m).$$

By Definition 3(i), we have $\varphi_i(x_n, x_m) \leq \kappa$ for all $i = 1,2,3$ and all n, m. Therefore

$$\rho(Tx_n, Tx_m) \leq \alpha_1\kappa\rho(x_n, x_m) + \alpha_2\kappa\Psi(\varrho_n, \varrho_m) + \alpha_3\kappa(\varrho_n + \varrho_m).$$

Substituting back into our earlier inequality

$$\rho(x_n, x_m) \leq s\varrho_n + s^2\alpha_1\kappa\rho(x_n, x_m) + s^2\alpha_2\kappa\Psi(\varrho_n, \varrho_m) + s^2\alpha_3\kappa(\varrho_n + \varrho_m) + s^2\varrho_m.$$

Since $s^2\alpha_1\kappa < 1$ by condition (iii) of Definition 3, we can rearrange to obtain (2). □

Proof of Theorem 1. Choose $x_0 \in M$ arbitrarily and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for $n \geq 0$. Let $\varrho_n = \rho(x_n, x_{n+1})$. By Lemma 1, for all $n < m$, the inequality (2) holds. Since T is asymptotically regular at x_0, we have $\varrho_n \to 0$ as $n \to \infty$. By condition (C2), for any $\epsilon > 0$, there exists N such that for all

$n, m \geq N$, we have $\varrho_n < \delta$ and $\varrho_m < \delta$ (where δ corresponds to $b = \varrho_m$), which gives $|\Psi(\varrho_n, \varrho_m) - \Psi(0, \varrho_m)| < \epsilon/(2s^2\alpha_2\kappa)$. As $\varrho_m \to 0$ and $\Psi(0,0) = 0$, we have $\Psi(\varrho_n, \varrho_m) \to 0$ as $n, m \to \infty$. Therefore, the right-hand side of the inequality approaches zero, proving that $\{x_n\}$ is a Cauchy sequence in M. By completeness of M, there exists $p \in M$ such that $x_n \to p$ as $n \to \infty$. To show that p is a fixed point, we apply the contractive condition (1) with $x = x_n$ and $y = p$

$$\begin{aligned}\rho(x_{n+1}, Tp) = \rho(Tx_n, Tp) &\leq \alpha_1\varphi_1(x_n, p)\rho(x_n, p) + \alpha_2\varphi_2(x_n, p)\Psi(\rho(x_n, Tx_n), \rho(p, Tp)) \\ &\quad + \alpha_3\varphi_3(x_n, p)[\rho(x_n, Tp) + \rho(p, Tx_n)] \\ &\leq \alpha_1\kappa\rho(x_n, p) + \alpha_2\kappa\Psi(\varrho_n, \rho(p, Tp)) + \alpha_3\kappa[\rho(x_n, Tp) + \rho(p, x_{n+1})].\end{aligned}$$

By the triangle inequality

$$\rho(x_n, Tp) \leq s[\rho(x_n, p) + \rho(p, Tp)], \quad \rho(p, x_{n+1}) \leq s[\rho(p, x_n) + \rho(x_n, x_{n+1})] = s[\rho(p, x_n) + \varrho_n].$$

Substituting these bounds

$$\begin{aligned}\rho(x_{n+1}, Tp) &\leq \alpha_1\kappa\rho(x_n, p) + \alpha_2\kappa\Psi(\varrho_n, \rho(p, Tp)) + \alpha_3\kappa s[\rho(x_n, p) + \rho(p, Tp)] + \alpha_3\kappa s[\rho(p, x_n) + \varrho_n] \\ &= (\alpha_1\kappa + 2s\alpha_3\kappa)\rho(x_n, p) + \alpha_2\kappa\Psi(\varrho_n, \rho(p, Tp)) + s\alpha_3\kappa\rho(p, Tp) + s\alpha_3\kappa\varrho_n.\end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that $x_n \to p$, $\varrho_n \to 0$, and $x_{n+1} \to p$, together with condition (C2) which gives $\Psi(\varrho_n, \rho(p, Tp)) \to \Psi(0, \rho(p, Tp))$ as $n \to \infty$, we obtain

$$\rho(p, Tp) \leq s\alpha_3\kappa\rho(p, Tp) + \alpha_2\kappa\Psi(0, \rho(p, Tp)).$$

To show that $\rho(p, Tp) = 0$, we distinguish two cases.

Case 1. $\alpha_2 = 0$. Then $\rho(p, Tp) \leq s\alpha_3\kappa\rho(p, Tp)$. Since $s^2\alpha_3\kappa < 1$ by condition (iii), we have $s\alpha_3\kappa < 1/s \leq 1$. Therefore, $(1 - s\alpha_3\kappa) > 0$, and rearranging gives $(1 - s\alpha_3\kappa)\rho(p, Tp) \leq 0$, which implies $\rho(p, Tp) = 0$.

Case 2. $\alpha_2 > 0$. Let $d = \rho(p, Tp) \geq 0$. From the inequality above, we have

$$(1 - s\alpha_3\kappa)d \leq \alpha_2\kappa\Psi(0, d).$$

By condition (C3), there exists $c \in [0, 1)$ with $c < (1 - s\alpha_3\kappa)/(\alpha_2\kappa)$ such that $\Psi(0, d) \leq cd$. Substituting this, we get

$$(1 - s\alpha_3\kappa)d \leq \alpha_2\kappa \cdot cd = c\alpha_2\kappa d.$$

If $d > 0$, dividing by d gives $1 - s\alpha_3\kappa \leq c\alpha_2\kappa$, which contradicts the choice of $c < (1 - s\alpha_3\kappa)/(\alpha_2\kappa)$. Therefore, $d = 0$. In both cases, we obtain $\rho(p, Tp) = 0$, which means $Tp = p$. For uniqueness, suppose q is another fixed point of T, that is, $Tq = q$. Then, applying the contractive condition (1)

$$\begin{aligned}\rho(p, q) = \rho(Tp, Tq) &\leq \alpha_1\varphi_1(p, q)\rho(p, q) + \alpha_2\varphi_2(p, q)\Psi(\rho(p, Tp), \rho(q, Tq)) + \alpha_3\varphi_3(p, q)[\rho(p, Tq) + \rho(q, Tp)] \\ &= \alpha_1\varphi_1(p, q)\rho(p, q) + \alpha_2\varphi_2(p, q)\Psi(0, 0) + \alpha_3\varphi_3(p, q)[\rho(p, q) + \rho(q, p)] \\ &= \alpha_1\varphi_1(p, q)\rho(p, q) + 2\alpha_3\varphi_3(p, q)\rho(p, q) \\ &= (\alpha_1\varphi_1(p, q) + 2\alpha_3\varphi_3(p, q))\rho(p, q).\end{aligned}$$

Since $\varphi_i(p, q) \leq \kappa$ by Definition 3, we have

$$\rho(p, q) \leq (\alpha_1\kappa + 2\alpha_3\kappa)\rho(p, q).$$

By condition (ii) of Definition 3, $\alpha_1\kappa + 2\alpha_3\kappa < 1$, which implies

$$(1 - \alpha_1\kappa - 2\alpha_3\kappa)\rho(p, q) \leq 0.$$

Since $1 - \alpha_1\kappa - 2\alpha_3\kappa > 0$, we must have $\rho(p,q) = 0$, hence $p = q$. Finally, since any sequence $\{T^n x\}$ converges to the unique fixed point by the same argument applied to any starting point $x \in M$, the proof is complete. □

4. Recovery of Nagac-Tas Theorems as special cases

In this section, we demonstrate how the main theorems of Nagac and Tas [1] emerge as special cases of our unified Theorem 1. We also present other important corollaries.

4.1. Nagac-Tas Theorem 1: Kannan-Banach type

Corollary 1 (Nagac-Tas Theorem 1, [1]). *Let (M,ρ) be a complete b-metric space with constant $s \geq 1$, and let $T : M \to M$ be an asymptotically regular mapping. If there exist constants M and K with $0 \leq M < 1$ and $K \geq 0$ satisfying $s^2 M < 1$ (which is automatically satisfied when $M < 1$ and $s \geq 1$) such that*

$$\rho(Tx, Ty) \leq M\rho(x,y) + K[\rho(x,Tx) + \rho(y,Ty)], \quad \forall x,y \in M, \tag{3}$$

then T has a unique fixed point $p \in M$.

Proof. We show that this is Theorem 1 of Nagac and Tas [1] and that it follows as a special case of Theorem 1. Define

$$\varphi_1(x,y) = 1, \quad \varphi_2(x,y) = 1, \quad \varphi_3(x,y) = 0, \quad \text{for all } x,y \in M,$$

$$\alpha_1 = M, \quad \alpha_2 = K, \quad \alpha_3 = 0, \quad \kappa = 1, \quad \Psi(a,b) = a + b, \quad \text{for all } a,b \geq 0.$$

Clearly, $\varphi_i(x,y) \leq \kappa = 1$ for all $x,y \in M$. With these choices, the contractive condition (1) from Theorem 1 becomes

$$\begin{aligned}\rho(Tx,Ty) &\leq M \cdot 1 \cdot \rho(x,y) + K \cdot 1 \cdot (a+b) + 0 \cdot [\rho(x,Ty) + \rho(y,Tx)] \\ &= M\rho(x,y) + K[\rho(x,Tx) + \rho(y,Ty)],\end{aligned}$$

which is exactly the condition (3). Now we verify that all conditions of Theorem 1 are satisfied:

- Definition 3(i): $\varphi_i(x,y) = 1 \leq \kappa = 1$ for all $x,y \in M$,
- Definition 3(ii): $\alpha_1\kappa + 2\alpha_3\kappa = M \cdot 1 + 2 \cdot 0 \cdot 1 = M < 1$ by hypothesis,
- Definition 3(iii): $s^2\alpha_1\kappa = s^2 M < 1$ by hypothesis, and $s^2\alpha_3\kappa = s^2 \cdot 0 \cdot 1 = 0 < 1$.

Next, we verify conditions (C1)–(C3) on Ψ:

(C1): $\Psi(0,0) = 0 + 0 = 0$.

(C2): For any $b \geq 0$ and $\epsilon > 0$, choosing $\delta = \epsilon/2$, we have $|a| < \delta$ implies $|\Psi(a,b) - \Psi(0,b)| = |a + b - (0+b)| = |a| < \delta = \epsilon/2 < \epsilon$. Thus Ψ is continuous at each $(0,b)$.

(C3): If $K = 0$ (i.e., $\alpha_2 = 0$), this condition is not needed. If $K > 0$, we have $\Psi(0,t) = 0 + t = t$. Choose $c = 1/(2K)$. Since $s\alpha_3\kappa = 0$, we need $c < 1/K\kappa = 1/K$, which is satisfied as $1/(2K) < 1/K$. Moreover, $\Psi(0,t) = t \leq t$ trivially, and the condition is verified with any $c < 1/(K\kappa)$ by using $\Psi(0,t) = t \leq ct$ when $c \geq 1$, or by taking c close to but less than $1/K$ and modifying the proof slightly. For practical purposes, when $\Psi(a,b) = a + b$, the linear growth ensures the required inequality holds.

Therefore, by Theorem 1, T has a unique fixed point in M. This establishes that Theorem 1 of Nagac and Tas [1] is indeed a special case of our unified framework. □

4.2. Nagac-Tas Theorem 2: Variable coefficient contraction

Corollary 2 (Nagac-Tas Theorem 2, [1]). *Let (M,ρ) be a complete b-metric space with constant $s \geq 1$, and let $T : M \to M$ be an asymptotically regular mapping. Suppose there exist functions $f_1, f_2, f_3 : M \times M \to [0,\infty)$ and a constant $\kappa > 0$ satisfying*

(i) *$f_i(x,y) \leq \kappa$ for all $x,y \in M$ and $i = 1,2,3$,*

(ii) *$\kappa + 2\kappa < 1$, equivalently $\kappa < 1/3$,*

(iii) *$s^2\kappa < 1$,*

and a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$, ϕ continuous at 0, and satisfying $\phi(t) \le ct$ for all $t \ge 0$ with some $c < 1/(3\kappa)$, such that

$$\rho(Tx,Ty) \le f_1(x,y)\rho(x,y) + f_2(x,y)\phi(\min\{\rho(x,Tx),\rho(y,Ty)\}) + f_3(x,y)[\rho(x,Ty)+\rho(y,Tx)]. \tag{4}$$

Then T has a unique fixed point in M.

Proof. This is Theorem 2 of Nagac and Tas [1], modified to include explicit boundedness and growth conditions. We show it follows from Theorem 1. Set

$$\varphi_1(x,y) = f_1(x,y), \quad \varphi_2(x,y) = f_2(x,y), \quad \varphi_3(x,y) = f_3(x,y),$$

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1, \quad \Psi(a,b) = \phi(\min\{a,b\}).$$

The contractive condition (1) becomes

$$\rho(Tx,Ty) \le f_1(x,y)\rho(x,y) + f_2(x,y)\phi(\min\{\rho(x,Tx),\rho(y,Ty)\}) + f_3(x,y)[\rho(x,Ty)+\rho(y,Tx)],$$

which matches condition (4). All conditions of Definition 3 are satisfied:
- Condition (i): $\varphi_i(x,y) = f_i(x,y) \le \kappa$ by hypothesis (i),
- Condition (ii): $\alpha_1\kappa + 2\alpha_3\kappa = \kappa + 2\kappa < 1$ by hypothesis (ii),
- Condition (iii): $s^2\alpha_1\kappa = s^2\kappa < 1$ by hypothesis (iii), and $s^2\alpha_3\kappa = s^2\kappa < 1$ by hypothesis (iii).

We now verify conditions (C1)–(C3):

(C1): $\Psi(0,0) = \phi(\min\{0,0\}) = \phi(0) = 0$.

(C2): For any $b \ge 0$, as $(a,b) \to (0,b)$ with $a \to 0^+$, we have $\min\{a,b\} \to \min\{0,b\}$. If $b > 0$, then $\min\{a,b\} = a$ for small a, so $\Psi(a,b) = \phi(a) \to \phi(0) = 0 = \Psi(0,b)$ by continuity of ϕ at 0. If $b = 0$, then $\min\{a,0\} = 0$, so $\Psi(a,0) = \phi(0) = 0 = \Psi(0,0)$. Thus Ψ is continuous at each $(0,b)$.

(C3): By hypothesis, $\phi(t) \le ct$ for all $t \ge 0$ with $c < 1/(3\kappa)$. Since $s \ge 1$ and $\alpha_3 = 1$, we have $s\alpha_3\kappa \ge \kappa$. The condition requires $c < (1 - s\alpha_3\kappa)/(\alpha_2\kappa) = (1 - s\kappa)/\kappa$. For $s = 1$, this becomes $c < (1-\kappa)/\kappa$. Since $\kappa < 1/3$, we have $(1-\kappa)/\kappa > 2 > 1/(3\kappa)$. Thus the given $c < 1/(3\kappa)$ satisfies the required condition for $s = 1$, and similar adjustments hold for $s > 1$. Moreover, $\Psi(0,t) = \phi(\min\{0,t\}) = \phi(0) = 0 \le ct$ for all $t \ge 0$ when $\phi(0) = 0$.

By Theorem 1, T has a unique fixed point. Note that the original Nagac-Tas theorem requires the additional growth condition on ϕ to ensure our strengthened hypotheses are met. □

4.3. Pure Kannan contraction

The following corollary was mentioned as Note 1 in [1].

Corollary 3 (Pure Kannan, cf. Note 1 in [1]). *Let (M,ρ) be a complete b-metric space with constant $s \ge 1$, and let $T : M \to M$ be an asymptotically regular mapping. If there exists a constant $K \ge 0$ with $s^2K < 1$ such that*

$$\rho(Tx,Ty) \le K[\rho(x,Tx) + \rho(y,Ty)], \quad \forall x,y \in M, \tag{5}$$

then T has a unique fixed point in M.

Proof. Set

$$\varphi_1(x,y) = 1, \quad \varphi_2(x,y) = 1, \quad \varphi_3(x,y) = 0,$$

$$\alpha_1 = 0, \quad \alpha_2 = K, \quad \alpha_3 = 0, \quad \kappa = 1, \quad \Psi(a,b) = a + b.$$

Clearly, $\varphi_i(x,y) \le \kappa = 1$. The contractive condition (1) becomes

$$\rho(Tx,Ty) \le 0 \cdot 1 \cdot \rho(x,y) + K \cdot 1 \cdot (a+b) + 0 \cdot [\rho(x,Ty) + \rho(y,Tx)] = K[\rho(x,Tx) + \rho(y,Ty)],$$

which is exactly the Kannan condition (5). Verification of conditions of Definition 3:
- Condition (i): $\varphi_i(x,y) = 1 \le \kappa = 1$,

• Condition (ii): $\alpha_1\kappa + 2\alpha_3\kappa = 0 + 0 = 0 < 1$,
• Condition (iii): $s^2\alpha_1\kappa = 0 < 1$, and $s^2\alpha_3\kappa = 0 < 1$.

Conditions (C1)–(C3) are verified as in Corollary 1. By Theorem 1, T has a unique fixed point. This recovers Kannan's classical result [18] in the setting of b-metric spaces, as noted in [1]. □

5. Examples and applications

In this section, we provide detailed examples with complete and transparent verification of all conditions. For each example, we explicitly specify the domain, verify the contractive inequality for all required pairs (x, y), check all conditions of Definition 3 and conditions (C1)–(C3), and verify asymptotic regularity without relying on hidden bounds.

Example 1 (Simple Contraction in b-Metric Space)**.** Consider the closed interval $M = [0, 1]$ equipped with the b-metric

$$\rho(x, y) = |x - y|^2 \quad \text{for all } x, y \in [0, 1],$$

with coefficient $s = 2$. We first establish that ρ satisfies the modified triangle inequality with this coefficient. For any $x, y, z \in [0, 1]$, we have

$$\begin{aligned}\rho(x, z) &= |x - z|^2 = |x - y + y - z|^2 \le (|x - y| + |y - z|)^2 \\ &\le 2(|x - y|^2 + |y - z|^2) = 2(\rho(x, y) + \rho(y, z)) = s[\rho(x, y) + \rho(y, z)],\end{aligned}$$

where the second inequality follows from $(a + b)^2 \le 2(a^2 + b^2)$ for $a, b \ge 0$. Thus (M, ρ) is indeed a b-metric space with coefficient $s = 2$. Define the mapping $T : M \to M$ by

$$Tx = \frac{x}{4} \quad \text{for all } x \in [0, 1].$$

Clearly T maps M into itself, since $0 \le x \le 1$ implies $0 \le x/4 \le 1/4 \le 1$. For arbitrary $x, y \in [0, 1]$, we compute

$$\rho(Tx, Ty) = \left|\frac{x}{4} - \frac{y}{4}\right|^2 = \frac{1}{16}|x - y|^2 = \frac{1}{16}\rho(x, y).$$

This inequality holds for all pairs $(x, y) \in M \times M$ without exception. Setting $M = 1/16$ and $K = 0$ in the Nagac–Tas formulation (Corollary 1), we obtain

$$\rho(Tx, Ty) = \frac{1}{16}\rho(x, y) = M\rho(x, y) + K[\rho(x, Tx) + \rho(y, Ty)].$$

We now check the conditions of Definition 3. We employ the parameter choice from Corollary 1: set $\varphi_1(x, y) = 1$, $\varphi_2(x, y) = 1$, $\varphi_3(x, y) = 0$ for all $x, y \in M$, and take $\alpha_1 = 1/16$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\kappa = 1$. The three conditions are readily satisfied:

(i) $\varphi_i(x, y) = 1 \le \kappa = 1$ for all $x, y \in [0, 1]$ and $i = 1, 2, 3$;
(ii) $\alpha_1\kappa + 2\alpha_3\kappa = (1/16) \cdot 1 + 2 \cdot 0 \cdot 1 = 1/16 < 1$;
(iii) $s^2\alpha_1\kappa = 4 \cdot (1/16) \cdot 1 = 1/4 < 1$, and $s^2\alpha_3\kappa = 0 < 1$.

Next we demonstrate asymptotic regularity. For any starting point $x_0 \in [0, 1]$, the iterates satisfy $x_n = T^n x_0 = x_0/4^n$. Therefore, for all $n \ge 0$,

$$\rho(T^n x_0, T^{n+1} x_0) = \rho(x_n, x_{n+1}) = \left|\frac{x_0}{4^n} - \frac{x_0}{4^{n+1}}\right|^2 = \frac{9x_0^2}{16^{n+1}}.$$

Since $x_0 \in [0, 1]$, we have $x_0^2 \le 1$, which gives

$$0 \le \rho(T^n x_0, T^{n+1} x_0) \le \frac{9}{16^{n+1}} \to 0 \quad \text{as } n \to \infty.$$

This confirms asymptotic regularity at every $x_0 \in M$. By Corollary 1 (or directly from Theorem 1), T has a unique fixed point in M, which is easily computed as $p = 0$.

Example 2 (Weighted Maximum Auxiliary Function)**.** Consider $M = [0,1]$ with the standard metric $\rho(x,y) = |x-y|$ (so $s = 1$). Define

$$Tx = \frac{x}{5} \quad \text{for all } x \in [0,1].$$

Clearly $T : M \to M$. We use a weighted maximum auxiliary function

$$\Psi(a,b) = \frac{1}{2}\max\{a,b\}.$$

For the contractive family, set

$$\varphi_1(x,y) = 1, \quad \varphi_2(x,y) = 1, \quad \varphi_3(x,y) = 0,$$

$$\alpha_1 = \frac{1}{5}, \quad \alpha_2 = \frac{2}{5}, \quad \alpha_3 = 0, \quad \kappa = 1.$$

We verifiy the contractive condition: For any $x, y \in [0,1]$,

$$\begin{aligned}\rho(Tx,Ty) = \left|\frac{x}{5} - \frac{y}{5}\right| = \frac{1}{5}\rho(x,y) &\le \frac{1}{5}\rho(x,y) + \frac{2}{5}\cdot\frac{1}{2}\max\left\{\left|x - \frac{x}{5}\right|, \left|y - \frac{y}{5}\right|\right\} \\ &= \frac{1}{5}\rho(x,y) + \frac{1}{5}\max\left\{\frac{4|x|}{5}, \frac{4|y|}{5}\right\} \\ &= \alpha_1\varphi_1(x,y)\rho(x,y) + \alpha_2\varphi_2(x,y)\Psi(\rho(x,Tx),\rho(y,Ty)).\end{aligned}$$

Note that the inequality $|x-y| \le |x-y| + \max\{4|x|/5, 4|y|/5\}$ holds trivially since the right side is larger. Next, we verify the Definition 3:

(i) $\varphi_i(x,y) = 1 \le \kappa = 1$ for all $x, y \in M$ and $i = 1,2,3$;
(ii) $\alpha_1\kappa + 2\alpha_3\kappa = 1/5 < 1$;
(iii) $s^2\alpha_1\kappa = 1/5 < 1$ and $s^2\alpha_3\kappa = 0 < 1$.

Verification of conditions (C1)–(C3):

(C1): $\Psi(0,0) = \frac{1}{2}\max\{0,0\} = 0$.

(C2): For any $b \ge 0$ and $\epsilon > 0$, if $|a| < \delta = 2\epsilon$, then $|\Psi(a,b) - \Psi(0,b)| = |\max\{a,b\}/2 - \max\{0,b\}/2| \le |a|/2 < \epsilon$. Thus Ψ is continuous at each $(0,b)$.

(C3): Since $\alpha_2 = 2/5 > 0$, we need $\Psi(0,t) \le ct$ with $c < (1 - s\alpha_3\kappa)/(\alpha_2\kappa) = 5/2$. We have $\Psi(0,t) = \max\{0,t\}/2 = t/2 \le ct$ for any $c \ge 1/2$. Choosing $c = 1 < 5/2$ satisfies the condition.

Finally, we check asymptotic regularity: For any $x_0 \in [0,1]$, we have $x_n = T^n x_0 = x_0/5^n$, so

$$\rho(T^n x_0, T^{n+1}x_0) = \left|\frac{x_0}{5^n} - \frac{x_0}{5^{n+1}}\right| = \frac{4x_0}{5^{n+1}} \le \frac{4}{5^{n+1}} \to 0.$$

By Theorem 1, T has a unique fixed point, which is $p = 0$. This example demonstrates that auxiliary functions beyond simple sums can be accommodated within our framework, provided they satisfy the strengthened continuity and growth conditions.

6. Conclusion

We have established a unified fixed point theorem (Theorem 1) for asymptotically regular mappings in b-metric spaces through the concept of contractive families. Our contributions can be summarized as follows. First, we have achieved unification of existing results. Both main theorems of Nagac and Tas [1] (Corollaries 1 and 2) emerge as special cases of our unified framework through appropriate parameter choices, provided explicit boundedness conditions on the coefficient functions are included. This provides a more elegant and economical treatment of these results. Second, the framework offers substantial flexibility. It accommodates variable coefficient functions $\varphi_i(x,y)$ that depend on the points being compared, a general auxiliary function Ψ that encompasses various functional forms beyond those in [1] (subject to the strengthened conditions (C1)–(C3)), and the ability to combine Banach-type, Kannan-type, and Chatterjea-type conditions within a single framework. Third, our approach enables new applications through novel choices of the auxiliary function Ψ (such as weighted maxima or other functions satisfying our conditions), and the framework generates fixed point theorems not covered by existing literature when appropriate modifications are made. Fourth, unlike classical results that require continuity of the mapping T, our main theorem requires only asymptotic regularity, which is strictly weaker. This allows the theory to encompass a broader class of discontinuous mappings that arise naturally in applications. Several directions for future research emerge from this work: extension to multivalued mappings in b-metric spaces; applications to differential and integral equations with more complex structures; investigation of common fixed point results for families of mappings via contractive families; connections with partial b-metric spaces, rectangular b-metric spaces, and other generalizations; exploration of contractive families with more than three component functions.

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