Open Journal of Mathematical Sciences

Article

Generalized operators and their associated inequalities within the framework of r-Fock spaces

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Academic Editor: Waqas Nazeer
Received: 04 December 2025; Revised: 05 January 2026; Accepted: 30 January 2026; Published: 13 February 2026

Abstract: We introduce r-Fock space $\mathscr{F}_r$ which generalizes some previously known Hilbert spaces, and study the r-derivative operator $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and the multiplication operator by z^r. A general uncertainty inequality of Heisenberg-type is obtained. We also consider the extremal functions for the r-difference operator D_r on the space and obtain approximate inversion formulas.

Keywords: r-Fock spaces; uncertainty inequalities; extremal functions

MSC: 30H20; 32A15; 44A20.

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1. Introduction

In [1], Bargmann obtained the realization of the classical Fock space $\mathscr{F}$ called also Segal-Bargmann space [2] as a space of entire functions on $\mathbb{C}$, equipped with the norm

$$\|f\|_{\mathscr{F}} := \left[\frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\mathrm{d}z\right]^{1/2}.$$

For $f \in \mathscr{F}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|_{\mathscr{F}} = \left[\sum_{n=0}^{\infty} n!|a_n|^2\right]^{1/2}.$$

The ordinary Fock space also has a reproducing kernel $K_z(w) = e^{w\bar{z}}$. The space $\mathscr{F}$ was applied in many works [3–8]. Precisely, the derivative operator $\frac{\mathrm{d}}{\mathrm{d}z}$ and the multiplication operator by z are closed, densely defined operators on $\mathscr{F}$ such that $\frac{\mathrm{d}}{\mathrm{d}z}$ is the adjoint operator of z, and $\mathrm{Dom}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right) = \mathrm{Dom}(z.)$, see [1, page 210]. These operators also satisfy the commutation relation

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}, z\right] = I,$$

where I is the identity. From this relation and by an application of the general result of functional analysis [17, Proposition 2.1], Chen and Zhu [4] proved the following uncertainty principle for the Fock space $\mathscr{F}$. Let $f \in \mathscr{F}$, and let $a, b \in \mathbb{C}$, then

$$\left\|\left(\frac{\mathrm{d}}{\mathrm{d}z} + z - a\right)f\right\|_{\mathscr{F}} \left\|\left(\frac{\mathrm{d}}{\mathrm{d}z} - z - b\right)f\right\|_{\mathscr{F}} \geq \|f\|^2_{\mathscr{F}}.$$

The authors Chen and Zhu [4] also proved that this inequality is optimal. The equality holds if and only if there exist $c \in \mathbb{R}$, $c \neq -1$ and $C \in \mathbb{C}$:

$$f(z) = C\exp\left(-\frac{1-c}{2(1+c)}z^2 + \frac{a-ibc}{1+c}z\right), \quad z \in \mathbb{C}.$$

Recently in [5,6] the author of the paper studied the properties of the difference operator $Df(z) := \frac{1}{z}(f(z) - f(0))$ on the Fock space $\mathscr{F}$, by examining the theory of extremal functions in the context of a Tikhonov-regularized extremal problem.

Let r be a positive integer such that $r \geq 2$. In this work we search a realisation of r-Fock space denoted $\mathscr{F}_r$, for which we study the r-derivative operator $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$, the multiplication operator by z^r and the r-difference operator $D_r f(z) := \frac{1}{z^r}(f(z) - f(0))$.

The r-Fock space $\mathscr{F}_r$ is the Hilbert space of entire functions f on $\mathbb{C}$, r-even, such that

$$\|f\|_{\mathscr{F}_r} := \left[\frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\mathrm{d}z\right]^{1/2} < \infty.$$

For $f \in \mathscr{F}_r$ with $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$, we have

$$\|f\|_{\mathscr{F}_r} = \left[\sum_{n=0}^{\infty} (rn)!|a_n|^2\right]^{1/2}.$$

The space $\mathscr{F}_r$ is a reproducing kernel Hilbert space (RKHS) that is $\mathscr{F}_r \subset \mathscr{F}$ with

$$\|f\|_{\mathscr{F}} \leq \|f\|_{\mathscr{F}_r}.$$

The r-derivative operator $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and the multiplication operator by z^r are closed, densely defined operators on $\mathscr{F}_r$ such that $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ is the adjoint operator of z^r. These operators also satisfy the commutation relation

$$\left[\frac{\mathrm{d}^r}{\mathrm{d}z^r}, z^r\right] = r!I + E_r,$$

where E_r is the r-Euler operator. The r-Euler operator E_r yields new structural information, and its positivity ($\langle E_r f, f\rangle_{\mathscr{F}_r} \geq 0, \forall\, f \in \mathscr{F}_r$) proved the following uncertainty inequality for $f \in \mathrm{Dom}(E_r)$:

$$\left\|\left(z^r + \frac{\mathrm{d}^r}{\mathrm{d}z^r} - a\right)f\right\|_{\mathscr{F}_r} \left\|\left(z^r - \frac{\mathrm{d}^r}{\mathrm{d}z^r} - b\right)f\right\|_{\mathscr{F}_r} \geq r!\|f\|^2_{\mathscr{F}_r}, \quad a, b \in \mathbb{C}.$$

However, I do not have an answer to the question of whether this inequality is optimal or not.

Building on the ideas of Saitoh et al. [9–11], we find the minimizer (denoted by $F^*_{\lambda,D_r}(h)$) for the extremal problem:

$$\inf_{f\in\mathscr{F}_r}\left\{\lambda\|f\|^2_{\mathscr{F}_r} + \|D_r f - h\|^2_{\mathscr{F}_r}\right\},$$

where $h \in \mathscr{F}_r$ and $\lambda > 0$. We prove that the extremal function $F^*_{\lambda,D_r}(h)$ (which exists from Saitoh-type extremal formulations) is given by

$$F^*_{\lambda,D_r}(h)(z) = \langle h, \Psi_z\rangle_{\mathscr{F}_r},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z})^{rn+r} w^{rn}}{\lambda(rn+r)! + (rn)!}, \quad w \in \mathbb{C}.$$

Moreover, we establish approximate inversion formulas for the r-difference operator D_r on the r-Fock space $\mathscr{F}_r$. A pointwise approximate inversion formulas for the operator D_r are also discussed.

The paper is organized as follows. In §2 we introduce the r-Fock space $\mathscr{F}_r$; and we establish a generalized uncertainty inequality of Heisenberg-type for the space $\mathscr{F}_r$. In §3 we examine the extremal functions for the r-difference operator D_r; and we give approximate inversion formulas for the operator D_r on the r-Fock space $\mathscr{F}_r$.

2. Generalized uncertainty principle

Heisenberg [12] showed that we can not determine simultaneously the position and the momentum of a particle with an arbitrary precision. This principle has been formulated by the following inequality

$$\sigma_x \sigma_p \geq \frac{h}{4\pi},$$

where h represents Planck's constant and σ_x, σ_p denote the standard deviations of the position and the momentum of the particle respectively. There exist many similar uncertainty principles, in quantum physics and in mathematics [4,13–15]. In this section we are going to prove a generalized uncertainty principle of Heisenberg-type for the r-Fock space $\mathscr{F}_r$.

We begin by recalling some results about trigonometric functions of r-order [16].

Let ω_k, $k = 1, \ldots, r$, the r-th roots of unity

$$\omega_k = e^{2i\pi(k-1)/r}.$$

Let $z \in \mathbb{C}$. A function $f(z)$ is called r-even if

$$f(\omega_k z) = f(z), \quad k = 1, \ldots, r.$$

For example, the r-hyperbolic cosine [16] given by

$$\cosh_r(z) = \sum_{n=0}^{\infty} \frac{z^{rn}}{(rn)!}, \tag{1}$$

is r-even, entire function on $\mathbb{C}$ and satisfies $|\cosh_r(z)| \leq e^{|z|}$.

The r-Fock space $\mathscr{F}_r$ is the set of all entire functions f on $\mathbb{C}$, r-even, such that

$$\|f\|^2_{\mathscr{F}_r} := \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \mathrm{d}z < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{\mathscr{F}_r} := \frac{1}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2} \mathrm{d}x\mathrm{d}y, \quad f, g \in \mathscr{F}_r.$$

For $f, g \in \mathscr{F}_r$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^{rn}$, we have

$$\|f\|^2_{\mathscr{F}_r} = \sum_{n=0}^{\infty} (rn)! |a_n|^2,$$

and

$$\langle f, g \rangle_{\mathscr{F}_r} = \sum_{n=0}^{\infty} (rn)! a_n \overline{b_n}.$$

The set $\left\{ \frac{z^{rn}}{\sqrt{(rn)!}} \right\}_{n \in \mathbb{N}}$ forms a Hilbert's basis for the space $\mathscr{F}_r$; and each $f \in \mathscr{F}_r$ can be written as

$$f(z) = \sum_{n=0}^{\infty} \frac{\langle f, z^{rn} \rangle_{\mathscr{F}_r}}{(rn)!} z^{rn},$$

and

$$\|f\|_{\mathscr{F}_r}^2 = \sum_{n=0}^{\infty} \frac{\left|\langle f, z^{rn}\rangle_{\mathscr{F}_r}\right|^2}{(rn)!}.$$

The function $K_{r,z}$, $z \in \mathbb{C}$, given by

$$K_{r,z}(w) := \cosh_r(w\bar{z}), \quad w \in \mathbb{C},$$

is a reproducing kernel for the r-Fock space $\mathscr{F}_r$, that is for (1) we have

(i) $K_{r,z} \in \mathscr{F}_r$, $z \in \mathbb{C}$,

(ii) $\langle f, K_{r,z}\rangle_{\mathscr{F}_r} = f(z)$, $f \in \mathscr{F}_r$.

So that, for $f \in \mathscr{F}_r$ and $z \in \mathbb{C}$, we have

$$|f(z)| \le \|K_{r,z}\|_{\mathscr{F}_r}\|f\|_{\mathscr{F}_r} \le e^{|z|^2/2}\|f\|_{\mathscr{F}_r}.$$

We conclude also that the family $\{K_{r,z}, z \in \mathbb{C}\}$ is dense in the space $\mathscr{F}_r$.

Let $f \in \mathscr{F}_r$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$. From [1], we have

$$\|f\|_{\mathscr{F}}^2 = \sum_{n=0}^{\infty} n!|a_n|^2.$$

Using the inequality $n! \le (rn)!$, we obtain

$$\|f\|_{\mathscr{F}} \le \left[\sum_{n=0}^{\infty}(rn)!|a_n|^2\right]^{1/2} = \|f\|_{\mathscr{F}_r}.$$

Therefore the inclusion $\mathscr{F}_r \subset \mathscr{F}$ is continuous embedding.

In the following we consider the r-derivative operator $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and the multiplication operator by z^r. We denote by

$$\mathrm{Dom}\left(\frac{\mathrm{d}^r}{\mathrm{d}z^r}\right) := \{f \in \mathscr{F}_r : f^{(r)} \in \mathscr{F}_r\}, \quad \mathrm{Dom}(z^r.) := \{f \in \mathscr{F}_r : z^r f \in \mathscr{F}_r\},$$

their domains. Each domain contains the elements of the basis $\left\{\frac{z^{rn}}{\sqrt{(rn)!}}\right\}_{n\in\mathbb{N}}$. Thus, $\mathrm{Dom}\left(\frac{\mathrm{d}^r}{\mathrm{d}z^r}\right)$ and $\mathrm{Dom}(z^r.)$ are dense in the space $\mathscr{F}_r$.

Lemma 1. *The r-derivative operator $\frac{d^r}{dz^r}$ and the multiplication operator by z^r are closed, densely defined operators on $\mathscr{F}_r$ and satisfy the commutation rule*

$$\left[\frac{d^r}{dz^r}, z^r\right] = r!I + E_r, \tag{2}$$

where E_r is the operator given by

$$E_r f(z) = \sum_{j=1}^{r-1} \frac{(r!)^2}{(r-j)!(j!)^2} z^j f^{(j)}(z).$$

Proof. Clearly the operators $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and z^r are densely defined (the set of $\left\{\frac{z^{rn}}{\sqrt{(rn)!}}\right\}_{n\in\mathbb{N}}$ is contained in each of their domains). As in the same of [1, page 210] we prove that the operators $\frac{\mathrm{d}^r}{\mathrm{d}z^r}$ and z^r are closed. Let now $f \in \mathscr{F}_r$, we have

$$\left[\frac{\mathrm{d}^r}{\mathrm{d}z^r}, z^r\right] = \frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^r f(z)) - z^r \frac{\mathrm{d}^r}{\mathrm{d}z^r}(f(z)).$$

But

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^r f(z)) = \sum_{j=0}^{r} \frac{r!}{j!(r-j)!}(z^r)^{(j)} f^{(r-j)}(z) = \sum_{j=0}^{r} \frac{(r!)^2}{(r-j)!(j!)^2} z^j f^{(j)}(z).$$

Thus
$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^r f(z)) = r!f(z) + E_r f(z) + z^r f^{(r)}(z).$$
Therefore we obtain
$$\left[\frac{\mathrm{d}^r}{\mathrm{d}z^r}, z^r\right] f(z) = r!f(z) + E_r f(z).$$
The lemma is proved. □

We define the Hilbert space $\mathscr{U}_r^{(1)}$ as the space of all function $f \in \mathscr{F}_r$ such that
$$\|f\|_{\mathscr{U}_r^{(1)}} := \|z^r f\|_{\mathscr{F}_r} < \infty.$$
Then $\mathrm{Dom}(z^r.) = \mathscr{U}_r^{(1)}$ and if $f \in \mathscr{U}_r^{(1)}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$ we have
$$\|f\|^2_{\mathscr{U}_r^{(1)}} = \sum_{n=0}^{\infty} (rn+r)!|a_n|^2.$$
We define also the Hilbert space $\mathscr{U}_r^{(2)}$ as the space of all function $f \in \mathscr{F}_r$ such that
$$\|f\|^2_{\mathscr{U}_r^{(2)}} := r!|f(0)|^2 + \|f^{(r)}\|^2_{\mathscr{F}_r} < \infty.$$
Then $\mathrm{Dom}\left(\frac{\mathrm{d}^r}{\mathrm{d}z^r}\right) = \mathscr{U}_r^{(2)}$ and if $f \in \mathscr{U}_r^{(2)}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$ we have
$$\|f\|^2_{\mathscr{U}_r^{(2)}} = r!|a_0|^2 + \sum_{n=1}^{\infty} \frac{((rn)!)^2}{(rn-r)!}|a_n|^2.$$
It is easy to see the inequality
$$\|f\|_{\mathscr{U}_r^{(2)}} \leq \|f\|_{\mathscr{U}_r^{(1)}}.$$
Therefore, we have the continuous inclusion $\mathscr{U}_r^{(1)} \subset \mathscr{U}_r^{(2)}$.

Lemma 2. *We have $(z^r.)^* = \frac{d^r}{dz^r}$ and $\left(\frac{d^r}{dz^r}\right)^* = z^r$.*

Proof. Let $g(z) = \sum\limits_{n=0}^{\infty} b_n z^{rn}$. We put $(z^r.)^* g(z) = \sum\limits_{n=0}^{\infty} c_n z^{rn}$. Set $f(z) = z^{rn}$, then $\langle f, (z^r.)^* g\rangle_{\mathscr{F}_r} = \langle z^r f, g\rangle_{\mathscr{F}_r}$ implies that $c_n = \frac{(rn+r)!}{(rn)!} b_{n+1}$, i.e.
$$(z^r.)^* g(z) = \sum_{n=0}^{\infty} \frac{(rn+r)!}{(rn)!} b_{n+1} z^{rn} = g^{(r)}(z).$$
Hence $(z^r.)^* \subseteq \frac{\mathrm{d}^r}{\mathrm{d}z^r}$. Conversely, let $f \in \mathscr{U}_r^{(1)}$ and $g \in \mathscr{U}_r^{(2)}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^{rn}$, we have
$$z^r f(z) = \sum_{n=1}^{\infty} a_{n-1} z^{rn}, \tag{3}$$
$$g^{(r)}(z) = \sum_{n=0}^{\infty} \frac{(rn+r)!}{(rn)!} b_{n+1} z^{rn}, \tag{4}$$
and
$$\langle z^r f, g\rangle_{\mathscr{F}_r} = \sum_{n=1}^{\infty} (rn)! a_{n-1} \overline{b_n} = \sum_{n=0}^{\infty} (rn+r)! a_n \overline{b_{n+1}} = \langle f, g^{(r)}\rangle_{\mathscr{F}_r}.$$

This proves that $\frac{\mathrm{d}^r}{\mathrm{d}z^r} \subseteq (z^r.)^*$, and thus $(z^r.)^* = \frac{\mathrm{d}^r}{\mathrm{d}z^r}$. The relation $\left(\frac{\mathrm{d}^r}{\mathrm{d}z^r}\right)^* = z^r$ follows immediately. The lemma is proved. □

We define the Hilbert space $\mathscr{S}_r^{(1)}$ as the space of all function $f \in \mathscr{F}_r$ such that

$$\|f\|_{\mathscr{S}_r^{(1)}} := \|(z^r f)^{(r)}\|_{\mathscr{F}_r} < \infty.$$

Then $\mathrm{Dom}\left(\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^r.)\right) = \mathscr{S}_r^{(1)}$ and if $f \in \mathscr{S}_r^{(1)}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$, from (3) and (4) we have

$$(z^r f)^{(r)}(z) = \sum_{n=0}^{\infty} \frac{(rn+r)!}{(rn)!} a_n z^{rn}.$$

Thus

$$\|f\|^2_{\mathscr{S}_r^{(1)}} = \sum_{n=0}^{\infty} \frac{((rn+r)!)^2}{(rn)!} |a_n|^2.$$

We define also the Hilbert space $\mathscr{S}_r^{(2)}$ as the space of all function $f \in \mathscr{F}_r$ such that

$$\|f\|^2_{\mathscr{S}_r^{(2)}} := (r!)^2 |f(0)|^2 + \|z^r f^{(r)}\|^2_{\mathscr{F}_r} < \infty.$$

Then $\mathrm{Dom}\left(z^r \frac{\mathrm{d}^r}{\mathrm{d}z^r}\right) = \mathscr{S}_r^{(2)}$ and if $f \in \mathscr{S}_r^{(2)}$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$, we have

$$z^r f^{(r)}(z) = \sum_{n=1}^{\infty} \frac{(rn)!}{(rn-r)!} a_n z^{rn}.$$

Thus

$$\|f\|^2_{\mathscr{S}_r^{(2)}} := (r!)^2 |a_0|^2 + \sum_{n=1}^{\infty} \frac{((rn)!)^3}{((rn-r)!)^2} |a_n|^2 < \infty.$$

It is easy to see the inequalities

$$\|f\|_{\mathscr{S}_r^{(2)}} \leq \|f\|_{\mathscr{S}_r^{(1)}}, \quad \|f\|_{\mathscr{U}_r^{(1)}} \leq \frac{1}{\sqrt{r!}} \|f\|_{\mathscr{S}_r^{(1)}}.$$

Therefore, we have the continuous inclusions $\mathscr{S}_r^{(1)} \subset \mathscr{S}_r^{(2)}$ and $\mathscr{S}_r^{(1)} \subset \mathscr{U}_r^{(1)}$.

Lemma 3. *We have* $\mathrm{Dom}\left(\left[\frac{d^r}{dz^r}, z^r\right]\right) = \mathscr{S}_r^{(1)}$.

Lemma 4. (See [17, *Proposition 2.1*]). *Let* A *and* B *be self-adjoint operators on a Hilbert space* $\mathscr{H}$, *then*

$$\|(A-a)f\|_{\mathscr{H}} \|(B-b)f\|_{\mathscr{H}} \geq \frac{1}{2} \left| \langle [A,B]f, f \rangle_{\mathscr{H}} \right|,$$

for all $f \in \mathrm{Dom}([A,B])$ *and all* $a, b \in \mathbb{C}$.

Theorem 1. *Let* $f \in \mathscr{S}_r^{(1)}$. *For all* $a, b \in \mathbb{C}$, *we have*

$$\|(z^r + \frac{d^r}{dz^r} - a)f\|_{\mathscr{F}_r} \|(z^r - \frac{d^r}{dz^r} - b)f\|_{\mathscr{F}_r} \geq r! \|f\|^2_{\mathscr{F}_r}. \tag{5}$$

Proof. Let $f \in \mathscr{S}_r^{(1)}$. Now, let A and B be the operators defined for $f \in \mathscr{S}_r^{(1)}$ by

$$Af(z) := \left(z^r + \frac{\mathrm{d}^r}{\mathrm{d}z^r}\right) f(z), \quad Bf(z) := i \left(z^r - \frac{\mathrm{d}^r}{\mathrm{d}z^r}\right) f(z).$$

By (2), Lemmas 2 and 3, the operators A and B possess the following properties.

(i) $A^* = A$ and $B^* = B$,
(ii) $[A, B] = -2i \left[z^r, \frac{\mathrm{d}^r}{\mathrm{d}z^r}\right] = 2i(r!I + E_r)$,
(iii) $\mathrm{Dom}([A, B]) = \mathscr{S}_r^{(1)}$.
On the other hand, let $f(z) = \sum_{n=0}^{\infty} a_n z^{rn}$, then

$$E_r f(z) = \sum_{n=0}^{\infty} \alpha_n(r) a_n z^{rn},$$

with

$$\alpha_n(r) = \sum_{j=1}^{r-1} \frac{(rn)!(r!)^2}{(r-j)!(j!)^2(rn-j)!} \geq 0.$$

Thus, the inequality (5) follows from Lemma 4 and the fact that

$$\langle E_r f, f \rangle_{\mathscr{F}_r} = \sum_{n=0}^{\infty} \alpha_n(r)|a_n|^2 \geq 0.$$

This completes the proof of the theorem. □

Remark 1. I do not have an answer to the question of whether this inequality is optimal or not.

3. Approximate inversion formulas

Tikhonov regularization in statistics is the method of ridge regression. In general, this method related to the Levenberg-Marquardt algorithm for solving nonlinear least squares problems. Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations [18,19].

Let $\mathscr{H}$ be a Hilbert space, and let $T : \mathscr{F}_r \to \mathscr{H}$ be a bounded linear operator from $\mathscr{F}_r$ into $\mathscr{H}$. Let $\lambda > 0$. We denote by $\langle ., . \rangle_{\lambda, \mathscr{F}_r}$ the inner product defined on the space $\mathscr{F}_r$ by

$$\langle f, g \rangle_{\lambda, \mathscr{F}_r} := \lambda \langle f, g \rangle_{\mathscr{F}_r} + \langle Tf, Tg \rangle_{\mathscr{H}}.$$

The two norms $\|.\|_{\mathscr{F}_r}$ and $\|.\|_{\lambda, \mathscr{F}_r}$ are equivalent. In particular, we have

$$|f(z)| \leq \frac{e^{|z|^2/2}}{\sqrt{\lambda}} \|f\|_{\lambda, \mathscr{F}_r}, \quad f \in \mathscr{F}_r, z \in \mathbb{C}.$$

Then the space $\mathscr{F}_r$, equipped with the norm $\|.\|_{\lambda, \mathscr{F}_r}$ has a reproducing kernel $K_{\lambda,r,z}$. Therefore, we have the functional equation

$$(\lambda I + T^*T) K_{\lambda,r,z} = K_{r,z}, \quad z \in \mathbb{C}, \tag{6}$$

where I is the unit operator and $T^* : \mathscr{H} \longrightarrow \mathscr{F}_r$ is the adjoint of T.

For any $h \in \mathscr{H}$ and for any $\lambda > 0$, we define the extremal function $F^*_{\lambda,T}(h)$ by

$$F^*_{\lambda,T}(h)(z) = \langle h, TK_{\lambda,r,z} \rangle_{\mathscr{H}}, \quad z \in \mathbb{C}.$$

Then by (6) we deduce that

$$\begin{aligned} F^*_{\lambda,T}(h)(z) &= \langle T^*h, K_{\lambda,r,z} \rangle_{\mathscr{F}_r} \\ &= \langle T^*h, (\lambda I + T^*T)^{-1} K_{r,z} \rangle_{\mathscr{F}_r} \\ &= \langle (\lambda I + T^*T)^{-1} T^*h, K_{r,z} \rangle_{\mathscr{F}_r}. \end{aligned}$$

Hence

$$F^*_{\lambda,T}(h)(z) = (\lambda I + T^*T)^{-1} T^*h(z), \quad z \in \mathbb{C}. \tag{7}$$

The extremal function $F^*_{\lambda,T}(h)$ is the unique solution [9, Theorem 2.5, Section 2] of the Tikhonov regularization problem

$$\inf_{f\in\mathscr{F}_r}\Big\{\lambda\|f\|^2_{\mathscr{F}_r}+\|Tf-h\|^2_{\mathscr{H}}\Big\}.$$

Let D_r be the r-difference operator defined for $f\in\mathscr{F}_r$ by

$$D_rf(z):=\frac{1}{z^r}(f(z)-f(0)).$$

The r-difference operator D_r is also studied in [20–23]. For $f\in\mathscr{F}_r$ with $f(z)=\sum\limits_{n=0}^{\infty}a_nz^{rn}$ we have

$$D_rf(z):=\sum_{n=0}^{\infty}a_{n+1}z^{rn}. \tag{8}$$

We emphasize the non-invertibility due to the constants: $D_rf=0$ if and only if f is constant, and the inversion is meaningful only on $\{h:h(0)=0\}$ or modulo the constants.

Lemma 5. *The operator D_r maps continuously from $\mathscr{F}_r$ into $\mathscr{F}_r$, and*

$$\|D_rf\|_{\mathscr{F}_r}\le\frac{1}{\sqrt{r!}}\|f\|_{\mathscr{F}_r}\quad\text{and}\quad\|D_r\|=\frac{1}{\sqrt{r!}}.$$

Proof. Let $f\in\mathscr{F}_r$ with $f(z)=\sum\limits_{n=0}^{\infty}a_nz^{rn}$. We have

$$\|D_rf\|^2_{\mathscr{F}_r}=\sum_{n=0}^{\infty}(rn)!|a_{n+1}|^2=\sum_{n=1}^{\infty}\frac{(rn)!|a_n|^2}{(rn)(rn-1)\dots(rn-r+1)}.$$

Since

$$\sup_{n\ge1}\left[\frac{1}{(rn)(rn-1)\dots(rn-r+1)}\right]=\frac{1}{r!}.$$

Then

$$\|D_rf\|_{\mathscr{F}_r}\le\frac{1}{\sqrt{r!}}\|f\|_{\mathscr{F}_r}\quad\text{and}\quad\|D_r\|=\frac{1}{\sqrt{r!}}.$$

The lemma is proved. □

Building on the ideas of Saitoh [9–11] we examine the extremal function associated with the r-difference operator D_r.

Theorem 2. (i) *For $f\in\mathscr{F}_r$ with $f(z)=\sum\limits_{n=0}^{\infty}a_nz^{rn}$, we have*

$$D_r^*f(z)=\sum_{n=1}^{\infty}\frac{(rn-r)!}{(rn)!}a_{n-1}z^{rn},\quad D_r^*D_rf(z)=\sum_{n=1}^{\infty}\frac{(rn-r)!}{(rn)!}a_nz^{rn}.$$

(ii) *For any $h\in\mathscr{F}_r$ and for any $\lambda>0$, the problem*

$$\inf_{f\in\mathscr{F}_r}\Big\{\lambda\|f\|^2_{\mathscr{F}_r}+\|D_rf-h\|^2_{\mathscr{F}_r}\Big\}, \tag{9}$$

has a unique extremal function given by

$$F^*_{\lambda,D_r}(h)(z)=\langle h,\Psi_z\rangle_{\mathscr{F}_r},$$

where

$$\Psi_z(w)=\sum_{n=0}^{\infty}\frac{(\overline{z})^{rn+r}w^{rn}}{\lambda(rn+r)!+(rn)!},\quad w\in\mathbb{C}.$$

Proof. (i) If $f, g \in \mathscr{F}_r$ with $f(z) = \sum\limits_{n=0}^{\infty} a_n z^{rn}$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^{rn}$, then

$$\langle D_r f, g\rangle_{\mathscr{F}_r} = \sum_{n=0}^{\infty}(rn)!a_{n+1}\overline{b_n} = \sum_{n=1}^{\infty}(rn-r)!a_n\overline{b_{n-1}} = \langle f, D_r^* g\rangle_{\mathscr{F}_r},$$

where

$$D_r^* g(z) = \sum_{n=1}^{\infty}\frac{(rn-r)!}{(rn)!}b_{n-1}z^{rn}.$$

And therefore

$$D_r^* D_r f(z) = \sum_{n=1}^{\infty}\frac{(rn-r)!}{(rn)!}a_n z^{rn}.$$

(ii) We put $h(z) = \sum\limits_{n=0}^{\infty} h_n z^{rn}$ and $F^*_{\lambda,D_r}(h)(z) = \sum\limits_{n=0}^{\infty} f_n z^{rn}$. From (7) we have $(\lambda I + D_r^* D_r)F^*_{\lambda,D_r}(h)(z) = D_r^* h(z)$. By (i) we deduce that

$$f_0 = 0, \quad f_n = \frac{(rn-r)!h_{n-1}}{\lambda(rn)! + (rn-r)!}, \quad n \geq 1.$$

Thus

$$F^*_{\lambda,D_r}(h)(z) = \sum_{n=0}^{\infty}\frac{(rn)!h_n}{\lambda(rn+r)! + (rn)!}z^{rn+r} = \langle h, \Psi_z\rangle_{\mathscr{F}_r}, \tag{10}$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty}\frac{(\overline{z})^{rn+r}w^{rn}}{\lambda(rn+r)! + (rn)!}, \quad w \in \mathbb{C}.$$

The theorem is proved. □

In the next part of this section we establish the estimate properties of the extremal function $F^*_{\lambda,D_r}(h)(z)$, and we deduce approximate inversion formulas for the r-difference operator D_r. These formulas are the analogous of Calderón's reproducing formulas for the Fourier type transforms [24,25]. A pointwise approximate inversion formulas for the operator D_r are also discussed.

The extremal function $F^*_{\lambda,D_r}(h)$ given by (10) satisfies the following properties.

Lemma 6. *If $\lambda > 0$ and $h \in \mathscr{F}_r$, then*

(i) $|F^*_{\lambda,D_r}(h)(z)| \leq \dfrac{e^{|z|^2/2}}{\sqrt{4\lambda}}\|h\|_{\mathscr{F}_r}$,

(ii) $|D_r F^*_{\lambda,D_r}(h)(z)| \leq \dfrac{e^{|z|^2/2}}{\sqrt{4\lambda r!}}\|h\|_{\mathscr{F}_r}$,

(iii) $\|F^*_{\lambda,D_r}(h)\|_{\mathscr{F}_r} \leq \dfrac{1}{\sqrt{4\lambda}}\|h\|_{\mathscr{F}_r}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_r$ with $h(z) = \sum\limits_{n=0}^{\infty} h_n z^{rn}$. From (10) we have

$$|F^*_{\lambda,D_r}(h)(z)| \leq \|\Psi_z\|_{\mathscr{F}_r}\|h\|_{\mathscr{F}_r}.$$

Using the fact that $(x+y)^2 \geq 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{F}_r} = \sum_{n=0}^{\infty}(rn)!\left[\frac{|z|^{r(n+1)}}{\lambda(rn+r)! + (rn)!}\right]^2 \leq \frac{1}{4\lambda}\sum_{n=0}^{\infty}\frac{|z|^{2r(n+1)}}{(rn+r)!} \leq \frac{1}{4\lambda}\cosh_r(|z|^2).$$

This gives (i).

On the other hand, from (8) and (10) we have

$$D_r F^*_{\lambda,D_r}(h)(z) = \sum_{n=0}^{\infty} \frac{(rn)!h_n}{\lambda(rn+r)!+(rn)!} z^{rn} = \langle h, \Phi_z \rangle_{\mathscr{F}_r}, \tag{11}$$

where

$$\Phi_z(w) = \sum_{n=0}^{\infty} \frac{(w\bar{z})^{rn}}{\lambda(rn+r)!+(rn)!}.$$

Then

$$|D_r F^*_{\lambda,D_r}(h)(z)| \le \|\Phi_z\|_{\mathscr{F}_r} \|h\|_{\mathscr{F}_r}.$$

But

$$\|\Phi_z\|^2_{\mathscr{F}_r} = \sum_{n=0}^{\infty} (rn)! \left[\frac{|z|^{rn}}{\lambda(rn+r)!+(rn)!}\right]^2 \le \frac{1}{4\lambda} \sum_{n=0}^{\infty} \frac{|z|^{2rn}}{(rn+r)!} \le \frac{1}{4\lambda r!} \cosh_r(|z|^2).$$

This gives (ii).

Finally, from (10) we have

$$\|F^*_{\lambda,D_r}(h)\|^2_{\mathscr{F}_r} = \sum_{n=1}^{\infty} (rn)! \left[\frac{(rn-r)!|h_{n-1}|}{\lambda(rn)!+(rn-r)!}\right]^2.$$

Then we obtain

$$\|F^*_{\lambda,D_r}(h)\|^2_{\mathscr{F}_r} \le \frac{1}{4\lambda} \sum_{n=1}^{\infty} (rn-r)!|h_{n-1}|^2 = \frac{1}{4\lambda} \|h\|^2_{\mathscr{F}_r},$$

which gives (iii) and completes the proof of the lemma. □

We establish approximate inversion formulas for the operator D_r.

Theorem 3. *If $\lambda > 0$ and $h \in \mathscr{F}_r$, then*

(i) $\lim_{\lambda \to 0^+} \|D_r F^*_{\lambda,D_r}(h) - h\|_{\mathscr{F}_r} = 0$,

(ii) $\lim_{\lambda \to 0^+} \|F^*_{\lambda,D_r}(D_r h) - h_0\|_{\mathscr{F}_r} = 0$, *where $h_0(z) = h(z) - h(0)$.*

Proof. Let $\lambda > 0$ and $h \in \mathscr{F}_r$ with $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$. From (11) we have

$$D_r F^*_{\lambda,D_r}(h)(z) - h(z) = \sum_{n=0}^{\infty} \frac{-\lambda(rn+r)!h_n}{\lambda(rn+r)!+(rn)!} z^{rn}. \tag{12}$$

Therefore

$$\|D_r F^*_{\lambda,D_r}(h) - h\|^2_{\mathscr{F}_r} = \sum_{n=0}^{\infty} (rn)! \left[\frac{\lambda(rn+r)!|h_n|}{\lambda(rn+r)!+(rn)!}\right]^2.$$

Again, by dominated convergence theorem and the fact that

$$(rn)! \left[\frac{\lambda(rn+r)!|h_n|}{\lambda(rn+r)!+(rn)!}\right]^2 \le (rn)!|h_n|^2,$$

we deduce (i).

Finally, from (8) and (10) we have

$$F^*_{\lambda,D_r}(D_r h)(z) - h_0(z) = \sum_{n=1}^{\infty} \frac{-\lambda(rn)!h_n}{\lambda(rn)!+(rn-r)!} z^{rn}. \tag{13}$$

So, one has

$$\|F^*_{\lambda,D_r}(D_r h) - h_0\|^2_{\mathscr{F}_r} = \sum_{n=1}^{\infty} (rn)! \left[\frac{\lambda(rn)!|h_n|}{\lambda(rn)!+(rn-r)!}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$(rn)!\left[\frac{\lambda(rn)!|h_n|}{\lambda(rn)!+(rn-r)!}\right]^2 \le (rn)!|h_n|^2,$$

we deduce (ii). □

We deduce also pointwise approximate inversion formulas for the r-difference operator D_r.

Theorem 4. *If $\lambda > 0$ and $h \in \mathscr{F}_r$, then*
(i) $\lim_{\lambda\to 0^+} D_r F^*_{\lambda,D_r}(h)(z) = h(z)$,
(ii) $\lim_{\lambda\to 0^+} F^*_{\lambda,D_r}(D_r h)(z) = h_0(z) = h(z) - h(0)$.

Proof. Let $h \in \mathscr{F}_r$ with $h(z) = \sum_{n=0}^{\infty} h_n z^{rn}$. From (12) and (13), by using the dominated convergence theorem and the fact that

$$\frac{\lambda(rn+r)!|h_n|}{\lambda(rn+r)!+(rn)!}|z|^{rn}, \frac{\lambda(rn)!|h_n|}{\lambda(rn)!+(rn-r)!}|z|^{rn} \le |h_n||z|^{rn},$$

we obtain (i) and (ii). □

Remark 2. Let $h \in \mathscr{F}_r$, in the limit case $\lambda \to 0^+$, the problem (9) reduces to the Tikhonov problem

$$\inf_{f\in\mathscr{F}_r}\left\{\|D_r f - h\|^2_{\mathscr{F}_r}\right\},$$

and it's extremal function is defined as

$$F^*_{0,D_r}(h)(z) = \lim_{\lambda\to 0} F^*_{\lambda,D_r}(h)(z), \quad z \in \mathbb{C}.$$

And from Theorem 4(i), for $h \in \mathscr{F}_r$, we obtain

$$D_r F^*_{0,D_r}(h)(z) = \lim_{\lambda\to 0^+} D_r F^*_{\lambda,D_r}(h)(z) = h(z).$$

On the other hand, from Theorem 4(ii), for $h \in \mathscr{F}_r$, we obtain

$$F^*_{0,D_r}(D_r h)(z) = \lim_{\lambda\to 0^+} F^*_{\lambda,D_r}(D_r h)(z) = h_0(z) = h(z) - h(0).$$

We emphasize the pointwise inversion formula is meaningful only on $\{h : h(0) = 0\}$ or modulo the constants.

Conflicts of Interest: The author declares no competing interests.

Data Availability: There is no data used in this manuscript.

Funding Information: This research did not receive funding.

Acknowledgments: The author is very grateful to the referee and editor for many critical comments on this paper.

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