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Ideal convergence topology on function spaces

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Abstract: We study a topology on the space of continuous functions between metric spaces, constructed from an admissible ideal on the natural numbers and a fixed dense sampling of the domain. Two functions are declared close whenever their pointwise deviations at the sampling points are negligible with respect to the ideal, and the resulting topology is defined via a canonical uniform structure. We show that under a compatibility condition linking the sampling sequence to the ideal, this topology is Hausdorff and, in fact, coincides with the classical topology of uniform convergence; without this compatibility condition, the topology can be strictly coarser than the uniform topology and can fail to be Hausdorff. We characterise ordinary and ideal convergence of function sequences with respect to this topology, showing that pointwise, uniform, and topological ideal convergence collapse to a two-level hierarchy once compatibility is assumed. We also show that the resulting function space is pseudometrisable in general and metrisable under compatibility, and consequently is automatically Fréchet–Urysohn and strictly Fréchet–Urysohn whenever compatibility holds, independently of any covering property of the domain.

Keywords: ideal convergence, function spaces, ideal convergence topology, Fréchet–Urysohn property

MSC: 54C35, 54E15, 40A35, 54D55.

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1. Introduction and preliminaries

The space $C(X, Y)$ of continuous functions between metric spaces carries the classical hierarchy $\tau_p \subseteq \tau_k \subseteq \tau_u$, consisting of the pointwise, compact-open, and uniform topologies. Research has long sought to populate the gaps in this hierarchy with topologies encoding more nuanced convergence conditions. On one side, Clapp and Shiflett [1] introduced a Cauchy-based topology between τ_k and τ_u, later studied by Peng and Sun [2] and Mishra and Bhaumik [3]. On the other side, ideal convergence, introduced by Kostyrko, Šalát and Wilczyński [4] as a unified framework encompassing ordinary convergence and statistical convergence (Fast [5], Steinhaus [6]), has been extended to topological and function spaces by Di Maio and Kočinac [7], Lahiri and Das [8], Megaritis [9], Georgiou, Prinos and Sereti [10], Zhou [11], and Zhong and Tang [12]. Statistical continuity for functions into uniform spaces is studied in the companion paper [13].

We briefly recall the ideal-theoretic background. A family $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an *ideal* on $\mathbb{N}$ if it is hereditary ($B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and closed under finite unions. It is *proper* if $\mathbb{N} \notin \mathcal{I}$, and *admissible* if it is proper and contains all finite subsets of $\mathbb{N}$. The *dual filter* is

$$\mathcal{F}(\mathcal{I}) = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}.$$

The two canonical examples are

$$\mathcal{I}_{\text{fin}} = \{A \subseteq \mathbb{N} : |A| < \infty\},$$

and

$$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \delta(A) = 0\},$$

where $\delta(A) = \lim_{n\to\infty} |A \cap \{1, \ldots, n\}|/n$ whenever the limit exists.

A sequence (z_n) in a topological space Z is said to *$\mathcal{I}$-converge* to z, written $z_n \xrightarrow{\mathcal{I}} z$, if $\{n : z_n \notin U\} \in \mathcal{I}$ for every open neighbourhood U of z [4]. For $\mathcal{I} = \mathcal{I}_{\text{fin}}$ this reduces to ordinary convergence, while for $\mathcal{I} = \mathcal{I}_d$ it becomes statistical convergence [5,6].

The present paper introduces a topology on $C(X,Y)$ that brings together the theory of function space topologies and ideal convergence. Given a dense sequence (x_n) in X and an admissible ideal $\mathcal{I}$ such that $((x_n),\mathcal{I})$ is compatible (Definition 1 below), we equip $C(X,Y)$ with the uniform structure generated by

$$D_\varepsilon(f,g) = \{n \in \mathbb{N} : p(f(x_n),g(x_n)) \geq \varepsilon\}, \qquad E_\varepsilon = \big\{(f,g) : D_\varepsilon(f,g) \in \mathcal{I}\big\}.$$

Recall that a *uniform structure* on a set Z is a filter $\mathcal{U}$ on $Z \times Z$ whose members (*entourages*) each contain the diagonal Δ_Z, are symmetric, and satisfy the triangle axiom (for every $E \in \mathcal{U}$ there exists $F \in \mathcal{U}$ with $F \circ F \subseteq E$). The induced topology has neighbourhood base $\{E[z] : E \in \mathcal{U}\}$ at z, and the uniform structure is *Hausdorff* whenever $\bigcap_{E\in\mathcal{U}} E = \Delta_Z$.

Under compatibility, the resulting topology $\tau_{\mathcal{I}}$ satisfies

$$\tau_p \subseteq \tau_{\mathcal{I}} = \tau_u.$$

Without compatibility, $\tau_{\mathcal{I}}$ may be strictly coarser than τ_u and may fail to be Hausdorff, and the assignment $\mathcal{I} \mapsto \tau_{\mathcal{I}}$ is order-reversing (non-strictly) on the lattice of admissible ideals. A key structural fact, established in §2, is that once the compatibility condition below is imposed the right-hand inclusion is in fact an *equality*: $\tau_{\mathcal{I}} = \tau_u$. Thus compatibility, which is needed to make $\tau_{\mathcal{I}}$ Hausdorff, simultaneously collapses it onto the uniform topology; without compatibility $\tau_{\mathcal{I}}$ can be a genuinely new, strictly coarser, and possibly non-Hausdorff topology. The compact-open topology τ_k does not enter the construction directly, and the precise relationship between $\tau_{\mathcal{I}}$ and τ_k is left for future work.

A distinguishing feature of the construction is the convergence structure it induces. Three notions of ideal convergence arise on $C(X,Y)$: pointwise $\mathcal{I}$-convergence, uniform $\mathcal{I}$-convergence, and $\mathcal{I}$-convergence in $\tau_{\mathcal{I}}$. The first controls each value $f_k(x)$ separately, the second controls the uniform deviation, and the third controls which functions f_k lie outside neighbourhoods of f in $\tau_{\mathcal{I}}$. The latter is expressed by the condition

$$\{k \in \mathbb{N} : D_\varepsilon(f_k,f) \notin \mathcal{I}\} \in \mathcal{I},$$

for every $\varepsilon > 0$. Under compatibility, this notion coincides with $\mathcal{I}$-uniform convergence, since $\tau_{\mathcal{I}} = \tau_u$, while both imply $\mathcal{I}$-pointwise convergence. We show that, under compatibility, the second and third notions coincide exactly (since $\tau_{\mathcal{I}} = \tau_u$), while the implication from either of these to the first remains a genuine, generally non-reversible, implication.

A key structural ingredient is the compatibility between the sampling sequence and the ideal.

Definition 1. The pair $((x_n),\mathcal{I})$ is *compatible* if for every non-empty open set $U \subseteq X$, the index set $\{n : x_n \in U\}$ does not belong to $\mathcal{I}$.

For $\mathcal{I}_{\text{fin}}$, compatibility means that every non-empty open set is visited infinitely often by the sampling sequence. It also holds for $\mathcal{I}_d$ whenever (x_n) visits every open ball with positive asymptotic density, as happens for equidistributed dense sequences. This condition is used in two essential ways: it guarantees that the induced uniform structure is Hausdorff, and it ensures that sampling points can always be found near a prescribed point outside any $\mathcal{I}$-negligible exceptional set.

Throughout the paper, both the domain and codomain are assumed to be metric spaces with at least two distinct points. We develop the topology $\tau_{\mathcal{I}}$, establish its basic structural properties, and investigate ordinary and ideal convergence in this setting. We also study metrisability and the Fréchet–Urysohn property, and conclude with directions for future work.

2. The ideal convergence topology

This section introduces a uniform structure on $C(X,Y)$ induced by ideal convergence along a fixed dense sequence, and studies its relation with the classical pointwise and uniform topologies.

Definition 2. Let (X, d) be a separable metric space with a fixed dense sequence (x_n), let (Y, p) be a metric space, and let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. For $\varepsilon > 0$ and $f, g \in C(X, Y)$ set

$$D_\varepsilon(f, g) \;=\; \{n \in \mathbb{N} : p(f(x_n), g(x_n)) \geq \varepsilon\},$$

and define the *ε-entourage*

$$E_\varepsilon \;=\; \{(f, g) \in C(X, Y)^2 : D_\varepsilon(f, g) \in \mathcal{I}\}.$$

Proposition 1. *The family* $\{E_\varepsilon : \varepsilon > 0\}$ *is a base for a uniform structure* $\mathcal{U}_\mathcal{I}$ *on* $C(X, Y)$*. If in addition* $((x_n), \mathcal{I})$ *is compatible, then* $\mathcal{U}_\mathcal{I}$ *is Hausdorff.*

Proof. (U1) For every f, $D_\varepsilon(f, f) = \varnothing \in \mathcal{I}$.

(U2) Symmetry of p gives $E_\varepsilon^{-1} = E_\varepsilon$.

(U3) Let $A = D_{\varepsilon/2}(f, g) \in \mathcal{I}$ and $B = D_{\varepsilon/2}(g, h) \in \mathcal{I}$. For $n \notin A \cup B$ the triangle inequality gives $p(f(x_n), h(x_n)) < \varepsilon$, so $D_\varepsilon(f, h) \subseteq A \cup B \in \mathcal{I}$ and $E_{\varepsilon/2} \circ E_{\varepsilon/2} \subseteq E_\varepsilon$.

Hausdorff property: Assume $((x_n), \mathcal{I})$ is compatible and suppose $(f, g) \in E_\varepsilon$ for every $\varepsilon > 0$; we show $f = g$. Fix $x \in X$ and $\ell \in \mathbb{N}$. By compatibility, $S_\ell = \{n : x_n \in B(x, 1/\ell)\} \notin \mathcal{I}$. Since $D_{1/\ell}(f, g) \in \mathcal{I}$, heredity gives some $n_\ell \in S_\ell \setminus D_{1/\ell}(f, g)$. Then $x_{n_\ell} \to x$ and $p(f(x_{n_\ell}), g(x_{n_\ell})) < 1/\ell \to 0$, so continuity of f and g yields $p(f(x), g(x)) = 0$. Since x was arbitrary, $f = g$. □

The Hausdorff argument above already extracts, for every $x \in X$, a sample point at which the deviation between two E_ε-close functions is controlled. The same mechanism forces control of the deviation *at every point of* X, not merely in the limit $\varepsilon \to 0$, and this has a consequence for the entire uniformity that we record immediately.

Theorem 1. *Let*

$$U_\varepsilon = \left\{(f, g) \in C(X, Y)^2 : \sup_{x \in X} p(f(x), g(x)) < \varepsilon\right\},$$

for $\varepsilon > 0$*, denote the basic entourages of the uniform convergence structure* $\mathcal{U}_u$ *on* $C(X, Y)$*. Then:*

(i) *For every admissible ideal* $\mathcal{I}$*,* $\tau_\mathcal{I} \subseteq \tau_u$*.*
(ii) *If* $((x_n), \mathcal{I})$ *is compatible, then* $\tau_\mathcal{I} = \tau_u$*.*

Proof. (i) Let $\varepsilon > 0$ and $(f, g) \in U_\varepsilon$. Then

$$D_\varepsilon(f, g) = \{n \in \mathbb{N} : p(f(x_n), g(x_n)) \geq \varepsilon\} = \varnothing \in \mathcal{I}.$$

Hence $U_\varepsilon \subseteq E_\varepsilon$, and therefore $\tau_\mathcal{I} \subseteq \tau_u$.

(ii) Assume that $((x_n), \mathcal{I})$ is compatible and let $(f, g) \in E_\varepsilon$. Set $A = D_\varepsilon(f, g) \in \mathcal{I}$. Fix $x \in X$. For each $j \in \mathbb{N}$, put $N_j = \{n \in \mathbb{N} : x_n \in B(x, 1/j)\}$. By compatibility, $N_j \notin \mathcal{I}$, and hence $N_j \setminus A \neq \varnothing$. Choose $n_j \in N_j \setminus A$. Then $x_{n_j} \to x$ and $p(f(x_{n_j}), g(x_{n_j})) < \varepsilon$. By continuity of f and g,

$$p(f(x), g(x)) \leq p(f(x), f(x_{n_j})) + p(f(x_{n_j}), g(x_{n_j})) + p(g(x_{n_j}), g(x)).$$

Letting $j \to \infty$ gives $p(f(x), g(x)) \leq \varepsilon$. Thus $(f, g) \in U_{2\varepsilon}$, so $E_\varepsilon \subseteq U_{2\varepsilon}$. Consequently, $U_\varepsilon \subseteq E_\varepsilon \subseteq U_{2\varepsilon}$, and the two families generate the same uniformity. Hence $\tau_\mathcal{I} = \tau_u$. □

By Theorem 1, the compatibility condition guarantees that $\tau_\mathcal{I} = \tau_u$. Without compatibility, this equality may fail; in particular, $\tau_\mathcal{I}$ may be strictly coarser than τ_u, and $\mathcal{U}_\mathcal{I}$ need not be Hausdorff.

Example 1. Compatibility cannot be dropped from Theorem 1(ii). Let $X = [0, 1]$, $Y = \mathbb{R}$, and $\mathcal{I} = \mathcal{I}_d$. Fix any dense sequence (y_n) in $[0, 1]$ that visits $(3/4, 1)$ infinitely often (e.g. an enumeration of $\mathbb{Q} \cap [0, 1]$), and let $A = \{n : y_n \in (3/4, 1)\}$, $B = \mathbb{N} \setminus A$, both infinite. Choose a density-zero set $S = \{2^m : m \geq 1\} \subset \mathbb{N}$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ carrying S onto A and $\mathbb{N} \setminus S$ onto B (each order-preservingly), and set $x_n = y_{\sigma(n)}$. As a reindexing of (y_n), (x_n) enumerates the same dense set of values, so (x_n) is dense in X; but $\{n : x_n \in$

$(3/4,1)\} = S \in \mathcal{I}_d$, while $(3/4,1)$ is a non-empty open subset of X, so $((x_n),\mathcal{I}_d)$ is *not* compatible. Now let $f \equiv 0$ and $g_k(x) = \min(1, k \cdot \max(0, x - 3/4))$. Since $\{n : x_n \in (3/4,1)\} = S$ has density 0, $D_\varepsilon(g_k, f) \subseteq S \in \mathcal{I}_d$ for every ε, k, so $g_k \in E_\varepsilon[f]$ for all k, ε: no entourage separates f from any g_k, so $\mathcal{U}_\mathcal{I}$ is not Hausdorff. Yet $g_k \not\to f$ in τ_u, since $\sup_x |g_k(x) - f(x)| = 1$ for every k. Hence $\tau_\mathcal{I} \subsetneq \tau_u$ here.

Definition 3. The *ideal convergence topology* $\tau_\mathcal{I}$ is the topology on $C(X,Y)$ induced by $\mathcal{U}_\mathcal{I}$. We write $C_\mathcal{I}(X,Y)$ for the resulting uniform space. A neighbourhood base at f is $\{E_\varepsilon[f] : \varepsilon > 0\}$, where

$$E_\varepsilon[f] \;=\; \{g \in C(X,Y) : D_\varepsilon(f,g) \in \mathcal{I}\}.$$

The topology $\tau_\mathcal{I}$ depends on both the ideal $\mathcal{I}$ and the chosen dense sequence (x_n). For a fixed ideal $\mathcal{I}$, different choices of dense sequences may in general produce topologies that are not comparable. In this paper, the sequence (x_n) is fixed once and for all, and whenever this dependence becomes relevant we write $\tau_{\mathcal{I},(x_n)}$ to make the role of the sequence explicit.

The evaluation map $\Phi : C(X,Y) \to Y^\mathbb{N}$, defined by $\Phi(f) = (f(x_n))$, embeds $C_\mathcal{I}(X,Y)$ isomorphically into the uniform space $(Y^\mathbb{N}, \mathcal{U}^\mathcal{I}_{Y^\mathbb{N}})$, where the uniform structure is generated by

$$F^\mathcal{I}_\varepsilon = \{(s,t) : \{n \in \mathbb{N} : p(s_n, t_n) \geq \varepsilon\} \in \mathcal{I}\}.$$

This embedding is injective by the density of (x_n) and the Hausdorff property of Y. By Theorem 1, whenever $((x_n),\mathcal{I})$ is compatible the pulled-back topology on $\Phi(C(X,Y))$ agrees with τ_u regardless of the ideal $\mathcal{I}$ or of whether (x_n) is injective; injectivity of (x_n) plays no role in this equality; compatibility is the operative hypothesis.

Theorem 2. *If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ are admissible ideals (with the same fixed sampling sequence (x_n)), then $\tau_{\mathcal{I}_2} \subseteq \tau_{\mathcal{I}_1}$.*

Proof. If $D_\varepsilon(f,g) \in \mathcal{I}_1$, then $D_\varepsilon(f,g) \in \mathcal{I}_2$, so $E^{\mathcal{I}_1}_\varepsilon \subseteq E^{\mathcal{I}_2}_\varepsilon$. Hence $\mathcal{U}_{\mathcal{I}_1} \subseteq \mathcal{U}_{\mathcal{I}_2}$, giving $\tau_{\mathcal{I}_2} \subseteq \tau_{\mathcal{I}_1}$. $\square$

Remark 1. Monotonicity cannot be strengthened to strict inclusion under a compatibility hypothesis on both ideals. Indeed, if $((x_n),\mathcal{I}_1)$ and $((x_n),\mathcal{I}_2)$ are both compatible, Theorem 1(ii) gives $\tau_{\mathcal{I}_1} = \tau_u = \tau_{\mathcal{I}_2}$ regardless of how $\mathcal{I}_1$ and $\mathcal{I}_2$ compare as ideals, so $\tau_{\mathcal{I}_2} \subsetneq \tau_{\mathcal{I}_1}$ is impossible in that regime. A genuine strict-monotonicity phenomenon, if it exists at all, must therefore be sought outside the compatible regime, where the identification with τ_u breaks down (Example 1); we do not pursue this here, and leave a correct strict-monotonicity statement — together with a valid separating construction, since a naive Urysohn-lemma argument runs into the difficulties detailed in the referee report accompanying this revision (disjointness of $\{x_n : n \in A\}$ and its complement is not implied by $A \notin \mathcal{I}_1$, and a two-point real-valued separator does not extend to general metric Y) — for future work.

Theorem 3. *Assume that $((x_n),\mathcal{I})$ is compatible. Then $\tau_p \subseteq \tau_\mathcal{I} = \tau_u$.*

Proof. By Theorem 1(ii), $\tau_\mathcal{I} = \tau_u$. It remains to prove that $\tau_p \subseteq \tau_\mathcal{I}$.

Let $W = \{h \in C(X,Y) : p(f(x_0), h(x_0)) < \varepsilon\}$ be a basic τ_p-open set and let $g \in W$. Set $\delta = \varepsilon - p(f(x_0), g(x_0)) > 0$ and $\eta = \delta/5$.

By continuity of f and g at x_0, choose $r > 0$ so that

$$p(f(x), f(x_0)) < \frac{\delta}{5} \quad \text{and} \quad p(g(x), g(x_0)) < \frac{\delta}{5} \qquad \text{for all } x \in B(x_0, r).$$

By compatibility, $N_0 = \{n : x_n \in B(x_0, r)\} \notin \mathcal{I}$.

Now let $h \in E_\eta[g]$, so $C_h = D_\eta(g,h) \in \mathcal{I}$. By continuity of h at x_0, choose $r_h > 0$ so that $p(h(x), h(x_0)) < \delta/5$ on $B(x_0, r_h)$. Set $r^* = \min(r, r_h)$. By compatibility, $N^* = \{n : x_n \in B(x_0, r^*)\} \notin \mathcal{I}$. Since $C_h \in \mathcal{I}$, we have $N^* \setminus C_h \neq \varnothing$. Choose $n_* \in N^* \setminus C_h$. Then $x_{n_*} \in B(x_0, r^*)$ and $p(g(x_{n_*}), h(x_{n_*})) < \eta$.

Therefore,

$$
\begin{aligned}
p(f(x_0),h(x_0)) &\leq p(f(x_0),f(x_{n_*})) + p(f(x_{n_*}),g(x_{n_*})) + p(g(x_{n_*}),h(x_{n_*})) + p(h(x_{n_*}),h(x_0)) \\
&< \frac{\delta}{5} + \left(p(f(x_0),g(x_0)) + \frac{\delta}{5} + \frac{\delta}{5}\right) + \frac{\delta}{5} + \frac{\delta}{5} \\
&= p(f(x_0),g(x_0)) + \delta = \varepsilon.
\end{aligned}
$$

Hence $h \in W$, so $E_\eta[g] \subseteq W$. Thus W is $\tau_{\mathcal{I}}$-open and $\tau_p \subseteq \tau_{\mathcal{I}}$. □

The inclusion $\tau_p \subseteq \tau_{\mathcal{I}}$ need not be strict in general: if X is connected and Y is totally disconnected (e.g. $X = [0,1]$ and $Y = \{0,1\}$ with the discrete metric), every continuous $f : X \to Y$ is constant, so $C(X,Y)$ has only $|Y|$ elements and τ_p is already discrete there, forcing $\tau_p = \tau_u$ even though X is infinite. Under suitable hypotheses on X and Y, however, the inclusion can be strict, as the following classical example illustrates.

Example 2. Take $X = [0,1]$, $Y = \mathbb{R}$, and define $f_k(x) = k\,x\,(1-x)^k$. For $x = 0$ and $x = 1$, $f_k(x) = 0$ for every k; for $x \in (0,1)$, $f_k(x) \to 0$ as $k \to \infty$, since $(1-x)^k \to 0$ geometrically while the factor kx grows only linearly. Hence $f_k \to 0$ pointwise on all of $[0,1]$, so $f_k \to 0$ in τ_p. However, a direct computation shows f_k attains its maximum at $x = 1/(k+1)$, with

$$
\sup_{x\in[0,1]} f_k(x) = f_k\left(\frac{1}{k+1}\right) = \left(\frac{k}{k+1}\right)^{k+1} \xrightarrow[k\to\infty]{} \frac{1}{e} \neq 0,
$$

so $f_k \not\to 0$ in τ_u. Since $\tau_{\mathcal{I}} = \tau_u$ under compatibility by Theorem 1(ii), this example witnesses $\tau_p \subsetneq \tau_{\mathcal{I}}$.

3. Convergence and ideal convergence

This section describes convergence in $\tau_{\mathcal{I}}$ in terms of pointwise and uniform ideal convergence and establishes a hierarchy among these notions.

By the definition of the topology induced by $\mathcal{U}_{\mathcal{I}}$, a sequence $(f_k) \subseteq C(X,Y)$ converges to f in $\tau_{\mathcal{I}}$ if and only if, for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$
D_\varepsilon(f_k, f) \in \mathcal{I},
$$

for all $k \geq K$. Under compatibility, this condition is equivalent to uniform convergence by Theorem 1(ii).

Definition 4. Let $(f_k) \subseteq C(X,Y)$ and $f \in C(X,Y)$. Following Balcerzak, Dems and Komisarski [14], we say

(i) (f_k) *converges $\mathcal{I}$-pointwise* to f, written $f_k \xrightarrow{\mathcal{I}\text{-}p} f$, if for every $x \in X$ and $\varepsilon > 0$,

$$
(\forall\, x \in X)\ (\forall\, \varepsilon > 0)\ (\exists\, M \in \mathcal{I})\ (\forall\, k \notin M): \quad p(f_k(x), f(x)) < \varepsilon.
$$

(ii) (f_k) *converges $\mathcal{I}$-uniformly* to f, written $f_k \xrightarrow{\mathcal{I}\text{-}u} f$, if for every $\varepsilon > 0$,

$$
(\forall\, \varepsilon > 0)\ (\exists\, M \in \mathcal{I})\ (\forall\, k \notin M)\ (\forall\, x \in X): \quad p(f_k(x), f(x)) < \varepsilon.
$$

We introduce the following new notion.

(iii) (f_k) *$\mathcal{I}$-converges in $\tau_{\mathcal{I}}$* to f, written $f_k \xrightarrow{\mathcal{I}\text{-}\tau_{\mathcal{I}}} f$, if for every $\varepsilon > 0$,

$$
\{k \in \mathbb{N} : D_\varepsilon(f_k, f) \notin \mathcal{I}\} \in \mathcal{I}.
$$

Condition (iii) is the usual $\mathcal{I}$-convergence of (f_k) to f with respect to the topology $\tau_{\mathcal{I}}$. Under compatibility, it coincides with $\mathcal{I}$-uniform convergence.

Theorem 4. *Assume that $((x_n), \mathcal{I})$ is compatible. Then, for every $(f_k) \subseteq C(X,Y)$ and $f \in C(X,Y)$,*

$$f_k \xrightarrow{\mathcal{I}\text{-}u} f \iff f_k \xrightarrow{\mathcal{I}\text{-}\tau_{\mathcal{I}}} f \implies f_k \xrightarrow{\mathcal{I}\text{-}p} f.$$

Proof. By definition, $f_k \xrightarrow{\mathcal{I}\text{-}u} f$ if and only if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : \sup_{x \in X} p(f_k(x), f(x)) \geq \varepsilon\} \in \mathcal{I}.$$

Thus, $f_k \xrightarrow{\mathcal{I}\text{-}u} f$ is precisely $\mathcal{I}$-convergence of (f_k) to f in τ_u. On the other hand, $f_k \xrightarrow{\mathcal{I}\text{-}\tau_{\mathcal{I}}} f$ means that, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : D_\varepsilon(f_k, f) \notin \mathcal{I}\} \in \mathcal{I}.$$

By Theorem 1(ii), compatibility gives $\tau_{\mathcal{I}} = \tau_u$. Hence

$$f_k \xrightarrow{\mathcal{I}\text{-}u} f \iff f_k \xrightarrow{\mathcal{I}\text{-}\tau_{\mathcal{I}}} f.$$

It remains to prove that $f_k \xrightarrow{\mathcal{I}\text{-}\tau_{\mathcal{I}}} f$ implies $f_k \xrightarrow{\mathcal{I}\text{-}p} f$. Fix $x_0 \in X$ and $\varepsilon > 0$. Set

$$D = \{k \in \mathbb{N} : D_{\varepsilon/2}(f_k, f) \notin \mathcal{I}\} \in \mathcal{I}.$$

For each $k \notin D$, put

$$A_k = D_{\varepsilon/2}(f_k, f) \in \mathcal{I}.$$

For $r > 0$, let

$$N_r = \{n \in \mathbb{N} : x_n \in B(x_0, r)\}.$$

By compatibility, $N_r \notin \mathcal{I}$. Since $A_k \in \mathcal{I}$, we have $N_r \setminus A_k \neq \varnothing$. Choose $n_r \in N_r \setminus A_k$. Then $x_{n_r} \to x_0$ and

$$p(f_k(x_{n_r}), f(x_{n_r})) < \frac{\varepsilon}{2}.$$

Therefore,

$$p(f_k(x_0), f(x_0)) \leq p(f_k(x_0), f_k(x_{n_r})) + p(f_k(x_{n_r}), f(x_{n_r})) + p(f(x_{n_r}), f(x_0)).$$

Since f_k and f are continuous at x_0 and $x_{n_r} \to x_0$,

$$p(f_k(x_0), f_k(x_{n_r})) \to 0 \quad \text{and} \quad p(f(x_{n_r}), f(x_0)) \to 0.$$

Hence, letting $r \to 0$,

$$p(f_k(x_0), f(x_0)) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus

$$\{k \in \mathbb{N} : p(f_k(x_0), f(x_0)) \geq \varepsilon\} \subseteq D \in \mathcal{I}.$$

Since x_0 and ε were arbitrary, $f_k \xrightarrow{\mathcal{I}\text{-}p} f$. □

We briefly comment on the continuity of limits in the present setting. Since $\tau_{\mathcal{I}}$ is a topology on $C(X,Y)$, convergence in $\tau_{\mathcal{I}}$ is defined only for limits belonging to $C(X,Y)$. Thus, under compatibility, Theorem 1(ii) identifies $\tau_{\mathcal{I}}$ with τ_u, and the corresponding limit result is the classical uniform limit theorem.

If one extends $D_\varepsilon(f,g)$ to an arbitrary map $f : X \to Y$, without assuming $f \in C(X,Y)$, continuity of the limit can no longer be deduced from the sampled condition alone. Indeed, the following example shows that the condition may hold even for a discontinuous limit.

Example 3. Assume that $((x_n), \mathcal{I})$ is compatible and that $X \setminus \{x_n : n \in \mathbb{N}\} \neq \varnothing$. Let $f_k \equiv y_0 \in Y$ for all k, and choose $z \in X \setminus \{x_n : n \in \mathbb{N}\}$. Define $f : X \to Y$ by $f(x_n) = y_0$ for every n and $f(z) \neq y_0$, with $f = y_0$ elsewhere. Then f is discontinuous, whereas

$$D_\varepsilon(f_k, f) = \varnothing \in \mathcal{I},$$

for every k and every $\varepsilon > 0$. Hence the sampled entourage condition alone does not imply continuity of an arbitrary target map.

Thus, no new ideal analogue of the uniform limit theorem is obtained beyond the classical uniform limit theorem under the compatibility assumption.

4. The Fréchet–Urysohn property

This section establishes that, under compatibility, $C_{\mathcal{I}}(X, Y)$ is both Fréchet–Urysohn and strictly Fréchet–Urysohn as a direct consequence of metrisability. Throughout, $((x_n), \mathcal{I})$ is assumed compatible.

A topological space Z is *Fréchet–Urysohn* (FU) if for every $A \subseteq Z$ and every $z \in \overline{A}$ there exists a sequence in A converging to z. It is *strictly Fréchet–Urysohn* (SFU) if, whenever $z \in \bigcap_{k \geq 1} \overline{A_k}$, there exists a sequence (a_k) with $a_k \in A_k$ for each k and $a_k \to z$.

Proposition 2. *The uniform structure $\mathcal{U}_{\mathcal{I}}$ has the countable base $\{E_{1/m} : m \in \mathbb{N}\}$, so $C_{\mathcal{I}}(X, Y)$ is pseudometrisable; if $((x_n), \mathcal{I})$ is compatible, $\mathcal{U}_{\mathcal{I}}$ is Hausdorff and hence $C_{\mathcal{I}}(X, Y)$ is metrisable.*

Proof. $\{E_\varepsilon : \varepsilon > 0\}$ is already a base by Proposition 1; since $D_\varepsilon(f, g)$ is non-increasing in ε and every $\varepsilon > 0$ admits $1/m < \varepsilon$, the countable subfamily $\{E_{1/m}\}$ is cofinal, hence also a base. A uniform structure with a countable base is pseudometrisable, and Hausdorff pseudometrisable uniform spaces are metrisable. □

Theorem 5. *If $((x_n), \mathcal{I})$ is compatible, then $C_{\mathcal{I}}(X, Y)$ is Fréchet–Urysohn and strictly Fréchet–Urysohn.*

Proof. By Proposition 2, $C_{\mathcal{I}}(X, Y)$ is metrisable, hence first countable. Every first-countable space is Fréchet–Urysohn: if $f \in \overline{A}$, a countable neighbourhood base $\{E_{1/m}[f]\}$ at f meets A for every m, and choosing $f_m \in A \cap E_{1/m}[f]$ gives $f_m \to f$. The same diagonal argument gives SFU: if $f \in \bigcap_k \overline{A_k}$, choose $f_k \in A_k \cap E_{1/k}[f]$ for each k; then $f_k \to f$, since for every m and every $k \geq m$, $f_k \in E_{1/k}[f] \subseteq E_{1/m}[f]$. □

This settles the property unconditionally on X: no covering hypothesis on X is needed once compatibility holds, because compatibility already forces $\mathcal{U}_{\mathcal{I}}$ to be a Hausdorff uniformity with a countable base.

Remark 2. Under the compatibility assumption, the Fréchet–Urysohn and strictly Fréchet–Urysohn properties of $C_{\mathcal{I}}(X, Y)$ are automatic, since $C_{\mathcal{I}}(X, Y)$ is metrizable. Thus, these properties do not impose any additional covering condition on X. Consequently, an equivalence between FU (or SFU) and an $\mathcal{I}$-γ-cover property of X cannot hold in this setting. In particular, the reverse implications of a covering-based characterization cannot be obtained: the function-space properties follow from metrizability, independently of any $\mathcal{I}$-γ-cover property of X.

5. Conclusion and future work

We have introduced the topology $\tau_{\mathcal{I}}$ on $C(X, Y)$ through the uniform structure generated by the entourages $\{E_\varepsilon : \varepsilon > 0\}$. The main conclusion is that, under the compatibility condition, $\tau_{\mathcal{I}}$ coincides with the uniform topology τ_u. Consequently, the resulting function space is metrizable, Fréchet–Urysohn, and strictly Fréchet–Urysohn. The convergence results also show that $\mathcal{I}$-uniform convergence and $\mathcal{I}$-convergence in $\tau_{\mathcal{I}}$ coincide, while both imply $\mathcal{I}$-pointwise convergence.

The compatibility assumption is therefore essential for the structure developed in this paper. Without compatibility, the topology may be non-Hausdorff and strictly coarser than the uniform topology, and its properties are no longer determined solely by the uniform structure of $C(X, Y)$. This suggests that the non-compatible case provides the more genuinely new setting for further investigation.

In particular, it would be natural to study the convergence, separation, and Fréchet–Urysohn properties of $\tau_{\mathcal{I}}$ without compatibility, as well as the dependence of these properties on the interaction between the ideal $\mathcal{I}$ and the sampling sequence (x_n). Extending related statistical convergence and continuity results to this non-compatible setting is another possible direction for future work.

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